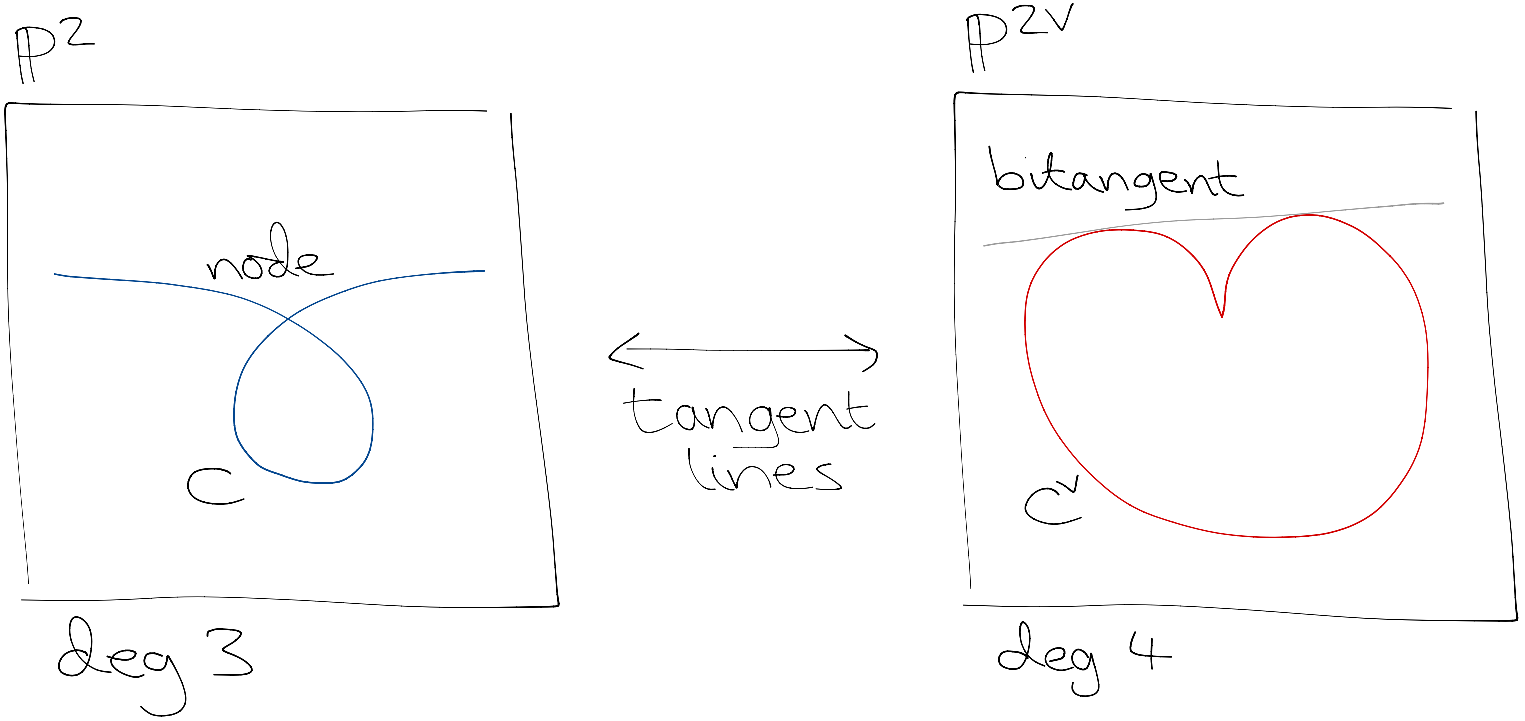
Windows on the Pfaffian-Grassmannian correspondence

Projective duality

Ex

[M. Monks]



Def $C^\vee = \{ L \in \mathbb{P}^{2\vee} \mid L \cap C \text{ singular} \}$

Pfaffian variety

$$\mathrm{Pf}_r = \{\operatorname{rk} M \le \underset{\text{even}}{r}\} \subset \{\text{skew-symm } n \times n \text{ matrices}\}$$

$$\cong \Lambda^2 V^\vee \qquad \dim V = n$$

$$=: W$$

Rem $\det M = (\operatorname{pf} M)^2$ Notation $\mathbb{P}\mathrm{Pf}_r \xleftarrow{\mathbb{P}} \mathrm{Pf}_r$

Projective duals: $\boxed{\mathbb{P}\mathrm{Pf}_r \subset \mathbb{P}W}$ $\boxed{\mathbb{P}W^\vee \supset \mathbb{P}\mathrm{Pf}_s}$

where $r + s (+1) = n$

Ex $\mathbb{P}\mathrm{Pf}_2 = \mathbb{P}\{f \wedge g\} = \mathrm{Gr}_2(V^\vee) \subset \mathbb{P}W$ (Plücker)

Note $\mathbb{P}\mathrm{Pf}_r$ singular for $r > 2$.

