Constant roll and primordial black holes

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based on *JCAP 03 (2020) 03, 002* with H. Motohashi and S. Mukohyama,

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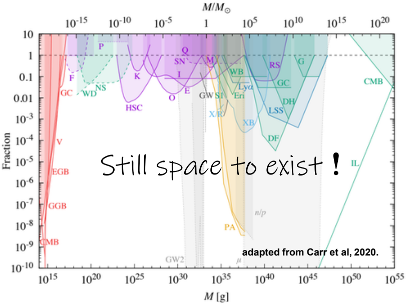
Introduction



The (non-)presence of **primordial black holes** may radically affect our understanding of cosmology, e.g.

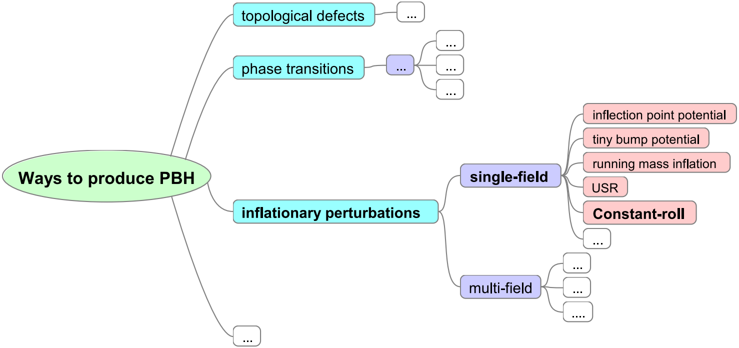
- as dark matter, LIGO-Virgo candidates,
- as sources via Hawking radiation, etc.

In this talk, we contribute a little to the following question:

What mechanism can produce PBH?

Ways to produce PBH

There are already several (review Sasaki et al. 2018): ...



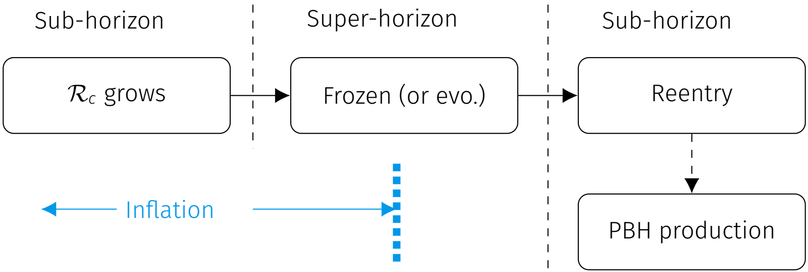
Here we focus on **single-field inflation models**.

PBH from inflation : formation threshold

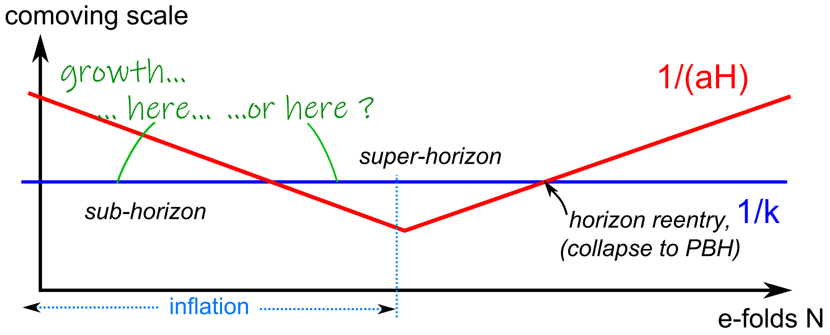
For a large enough density contrast

$$\Delta \equiv \frac{\rho - \bar{\rho}}{\bar{\rho}} \gtrsim \Delta_c \,, \qquad \Delta_c \sim 0.4 \text{ (from various studies)}, \qquad (1)$$

collapse may lead to PBH. This can be related to curvature perturbations ($\Delta \propto \nabla^2 \mathcal{R}_c$).



Amplification can occur either during sub-horizon or during super-horizon.



