Einstein-Gauss-Bonnet gravity in 4 dimensional space-time

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Ref: arXiv:1905.03601 by Drazen Glavan and CL

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大智若愚，大巧若拙！
---- 老子

Stupider is smarter!
----Laozi (601 BC -- unknown)

Overview

1. Lovelock Theorem
2. Einstein-Gauss-Bonnet gravity in $D = 4$
   - Maximally Symmetric Space-time
   - FLRW space-time
   - Spherical static solution
3. challenges that we are facing
4. conclusion

I. Lovelock Theorem

Lovelock gravity

D. Lovelock (1971,1972) :

1. metricity;
2. space-time diffeomorphism invariance;
3. 2nd order equation of motion;

The gravity theory is written

$$\mathcal{L} = \sqrt{-g}\ \sum_{n=0}^{t} \alpha_n\, \mathcal{R}^n, \qquad \mathcal{R}^n = \frac{1}{2^n}\delta^{\mu_1\nu_1\ldots\mu_n\nu_n}_{\alpha_1\beta_1\ldots\alpha_n\beta_n} \prod_{r=1}^{n} R^{\alpha_r\beta_r}{}_{\mu_r\nu_r}$$

$R^{\alpha\beta}{}_{\mu\nu}$: the Riemann tensor; δ: the antisymmetric product

$$\delta^{\mu_1\nu_1\ldots\mu_n\nu_n}_{\alpha_1\beta_1\cdots\alpha_n\beta_n} = n!\delta^{\mu_1}_{[\alpha_1}\delta^{\nu_1}_{\beta_1}\cdots\delta^{\mu_n}_{\alpha_n}\delta^{\nu_n}_{\beta_n]}$$

In D dimensions, only terms with $n < D/2$ are non-trivial.

Lovelock gravity

Expanding the product in $\mathcal{L}$,

$$\mathcal{L} = \sqrt{-g}\left[\alpha_0 + \alpha_1 R + \alpha_2 \underbrace{\left(R^2 + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 4R_{\mu\nu}R^{\mu\nu}\right)}_{Gauss-Bonnet\ \ term} + \alpha_3 \mathcal{O}(R^3)\right]$$

In $D = 4$, the Gauss-Bonnet term is a total derivative, so are the higher order Lovelock invariants, R^3 and so on.

Lovelock theorem

the General relativity (with a c.c) is unique if we assume:

1. 4 dimensional space-time;
2. metricity, metric only;
3. space-time diffeomorphism invariance.
4. 2nd order equation of motion;

A fundamental question

How to bypass this theorem?

Gauss-Bonnet term

$$S_{\mathrm{GB}}[g_{\mu\nu}] = \int d^D x \sqrt{-g}\, \alpha\, \mathcal{G} \,,$$

$\mathcal{G}$ is the Gauss-Bonnet invariant,

$$\mathcal{G} = R^{\mu\nu}{}_{\rho\sigma} R^{\rho\sigma}{}_{\mu\nu} - 4 R^{\mu}{}_{\nu} R^{\nu}{}_{\mu} + R^2 = 6 R^{\mu\nu}{}_{[\mu\nu} R^{\rho\sigma}{}_{\rho\sigma]}$$

Taking the variation w.r.t to $g_{\mu\nu}$,

$$H^{\mu}_{\nu} \equiv \frac{g_{\nu\rho}}{\sqrt{-g}} \frac{\delta S_{\mathrm{GB}}}{\delta g_{\mu\rho}} = 15\alpha \quad \underbrace{\delta^{\mu}{}_{[\nu} R^{\rho\sigma}{}_{\rho\sigma} R^{\alpha\beta}{}_{\alpha\beta]}}_{\text{anti-symmetrized over 5 indices}} \quad ,$$

It vanishes if $D = 4$. Taking a trace

$$\frac{g_{\mu\nu}}{\sqrt{-g}} \frac{\delta S_{\mathrm{GB}}}{\delta g_{\mu\nu}} = (D-4) \times \frac{\alpha}{2} \mathcal{G} \,,$$

The Gauss-Bonnet tensor $H^{\mu}_{\nu} \propto (D-4)$ in general,

- regardless of the space-time symmetries,
- true for all components (for $D > 4$).