Mirror symmetry

Fix $n = 7$ (skew symm 7×7), $r = 2$

Projective duals: $\boxed{Gr_2(7) \overset{(\text{Plücker})}{=} \mathbb{P}f_2 \subset \mathbb{P}W}$ $\boxed{\mathbb{P}W^\vee \supset \mathbb{P}f_4}$

Note $\deg K_{Gr} = -7$, so take $\underset{\text{generic}}{\wedge}$ subspace $\Pi \underset{\text{codim } 7}{\subset} W$

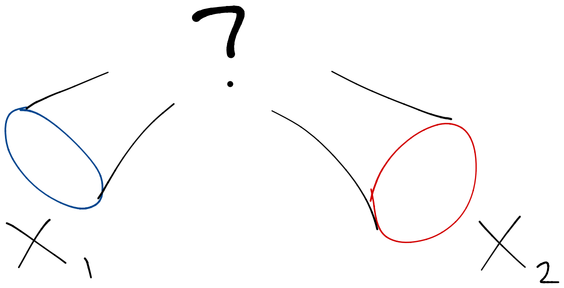
$$\xrightarrow{\text{cut}} \quad \overset{\text{dim } 10 - 7}{\boxed{Gr_2(7) \cap \mathbb{P}\Pi}} \qquad \overset{\text{dim } 6 - 3}{\boxed{\mathbb{P}\Pi^\perp \cap \mathbb{P}f_4}}$$

$$\qquad\qquad \uparrow \qquad\qquad\qquad\qquad \uparrow$$

Calabi-Yau $\underset{\text{smooth}}{\wedge}$ 3-folds $\qquad X_1 \qquad\qquad X_2$

Rem In general, CY $(n-4)$-folds

Conj X_i have same mirror,
(Rødland ~00)
correspond to LRLs, same moduli:



$X_1 = \boxed{Gr_2(7) \cap \mathbb{P}\Pi}$ $\quad X_2 = \boxed{\mathbb{P}\Pi^{\perp} \cap \mathbb{P}f_4}$

Conj X_i have same mirror

Rem X_i have $h^{1,1} = 1 \Rightarrow X_i$ not birational

Nevertheless, equiv B-brane categories:

Thm (Bor-Cal 06) $\Phi : \boxed{D(X_1)} \xleftarrow{\sim} \boxed{D(X_2)}$

$X_1 \supset \text{curve}_\psi \leftsquigarrow \psi \in \Lambda^2 V$

Hori-Tong 06 Obtain X_i from a non-abelian gauged linear σ-model

Add-Don-Seg 14 Reprove equiv with different Φ, inspired by

$X_1 = \boxed{Gr_2(7) \cap \mathbb{P}\Pi}$ $X_2 = \boxed{\mathbb{P}\Pi^{\perp} \cap \mathbb{P}f_4}$

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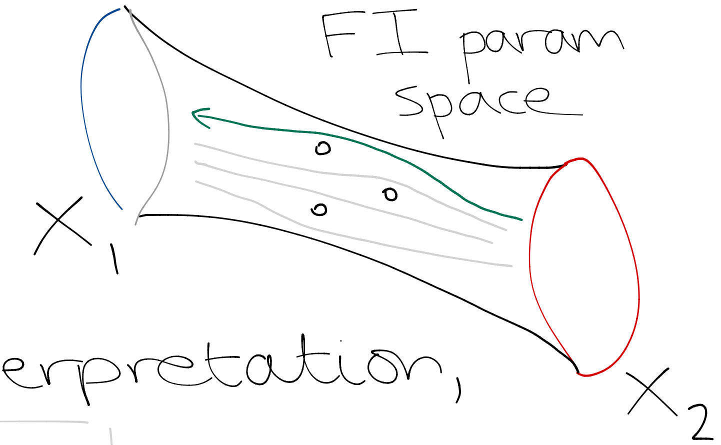
Hori, Eager-H-Knapp-Romo 16

Φ associated to $\boxed{\text{path}}$



This talk: explain math interpretation, and $\boxed{\text{other paths}}$

$X_1 = \boxed{Gr_2(7) \cap \mathbb{P}\Pi}$ $X_2 = \boxed{\mathbb{P}\Pi^\perp \cap \mathrm{Pf}_4}$

$$\boxed{\begin{array}{l} \underline{\text{Thm}} \text{ (Bor-Cal 06) } \underline{\Phi} : \boxed{D(X_1)} \xleftarrow{\sim} \boxed{D(X_2)} \\ \qquad\qquad\qquad\qquad X_1 \supset \text{curve}_\psi \leftsquigarrow \psi \in \Lambda^2 V \end{array}}$$