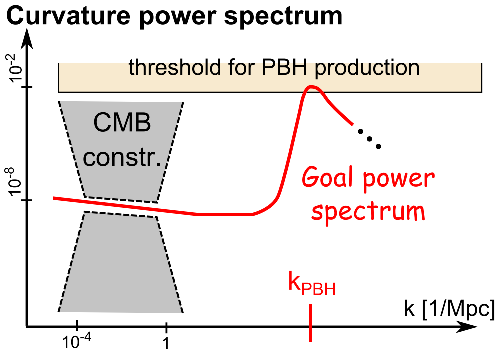
Power spectrum amplification, two dangers

One should be careful with growing curvature perturbation:

1. Power spectrum is constrained, e.g. at CMB scales.
2. Danger of overproduction of PBH.

Goal:

Have a **steep increase in the primordial curvature power spectrum**, on **scales smaller than CMB ones**, and **decrease** after that.



The threshold is at $\Delta_\zeta^2 \sim 10^{-2}$ (COBE-normalized).

In order to characterize inflation, we use here the **Hubble hierarchy**

$$\epsilon_1 \equiv -\frac{\dot{H}}{H^2}\,, \qquad \epsilon_{i+1} \equiv \frac{\dot{\epsilon}_i}{H\epsilon_i}\,, \tag{2}$$

These are important because H, ϵ_1, ϵ_2, and ϵ_3 fully characterize the background contribution to the inflationary perturbations.

- To have inflation, one needs at least $|\epsilon_1| \ll 1$.
- The slow-roll (SR) regime has $|\epsilon_i| \ll 1$.

Going beyond slow-roll

Even though in the slow-roll approximation we have

$$\Delta_\zeta^2 \approx \frac{H^2}{8\pi^2\epsilon_1^2} \tag{3}$$

one still needs to make the spectrum grow fast enough to have massive enough PBH. Motohashi and Hu 2017 suggest

$$\left|\frac{\Delta \ln \epsilon_1}{\Delta N}\right| \sim |\bar{\epsilon_2}| > 0.38 \tag{4}$$

by taking into account PBH large enough so that they do not evaporate fully and contribute to dark matter.

This indicates that the SR approximations has to be violated !

Ultra-slow-roll

A typical example (Germani and Prokopec 2017; Motohashi and Hu 2017) is an **ultra-slow-roll** (USR) (flat potential) stage of inflation.
The second SR condition is violated as

$$\epsilon_2 \equiv \frac{\dot{\epsilon}_1}{H\epsilon_1} \to -6\,, \tag{5}$$

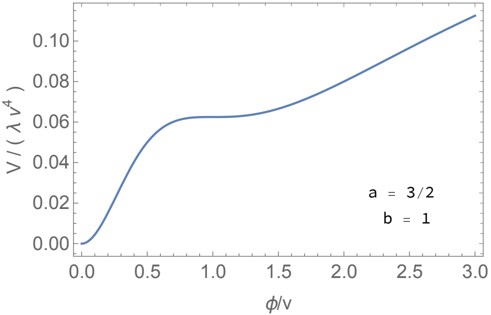
instead of $|\epsilon_2| \ll 1$ in SR ($\epsilon \equiv -\frac{\dot{H}}{H^2}$).

- To be precise, from $\ddot{\phi} = -3H\dot{\phi}$ in ultra slow-roll one has $\dot{\phi} \propto \sqrt{\epsilon_1} \propto a^{-3}$.
- $\epsilon_1 \propto a^{-6}$ also means that there is a growing mode in the super-horizon regime, so that the evaluation of the curvature power spectrum is more subtle (but tractable).

Example: single-field inflection models

$$\text{E.g.} \quad V(\phi) = \frac{\lambda}{12}\phi^2 v^2 \frac{6 - 4a\frac{\phi}{v} + 3\frac{\phi^2}{v^2}}{\left(1 + b\frac{\phi^2}{v^2}\right)^2}, \quad \text{(Garcia-Bellido and Morales 2017).} \qquad (6)$$

Inflection point on the potential causes a peak in the power spectrum.



This model well described by a transient USR stage, but has problems fitting both CMB data and produce enough PBH.

Constant-roll

Constant-roll

Our idea:
Use a blue-tilted **constant-roll** (CR) model (Martin et al. 2012).

$$\text{CR condition: } \frac{\ddot{\phi}}{H\dot{\phi}} \equiv \beta = \frac{1}{2}\epsilon_2 - \epsilon_1 \overset{!}{=} cst. \tag{7}$$

This is interesting because

- As USR, it allows for $|\epsilon_2| = \mathcal{O}(1)$.
- The model also includes an USR potential and a SR potential as limits ($\beta \to -3$ and $\beta \to 0$ respectively).