<u>Rennemo-Segal</u> 16

noncomm crepant resolution

$$\boxed{D(\widehat{\mathrm{Pf}_r \cap \mathbb{P}\Pi})} \quad \overset{\text{depending on } r,\, s}{\rightleftarrows} \quad \boxed{D(\widehat{\mathrm{Pf}_s \cap \mathbb{P}\Pi^\perp})}$$

<u>Note</u> Equiv if $\operatorname{codim}\Pi = \frac{1}{2} rn$ <u>Ex</u> Pf_2, $\operatorname{codim}\Pi = n$.

<u>Rem</u> Example of <u>homological</u> projective duality

Inspired by work of Hori

<u>Rem</u> Results for $\mathrm{Sym}^2 V$, compare Hosono-Takagi

Matrix factorizations

scheme $M \curvearrowleft \mathbb{C}^*$

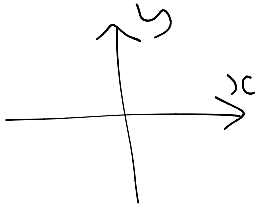
$\mathrm{id}_M \mapsfrom -1$

$f : M \to \mathbb{C}$ $\quad \mathrm{wt}_{\mathbb{C}^*} f = 2$

Def sheaf on (M, f) is $\mathbb{C}^*$-eqvt $\mathcal{O}_M$-mod $\mathcal{E}$ and $d : \mathcal{E} \circlearrowleft$ with $d^2 = f$

Obs $\mathcal{E} = \underset{\mathrm{wt}_{\mathbb{C}^*}\ \ \text{even}}{\mathcal{E}_0} \oplus \underset{\text{odd}}{\mathcal{E}_1}$ $\quad \mathrm{wt}_{\mathbb{C}^*} d = 1 \rightsquigarrow \quad \mathcal{E}_0 \underset{d}{\overset{d}{\rightleftarrows}} \mathcal{E}_1$

Ex $M = \mathbb{C}^2$ coords $\underset{\mathrm{wt}_{\mathbb{C}^*}\ 0}{x}$ $\underset{2}{y}$ $\quad f = xy$



(a) $\mathcal{O}_M \underset{y}{\overset{x}{\rightleftarrows}} \mathcal{O}_M[1]$ $\quad$ compare $\begin{pmatrix} & x \\ y & \end{pmatrix}\begin{pmatrix} & y \\ x & \end{pmatrix} = f\,\mathrm{id}$

(b) $\mathcal{O}_M \underset{f}{\overset{1}{\rightleftarrows}} \mathcal{O}_M[1]$

Def $D(M, f)$ Obj: sheaf $\mathcal{E}$ on (M, f) / acyclics

Def sheaf on (M,f) is $\mathbb{C}^*$-eqvt $\mathcal{O}_M$-mod $\mathcal{E}$ and $d : \mathcal{E} \circlearrowleft$ with $d^2 = f$

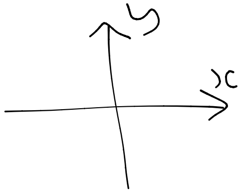
Def $D(M,f)$ Obj: sheaf $\mathcal{E}$ on (M,f) / acyclics

Ex $M \curvearrowleft \mathbb{C}^*$ trivial, $f \equiv 0$, get $D(M)$

Ex $f = xy$ (a) $\left(\mathcal{O}_M \underset{y}{\overset{x}{\rightleftarrows}} \mathcal{O}_M[1]\right) \cong \mathcal{O}_{\{y=0\}}, \mathcal{O}_{\{x=0\}}[1]$

(b) $\left(\mathcal{O}_M \underset{f}{\overset{1}{\rightleftarrows}} \mathcal{O}_M[1]\right) \cong \mathcal{O}_{Z_f} \cong 0$

Knorrer periodicity "$D(M,f)$ only sees Crit f"



$M = \mathbb{C}^{2n}$, $f = \sum_1^n x_i y_i$

$$\boxed{D(M,f) \cong D(pt)}$$

Rem compare $D_{sg}(Z_f)$

Ex $f = xy$ $\mathcal{E} = $ (a) generates, $\mathrm{Hom}(\mathcal{E},\mathcal{E}) = \mathbb{C}$

Knorrer periodicity "$D(M,f)$ only sees $\mathrm{Crit}\, f$"