One can build an analytic potential using the **Hamilton-Jacobi approach**. We review this approach by following Motohashi et al. 2014.

Hamilton-Jacobi construction of CR

By assuming that one can write (here we use $\varphi \equiv \phi/M_{\rm P}$)

$$\dot{H} = \frac{dH}{d\varphi}\dot{\varphi}\,, \tag{8}$$

one may then use the second Einstein equation $\dot{H} = -\frac{1}{2}\dot{\varphi}^2$ to write

$$\frac{dH}{d\varphi} = -\frac{1}{2}\dot{\varphi}\,. \tag{9}$$

Finally, using the CR condition (7), leads to

$$\frac{d^2H}{d\varphi^2} = -\frac{1}{2}\beta H\,, \tag{10}$$

solved by

$$H(\varphi) = C_1 \exp\left(\sqrt{-\frac{1}{2}\beta}\varphi\right) + C_2 \exp\left(-\sqrt{-\frac{1}{2}\beta}\varphi\right)\,. \tag{11}$$

Here we focus on the range $-3 < \beta < 0$ which is between SR and USR.

CR potentials

Using the Friedmann equation as well as (9), one can deduce several potentials. Choosing C_1 and C_2 leads to (some examples)

$$V_{\mathrm{CR1}} = M^2 M_{\mathrm{P}}^2 (\beta + 3) \exp\left(\sqrt{-2\beta}\varphi\right) , \tag{12}$$

$$V_{\mathrm{CR2}} = -3M^2 M_{\mathrm{P}}^2 \left\{1 - \frac{3+\beta}{6}\left[1 + \cosh\left(\sqrt{-2\beta}\varphi\right)\right]\right\}, \tag{13}$$

$$V_{\mathrm{CR3}} = 3M^2 M_{\mathrm{P}}^2 \left\{1 - \frac{3+\beta}{6}\left[1 - \cosh\left(\sqrt{-2\beta}\varphi\right)\right]\right\}, \tag{14}$$

$$\cdots$$

All these potentials are interesting per se (but do not always lead to the usual inflation, see Motohashi et al. 2014). **We however choose V_{CR3}.**

Analytic solution

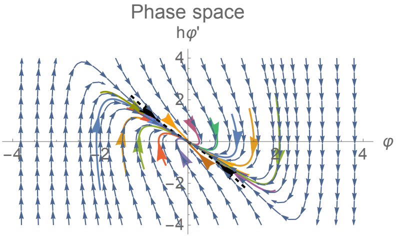
$$V_{\mathrm{CR}} = 3M^2 M_{\mathrm{P}}^2 \left\{1 - \frac{3+\beta}{6}\left[1 - \cosh\left(\sqrt{-2\beta}\varphi\right)\right]\right\}. \qquad (15)$$

Analytic background solution

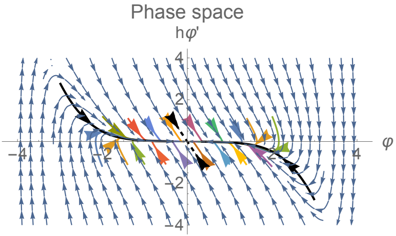
$$\begin{aligned} a &= a_0 \sinh^{-1/\beta}(-\beta M t) \\ \varphi &= \sqrt{-\frac{2}{\beta}} \ln\left[\coth\left(-\frac{\beta}{2} M t\right)\right] \\ &\simeq \sqrt{-\frac{2}{\beta}} \ln\left[1 + \left(\frac{a}{a_0}\right)^{\beta}\right] \quad @ a \gg a_0 \,, \end{aligned}$$

The analytic solution (dashed black) is an **attractor for** $\beta > -\frac{3}{2}$. For $\beta < -\frac{3}{2}$ they are a transient before a slow-roll phase (full black).

E.g. for $\beta = -\frac{7}{5}$,



But for $\beta = -3 + \frac{1}{50}$,



See also Lin et al. 2019 for more details on this.