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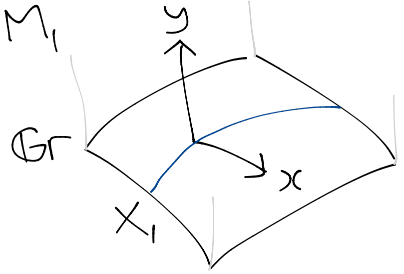
Calabi–Yau $X_1 = \boxed{Gr_2(7) \cap \mathbb{P}\Pi}$

$= $ zeroes of $x_1 \text{—} x_7$, sections of $\mathcal{O}_{Gr}(1)$

$$M_1 = \mathrm{Tot}\begin{pmatrix} \mathcal{O}(-1)^{\oplus 7} \\ \downarrow \\ Gr \end{pmatrix}$$

fibre coords y_i

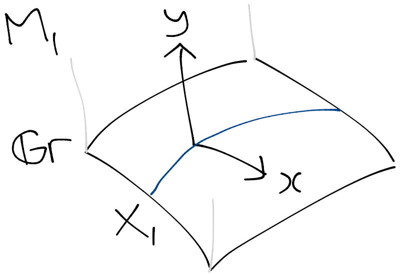
$f = \sum x_i y_i$



By family Knorrer (Shipman) $\boxed{D(M_1,f) \cong D(X_1)}$

$M_1 = \mathrm{Tot}\begin{pmatrix} \mathcal{O}(-1)^{\oplus 7} \\ \downarrow \\ \mathrm{Gr} \end{pmatrix}$ fibre coords: y_i

$f = \sum x_i y_i$



By family Knörrer (Shipman) $\boxed{D(M_1, f) \cong D(X_1)}$

Calabi-Yau $X_2 = \boxed{\mathbb{P}\Pi^{\perp} \cap \mathrm{Pf}_4} \subset \mathbb{P}^6$

GIT problem $\boxed{\mathcal{M} = [S^{\vee \oplus 7} \oplus \wedge^2 S^{\oplus 7} / GL(S)]}$ $\dim S = 2$

Quotients: M_1 and $M_2 = \mathrm{Tot}\begin{pmatrix} S^{\vee \oplus 7} \\ \downarrow \\ \mathcal{P} \end{pmatrix}$ where $\mathcal{P} = [\wedge^2 S^{\oplus 7} - 0 / GL(S)] \cong \mathbb{P}^6$ up to isotopy

Note f lifts to $\mathcal{M}$

Add-D-Seg $\boxed{D(M_2, f) \cong D(X_2)}$

Hori-Tong GL_2-gauged linear σ-model from $(\mathcal{M}, f) \rightsquigarrow X_i$

$$\boxed{\text{GIT quotients} \quad M_1 \hookrightarrow \mathcal{M} \hookleftarrow M_2}$$

Omitting f, have $\underline{D(X_1) \cong D(M_1)} \;\overset{?}{\cdots}\; \underline{D(M_2) \cong D(X_1)}$

Std technique: $D(M_1) \xleftarrow{res_1} D(\mathcal{M}) \xrightarrow{res_2} D(M_2)$

$\cup$

seek <u>window</u> $\mathcal{W}$ so $res_i|_{\mathcal{W}}$ equivs

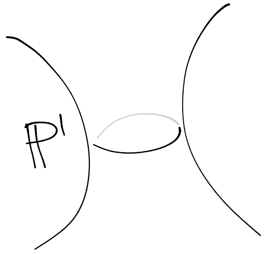
$$\boxed{\text{Notation} \quad \Psi_{\mathcal{W}} : D(M_1) \xrightarrow{\sim} D(M_2)}$$

<u>Ex</u> (simpler) $\mathcal{M} = [\mathbb{C}^4/\mathbb{C}^*]$ weights $1, 1, -1, -1$

$M_i \cong$ resolved conifold

$\mathcal{W} = \langle \mathcal{O}(-2), \mathcal{O}(-1) \rangle \qquad \mathcal{W}' = \langle \mathcal{O}(-1), \mathcal{O} \rangle$

mutation of exc coll on $\mathbb{P}^1$



$\Psi_{\mathcal{W}}^{-1} \Psi_{\mathcal{W}'} = \mathrm{Twist}_{\omega_{\mathbb{P}^1}} \leftarrow$ resolution $\mathcal{O}(-2) \leftarrow \mathcal{O}(-1)^{\oplus 2} \leftarrow \mathcal{O}$

Ex (simpler) $\mathcal{M} = [\mathbb{C}^4/\mathbb{C}^*]$ weights $1, 1, -1, -1$

$M_\iota \cong$ resolved conifold

$\mathcal{W} = \langle \mathcal{O}(-2), \mathcal{O}(-1) \rangle$ $\mathcal{W}' = \langle \mathcal{O}(-1), \mathcal{O} \rangle$