MS equation in constant roll

To study the curvature perturbations (we use the comoving gauge with $\zeta = \mathcal{R}_c$)

$$ds^2 = a^2(-d\tau^2(1+2\alpha) + \delta_{ij}(1+2\zeta)dx^i dx^j) \tag{16}$$

the Mukhanov-Sasaki equation

$$\frac{d^2 v_k}{d\tau^2} + \left(k^2 - \frac{1}{z}\frac{d^2 z}{d\tau^2}\right) v_k = 0\,, \quad v_k \equiv \sqrt{2} M_{\rm P} z \zeta_k\,, \quad z \equiv a\sqrt{\epsilon_1} \tag{17}$$

tells us that $\epsilon_{1,2,3}$ from the Hubble hierarchy are relevant, as

$$\frac{1}{z}\frac{d^2 z}{d\tau^2} = a^2 H^2 \left(2 - \epsilon_1 + \frac{3}{2}\epsilon_2 + \frac{1}{4}\epsilon_2^2 - \frac{1}{2}\epsilon_1\epsilon_2 + \frac{1}{2}\epsilon_2\epsilon_3\right)\,. \tag{18}$$

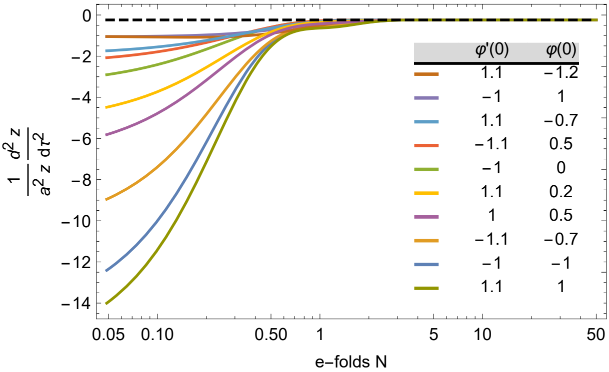
With the potential (15),

$$\epsilon_1 = \frac{1}{2}\epsilon_{2n+1} \simeq -\beta a^{2\beta}\,, \qquad \epsilon_2 = \epsilon_{2n} \simeq 2\beta \tag{19}$$

after some time, and hence $\frac{1}{a^2 H^2}\frac{1}{z}\frac{d^2 z}{d\tau^2} \simeq (1+\beta)(2+\beta)$.

How fast does one reach the attractor?

$\frac{1}{z}\frac{d^2z}{d\tau^2}$ gives a good criterion to estimate how fast one reaches the attractor starting from arbitrary initial conditions.



$\rightsquigarrow \mathcal{O}(1)$ e-folds, which will be useful for a **SR-to-CR transition**.

One may first show that, perturbations are frozen on super-horizon scales for $\beta > -\frac{3}{2}$.

After solving the MS equation as usual with Hankel functions, one finds

$$n_s - 1 = 3 - |2\beta + 3|\,, \quad \text{where} \quad \Delta_\zeta^2(k) = \frac{k^3}{2\pi^2}|\zeta_k|^2_{\text{@super-horizon}} \propto \left(\frac{k}{k_\star}\right)^{n_s - 1}\,, \qquad (20)$$

hence **a maximum of** $n_s - 1 \approx 3$ **for** $\beta \to -\frac{3}{2}$.

Summary

β		-3 (USR)	$-3/2$	0 (SR)	
n_s	red	$\vert$	blue	$\vert$	red
$k \ll aH$		growth	$\vert$	frozen	
attractor?		no	$\vert$	yes	

A note: USR from CR

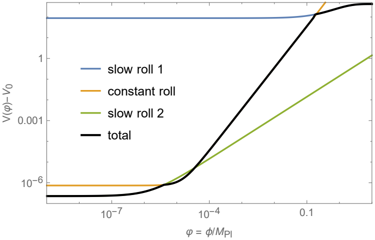
By taking $\beta = -3$, one obtains a constant potential ! Hence one may assume that CR contains USR.

According to the analysis of Pattison et al. 2018, CR analyses of USR instead miss the cases in which $\beta = -3$ is reached only asymptotically, and their analysis indicates that USR may be an attractor in some cases.

PBH formation setup

The 3-phase potential:

$$V(\varphi) \equiv \frac{1}{4} V_{\mathrm{CR}}(\varphi) \left[1 - \tanh\left(\frac{\varphi - \varphi_1}{d_1}\right)\right] \left[1 - \tanh\left(\frac{\varphi_2 - \varphi}{d_2}\right)\right]$$
$$+ \frac{1}{2} V_{\mathrm{Starobinsky}}(\varphi) \left[1 - \tanh\left(\frac{\varphi_1 - \varphi}{d_1}\right)\right]$$
$$+ \frac{1}{2} \left(W_{\mathrm{SR2}}\phi + \Lambda_{\mathrm{SR2}}\right) \left[1 - \tanh\left(\frac{\varphi - \varphi_2}{d_2}\right)\right], \qquad (21)$$



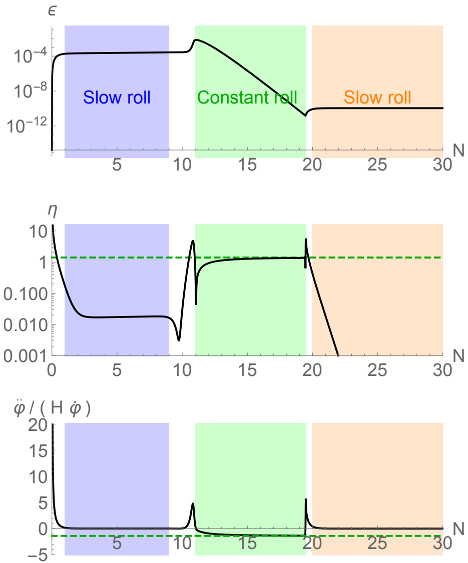
The 3 phases are built to last at least 5 − 10 e-folds. For the transitions tanh-type step functions are chosen, here with width

$$d_1 \sim 10^{-2}, \qquad d_2 \sim 10^{-7} \qquad (22)$$

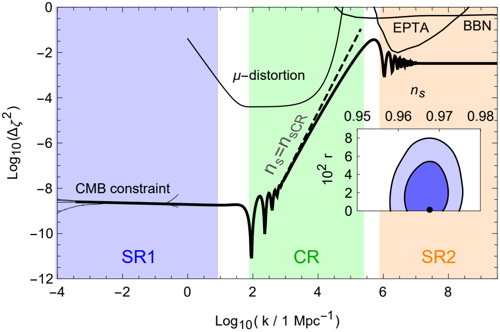
We plot ϵ_1, ϵ_2, as well as the **rate of roll**. The green dashed line corresponds to the pure constant roll values.

As desired, the SR and CR stages last for relatively long. Transitions here last a little less $\mathcal{O}(1)$ e-folds.

The transient periods will have an impact on the power spectrum.



Using the freedom to chose the scale of inflation, M, and a time to end inflation, we get the curvature power spectrum



Parameters:

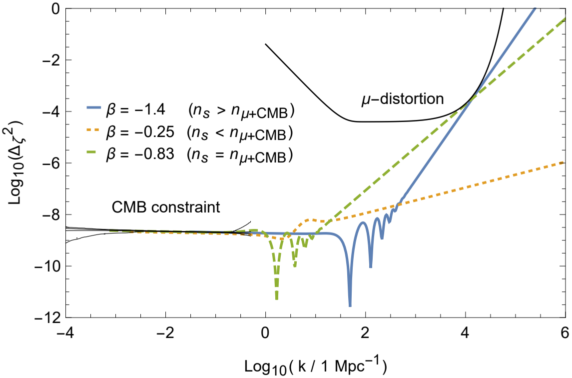
$$\beta = -1.4\,, \qquad \frac{m^2}{M^2} = 3.13,$$
$$\frac{W_{\rm SR2}}{V'_{\rm CR}(\varphi_2)} = 5\,, \quad \varphi_s = -5\,,$$
$$\varphi_1 = 0.175 \qquad \varphi_2 = 4.5 \times 10^{-6}\,,$$
$$d_1 = 10^{-2}, \qquad d_2 = 10^{-7}, \qquad (23)$$

Remember:

$$n_{s\rm CR} - 1 = -2\beta\,, \qquad -\frac{3}{2} < \beta < 0\,. \qquad (24)$$

Constraints on the power spectrum

In addition to CMB constraints, CMB spectral distortions are also very relevant.



A measurement of PBH masses can constrain the tilt n_s during CR (thus β). Note that other constraints exist at smaller scales (e.g. second order GW).