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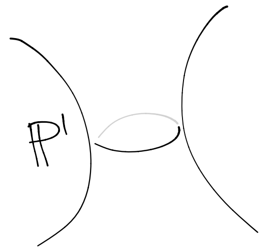


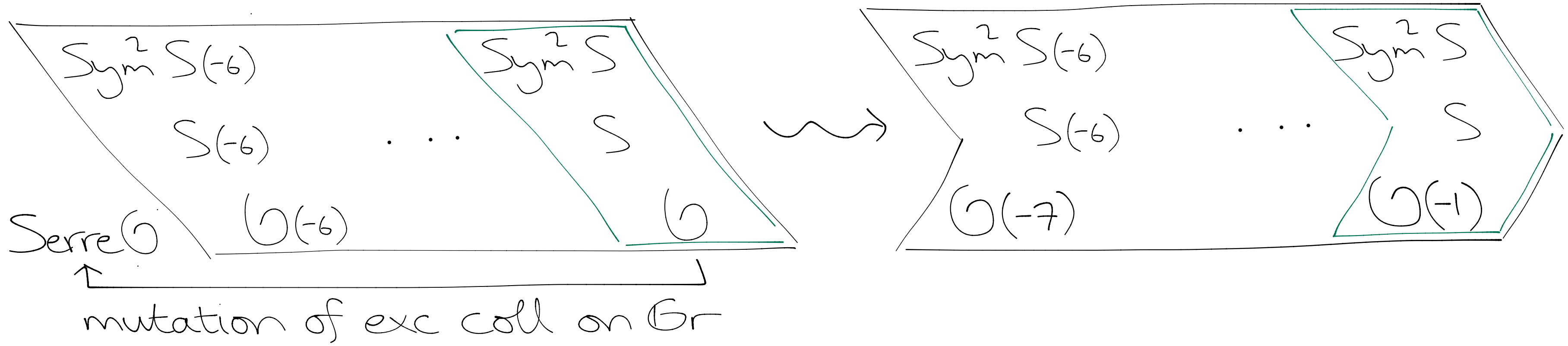
GIT problem $\boxed{\mathcal{M} = [S^{\vee \oplus 7} \oplus \Lambda^2 S^{\oplus 7} / GL(S)]}$ $\dim S = 2$

$\mathcal{W}$ on $\mathcal{M}$ generated by sheaves $\mathrm{Sym}^k S(l) \in$

$$\begin{array}{ccc} \mathrm{Sym}^2 S(-6) & & \mathrm{Sym}^2 S \\ S(-6) & \cdots & S \\ \mathcal{O}(-6) & & \mathcal{O} \end{array}$$

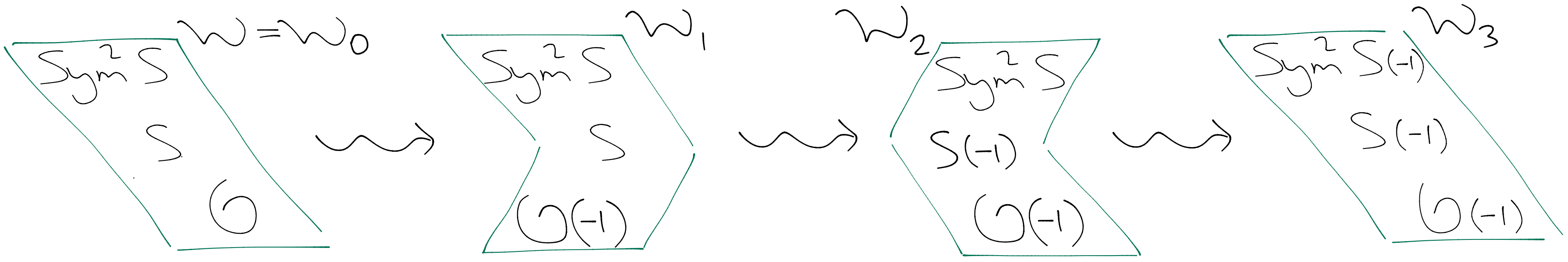
Note Same $GL(S)$-reps give exc coll on $Gr_2(7)$.