PBH abundance equations

We follow the standard treatment (see e.g. Inomata et al. 2017 and references therein) which relies on the variance of the matter contrast distribution,

$$\sigma^2(k) = \frac{16}{81} \int d\ln q\, W^2(q/k)\,(q/k)^4\,\Delta_\zeta^2(q)\,, \tag{25}$$

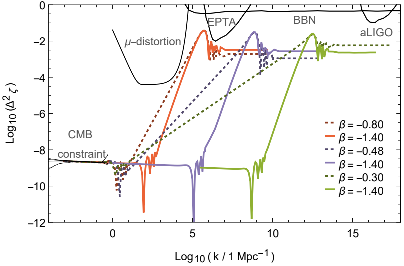
with the window function $W(x) = e^{-\frac{x^2}{2}}$. This then allows for an estimation of the PBH formation rate (Gaussian statistics assumed)

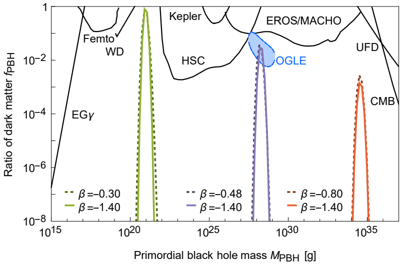
$$\beta_{\rm PBH}(M_{\rm PBH}) \simeq \frac{1}{\sqrt{2\pi}} \frac{1}{\delta_c/\sigma(M_{\rm PBH})} e^{-\frac{\delta_c^2}{2\sigma^2(M_{\rm PBH})}} \tag{26}$$

where the PBH mass $M_{\rm PBH}$ can be related to the wavenumber via

$$M_{\rm PBH}(k) \simeq 10^{20}\mathrm{g}\left(\frac{\gamma}{0.2}\right)\left(\frac{g_*}{106.75}\right)^{-\frac{1}{6}}\left(\frac{k}{7\cdot 10^{12}\mathrm{Mpc}^{-1}}\right)^{-2} \tag{27}$$

PBH mass function





- Lower PBH masses are achieved by transitioning later to CR or via lower slopes.
- Lower slopes can be interesting if one detects a small scale power spectrum amplification in the future, e.g. at the same scales as μ-distortion ones.

What makes CR different?

Conceptually:

- Several single-field models that aim at producing PBH rely on a **non-attractor transient phase** to amplify the power spectrum.
- In contrast, **the constant-roll stage is not a transient**; it lasts enough e-folds to reach an attractor behaviour.

Implementation:

- This scenario provides a **potential**, going a small step further than previous more phenomenological approaches with an $\epsilon_2 \geq -3$ stage.

Maybe:

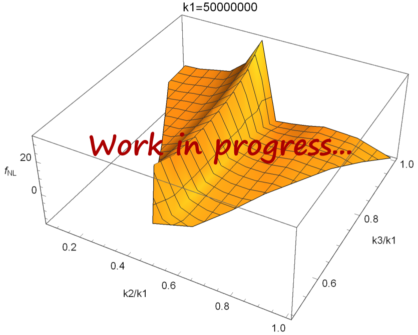
- This scenario may have a different behaviour with respect to either **non-Gaussianities** and **quantum diffusion**. Work in progress !

Outlook

We are currently working on: **non-Gaussianities**

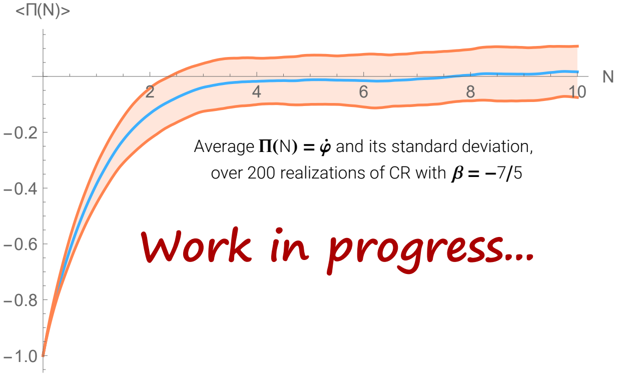
Non-Gaussianities can affect the the prediction for f_{PBH}. However is known (Passaglia et al. 2018) that in SR-USR-SR models they don't.

- ? How much NG during constant-roll phase?
- ? How much NG during transitions?



The mode $k_2 \approx 2 \times 10^7 M$ corresponds to the first transition.

We are currently working on: **quantum diffusion**
Quantum diffusion may also affect the the prediction for f_{PBH}.



Outlook

- Non gaussianities
- Quantum diffusion
- Relation with other works (e.g. Mishra and Sahni 2019; Byrnes et al. 2018)? Can we quantify the fine-tuning?
- Can we improve on the model?

☺ Thank you !

References

Byrnes, Christian T., Philippa S. Cole, and Subodh P. Patil (Nov. 27, 2018). "Steepest growth of the power spectrum and primordial black holes". In: *Journal of Cosmology and Astroparticle Physics* 2019.06, pp. 028–028. DOI: 10.1088/1475-7516/2019/06/028. arXiv: http://arxiv.org/abs/1811.11158v3 [astro-ph.CO].

Carrilho, Pedro, Karim A. Malik, and David J. Mulryne (July 11, 2019). "Dissecting the growth of the power spectrum for primordial black holes". In: arXiv: http://arxiv.org/abs/1907.05237v1 [astro-ph.CO].

Garcia-Bellido, Juan and Ester Ruiz Morales (Feb. 13, 2017). "Primordial black holes from single field models of inflation". In: *Physics Dark Universe 18 (2017) 47-54* 18, pp. 47–54. DOI: 10.1016/j.dark.2017.09.007. arXiv: http://arxiv.org/abs/1702.03901v6 [astro-ph.CO].

Germani, Cristiano and Tomislav Prokopec (June 13, 2017). "On primordial black holes from an inflection point". In: arXiv: http://arxiv.org/abs/1706.04226v3 [astro-ph.CO].

Hu, Wayne and Austin Joyce (Dec. 7, 2016). "Separate Universes beyond General Relativity". In: *Phys. Rev. D 95, 043529 (2017)* 95.4. DOI: 10.1103/PhysRevD.95.043529. arXiv: http://arxiv.org/abs/1612.02454v2 [astro-ph.CO].

Inomata, Keisuke et al. (Jan. 10, 2017). “Inflationary Primordial Black Holes as All Dark Matter”. In: *Phys. Rev. D 96, 043504 (2017)* 96.4. DOI: 10.1103/PhysRevD.96.043504. arXiv: http://arxiv.org/abs/1701.02544v3 [astro-ph.CO].

Lin, Wei-Chen, Michael J. P. Morse, and William H. Kinney (Apr. 12, 2019). “Dynamical Analysis of Attractor Behavior in Constant Roll Inflation”. In: arXiv: http://arxiv.org/abs/1904.06289v2 [astro-ph.CO].

Martin, Jerome, Hayato Motohashi, and Teruaki Suyama (Nov. 1, 2012). “Ultra Slow-Roll Inflation and the non-Gaussianity Consistency Relation”. In: *Phys.Rev. D87 (2013) 023514* 87.2. DOI: 10.1103/PhysRevD.87.023514. arXiv: http://arxiv.org/abs/1211.0083v1 [astro-ph.CO].

Mishra, Swagat S. and Varun Sahni (Oct. 31, 2019). “Primordial Black Holes from a tiny bump/dip in the Inflaton potential”. In: arXiv: http://arxiv.org/abs/1911.00057v3 [gr-qc].

Motohashi, Hayato and Wayne Hu (June 21, 2017). “Primordial Black Holes and Slow-Roll Violation”. In: *Phys. Rev. D 96, 063503 (2017)* 96.6. DOI: 10.1103/PhysRevD.96.063503. arXiv: http://arxiv.org/abs/1706.06784v2 [astro-ph.CO].

Motohashi, Hayato, Alexei A. Starobinsky, and Jun’ichi Yokoyama (Nov. 18, 2014). “Inflation with a constant rate of roll”. In: *JCAP 09 (2015) 018* 2015.09, pp. 018–018. DOI: 10.1088/1475-7516/2015/09/018. arXiv: http://arxiv.org/abs/1411.5021v3 [astro-ph.CO].

Passaglia, Samuel, Wayne Hu, and Hayato Motohashi (Dec. 19, 2018). "Primordial Black Holes and Local Non-Gaussianity in Canonical Inflation". In: *Phys. Rev. D 99, 043536 (2019)* 99.4. DOI: 10.1103/PhysRevD.99.043536. arXiv: http://arxiv.org/abs/1812.08243v2 [astro-ph.CO].