Get further new W by mutation:



mutation of exc coll on Gr

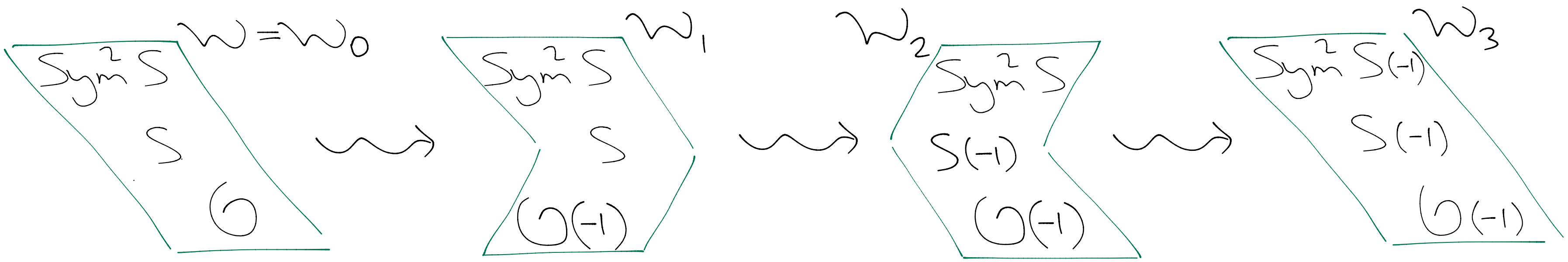
... determined by "blocks" as follows [thanks to Hori]



Check Each gives tilting bundle on M_1 and M_2

Thm Get $\Psi_i : D(M_1) \xrightarrow{\sim} D(M_2)$ from each W_i

$$\begin{tikzcd} \Psi_i : D(M_1) \arrow[r, "\sim"] \arrow[d, phantom, "\cong"] & D(M_2) \arrow[d, phantom, "\cong"] \\ \text{also } \Phi_i : D(X_1) & D(X_2) \end{tikzcd}$$



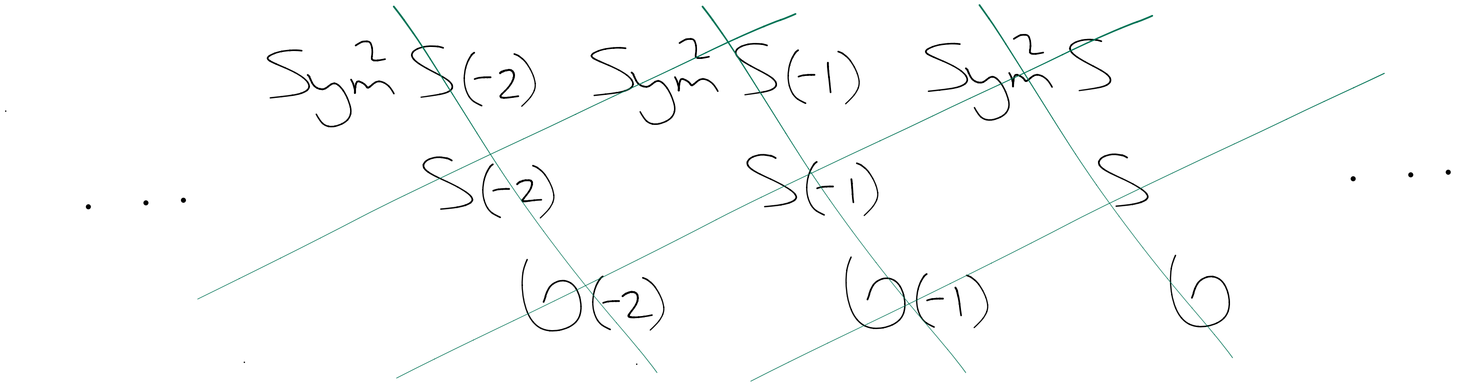
<u>Check</u> Each gives tilting bundle on M_1 and M_2

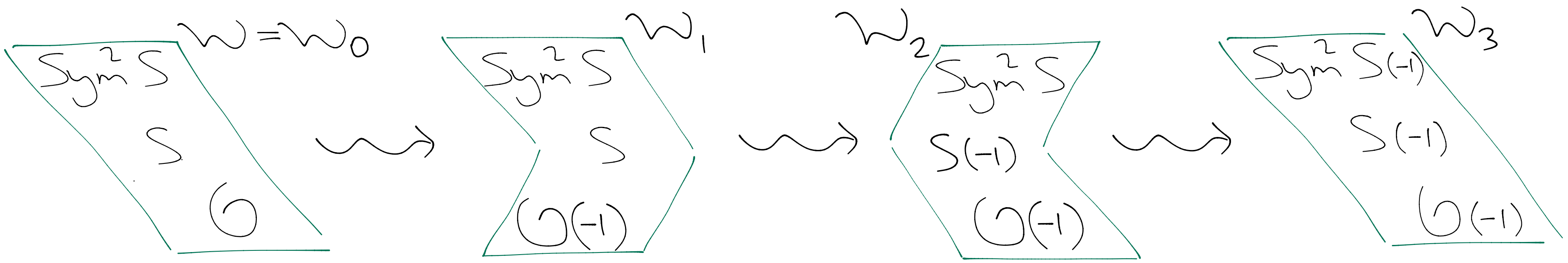
<u>Thm</u> Get $\Phi_i : D(M_1) \xrightarrow{\sim} D(M_2)$ from each $\mathcal{W}_i$