Pattison, Chris et al. (June 25, 2018). "The attractive behaviour of ultra-slow-roll inflation". In: *JCAP08(2018)048* 2018.08, pp. 048–048. DOI: 10.1088/1475-7516/2018/08/048. arXiv: http://arxiv.org/abs/1806.09553v2 [astro-ph.CO].

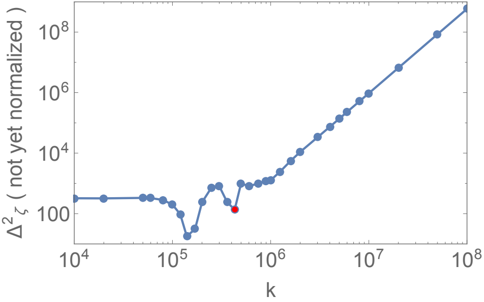
Sasaki, Misao et al. (Jan. 16, 2018). "Primordial Black Holes - Perspectives in Gravitational Wave Astronomy -". In: *Classical and Quantum Gravity* 35.6, p. 063001. DOI: 10.1088/1361-6382/aaa7b4. arXiv: http://arxiv.org/abs/1801.05235v1 [astro-ph.CO].

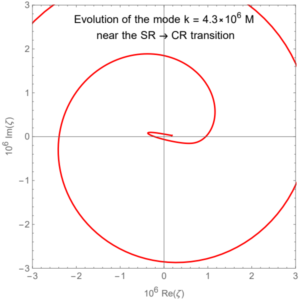
Transition SR → CR

The transition SR → CR will display **oscillations in the power spectrum**.
This may be understood from (Hu and Joyce 2016)

$$\frac{d}{dN}\zeta = -\frac{1}{a^3\epsilon H}\left[\int \frac{da}{a} a^3 \left(\frac{k}{aH}\right)^2 (\epsilon H)\,\zeta + cst.\right], \tag{28}$$

from a sharp increase of ϵ. This is followed by **the convergence of the integral @CR**.

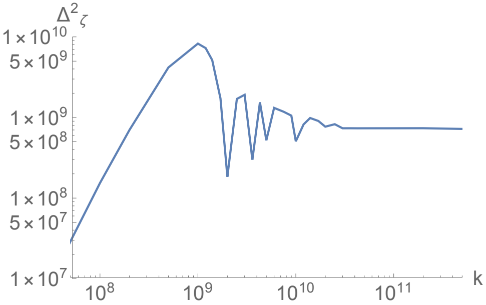




Transition CR $\rightarrow$ SR

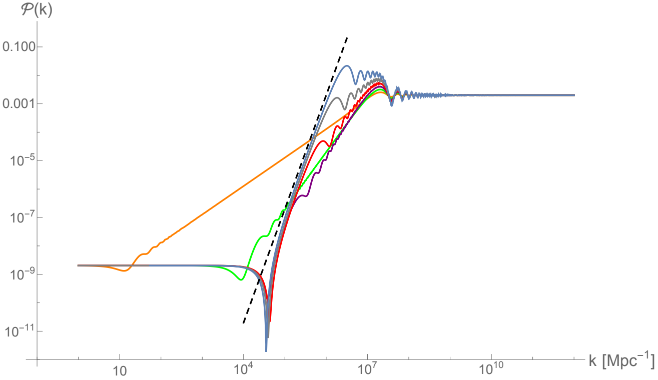
$$\frac{d}{dN}\zeta = -\frac{1}{a^3\epsilon H}\left[\int \frac{da}{a} a^3 \left(\frac{k}{aH}\right)^2 (\epsilon H)\,\zeta + cst.\right] \simeq -\frac{1}{a}\frac{k^2}{H^2a^2}\left[\int da\,\zeta + cst.\right], \tag{29}$$

The integral keeps growing in SR (since $\epsilon \sim cst.$). There are still some **oscillations**.



Other interesting works: Plot from (Byrnes et al. 2018)

Power spectrum based on a **more phenomenological model** (step function on ϵ_2), including USR stages (curves with a large dip) and simili-constant-roll stages (milder tilts).



Other interesting works: Plot from (Carrilho et al. 2019)

Simple SR-USR-SR setup.