also $\Phi_i : \overset{\cong}{D(X_1)} \qquad \overset{\cong}{D(X_2)}$

<u>Rem</u> $\mathcal{W}^\vee$ gives Add-D-Seg, $\mathcal{W}(7)$ gives Bor-Cal

<u>Key</u>: No "backward" RHom's between $\boxed{\text{lines}}$





Check Each gives tilting bundle on M_1 and M_2

Thm Get

$$\begin{tikzcd} \Psi_i : D(M_1) \arrow[r, "\sim"] \arrow[d, phantom, sloped, "\cong"] & D(M_2) \arrow[d, phantom, sloped, "\cong"] \\ \text{also } \Phi_i : D(X_1) & D(X_2) \end{tikzcd}$$

from each W_i

Prop $\Psi_1^{-1}\Psi_0 = \mathrm{Tw}_{\omega_{Gr}}$ on $D(M_1)$

Prop $\Phi_1^{-1}\Phi_0 = \mathrm{Tw}_{\mathcal{O}}$ on $D(X_1)$ Calabi-Yau

sim. $\Phi_2^{-1}\Phi_1 = \mathrm{Tw}_{S}$

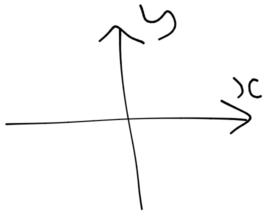
$\Phi_3^{-1}\Phi_2 = \mathrm{Tw}_{\mathrm{Sym}^2 S}$

sheaves on $X_1 \subset Gr_2(7)$

$$\boxed{\text{Prop } \Psi_1^{-1}\Psi_0 = \mathrm{Tw}_{\omega_{Gr}} \text{ on } D(M_1)}$$

Prop $\Phi_1^{-1}\Phi_0 = \mathrm{Tw}_{\mathcal{O}}$ on $D(X_1)$ Calabi-Yau

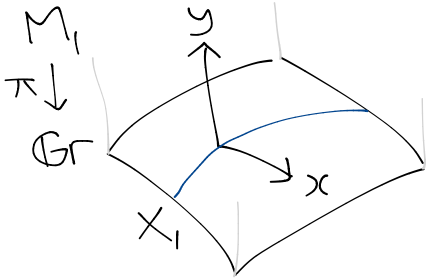
Ex $M = \mathbb{C}^2$, $f = xy$



$$D(pt) \xrightarrow{\sim} D(M, f)$$

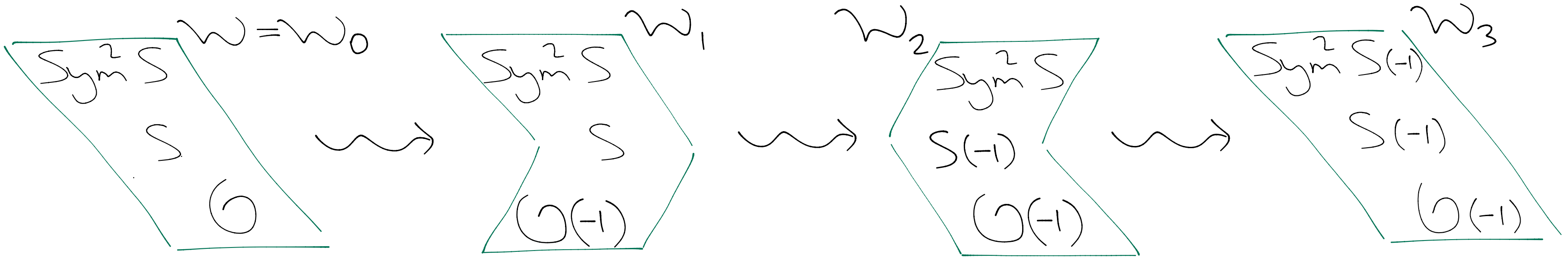
$$\mathcal{O}_{pt} \longmapsto \mathcal{O}_{\{x=0\}}[1] \cong \mathcal{O}_{\{y=0\}}$$

Similarly, have:



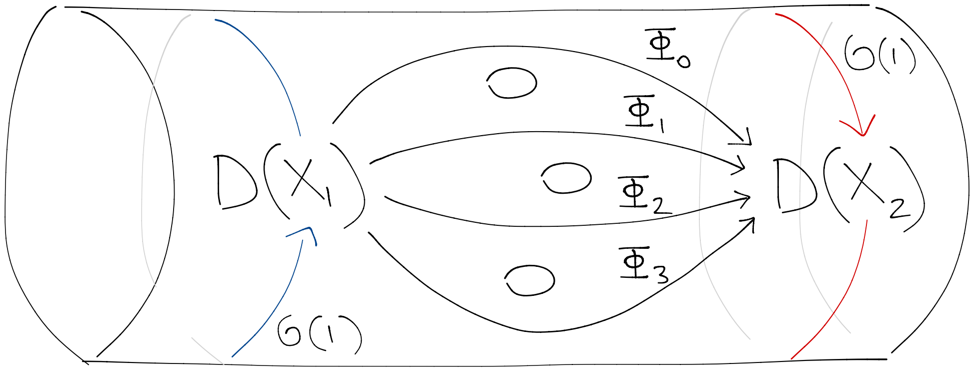
$$D(X_1) \xrightarrow{\sim} D(M_1, f)$$

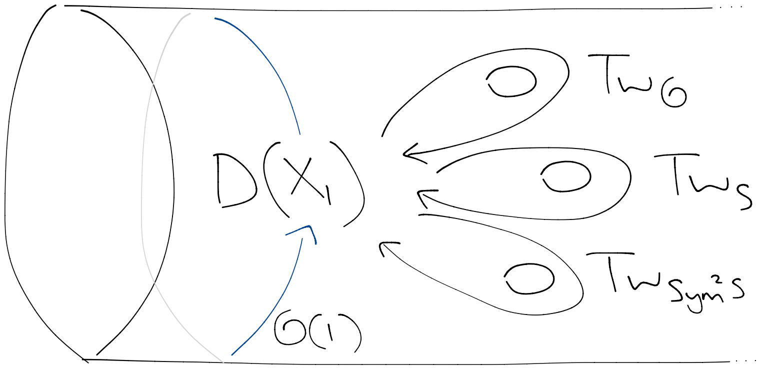
$$\mathcal{O}_{X_1} \longmapsto \pi^*\mathcal{O}_{X_1}[n] \cong \iota_*(\underbrace{\mathcal{O}_{Gr} \otimes \det M_1}_{\omega_{Gr}})$$



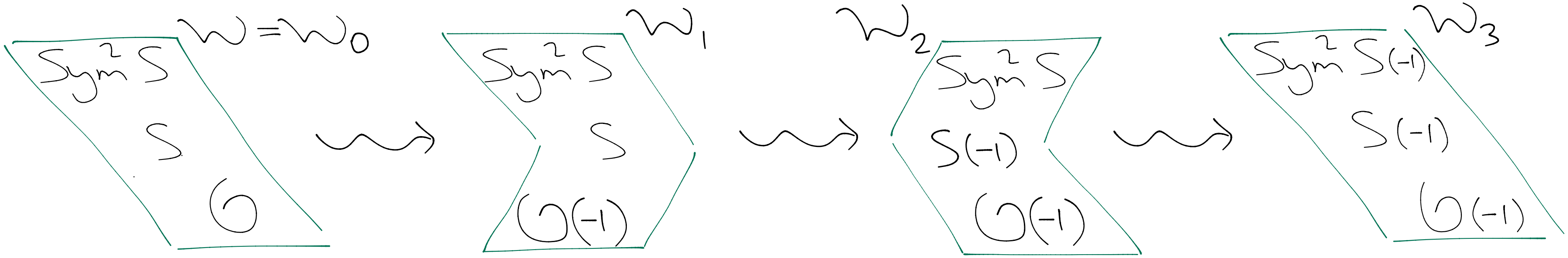
$X_1 = \boxed{\mathrm{Gr}_2(7) \cap \mathbb{P}\Pi}$ $\quad X_2 = \boxed{\mathbb{P}\Pi^{\perp} \cap \mathbb{P}f_4}$

Thm $\pi_1(\text{cylinder} - 3\text{pt})$ acts on $D(X_1)$ and $D(X_2)$:





Rem agrees with FI param space picture, Hori et al.



Recall Get $\Phi_i : D(X_1) \xrightarrow{\sim} D(X_2)$ from each W_i

with $\Phi_1^{-1}\Phi_0 = T_{W_{\mathcal{O}}}$, $\Phi_2^{-1}\Phi_1 = T_{W_S}$, . . .

Have homological projective dualities related by:

$$\begin{tikzcd} \mathrm{Gr} \supset X_1 & D(\mathrm{Gr}) \arrow[r, shift left] & D(X) \arrow[l, shift left] \\ & \text{Lefschetz collections} \arrow[r, bend right, "\text{mutations}"'] & \text{autos} \end{tikzcd}$$

Similar for Rennemo-Segal, Hosono-Takagi . . . ?

THANKS!