GB action in differential form

GB is the Euler density in $D=4$,

$$S_{\text{GB}} \sim \int \epsilon_{a_1 \dots a_4} R^{a_1 a_2} \wedge R^{a_3 a_4}$$

which is just a total derivative. In $D>4$

$$S_{\text{GB}} \sim \int \epsilon_{a_1 \dots a_D} R^{a_1 a_2} \wedge R^{a_3 a_4} \left(\wedge e^a\right)^{D-4}$$

taking variation w.r.t vielbeins,

$$\frac{\delta S_{GB}}{\delta e^{a_D}} = (D-4)\, \alpha \epsilon_{a_1 \dots a_D} R^{a_1, a_2} \wedge R^{a_3, a_4} \wedge e^{a_5} \wedge \dots \wedge e^{a_{D-1}} = 0.$$

Conjecture:

the Gauss-Bonnet tensor H^{μ}_{ν} approaches to zero in the same way as $(D-4)$ in the limit $D \to 4$.

what if

$$\alpha \to \frac{\alpha}{D-4}$$

and consider the limit

$$D \to 4\,?$$

$$\frac{g_{\mu\nu}}{\sqrt{-g}}\frac{\delta S_{\mathrm{GB}}}{\delta g_{\mu\nu}} = (D-4) \times \frac{\alpha}{2(D-4)}\mathcal{G}\,,$$

A non-trivial contribution to the Einstein's equation, isn't it?

$$M_p^2 G^\mu_{\ \nu} + \Lambda\delta^\mu_{\ \nu} - \underbrace{\frac{15\alpha}{D-4}\delta^\mu{}_{[\nu}R^{\rho\sigma}{}_{\rho\sigma}R^{\alpha\beta}{}_{\alpha\beta]}}_{\text{Gauss-Bonnet contribution}} = T^\mu_{\ \nu}$$

wait a minute...

Is it equivalent to GR?

II. novel 4D Einstein-Gauss-Bonnet gravity

II.1 Maximally Symmetric Space-time

Maximally Symmetric Space-time

Let us consider a pure gravity theory $S = S_{\rm EH} + S_{\rm GB}$,

$$S[g_{\mu\nu}] = \int d^D x \sqrt{-g} \left[\frac{M_{\rm P}^2}{2} R - \Lambda_0 + \frac{\alpha}{D-4} \mathcal{G} \right],$$

Riemann tensor:

$$R^{\mu\nu}{}_{\rho\sigma} = \frac{\left(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho\right)\Lambda}{M_{\rm P}^2 (D-1)} \qquad \Lambda : \text{effective CC}$$

The Gauss-Bonnet contribution evaluates to

$$\frac{g_{\nu\rho}}{\sqrt{-g}} \frac{\delta S_{\rm GB}}{\delta g_{\mu\rho}} = \frac{\alpha}{D-4} \times \frac{(D-2)(D-3)(D-4)}{2(D-1) M_{\rm P}^4} \times \Lambda^2 \delta^\mu_\nu ,$$

$$\overset{D\to 4}{=} \frac{\alpha\Lambda^2}{3M_p^4} \delta^\mu_\nu .$$

Einstein equation $\quad \left(\Lambda - \Lambda_0 + \frac{2\alpha\Lambda^2}{3M_p^4} \right) \delta^\mu{}_\nu = 0$

Maximally Symmetric Space-time

There are 2 branches of solutions to Einstein's equation.

$$\Lambda_+ \simeq \Lambda_0 \left(1 - \frac{2\alpha\Lambda_0}{3M_{\rm P}^4}\right) , \qquad \Lambda_- \simeq -\frac{3M_{\rm P}^4}{2\alpha} - \Lambda_0 ,$$

linearized graviton propagation

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

$$\underbrace{\left(M_p^2 + \frac{4\alpha}{3}\frac{\Lambda}{M_p^2}\right)}_{M_p^2 \text{ shifted by a constant}} \times \underbrace{\left[\nabla^\rho \nabla^\mu h_{\nu\rho} + \nabla_\nu \nabla_\rho h^{\mu\rho} + \ldots\right]}_{\text{identical to the one of GR}} = 0$$

- 2 degrees of freedom, no ghost;
- not distinguishable from GR at the level of perturbation theory;

II.2 FLRW space-time

FLRW space-time

add a scalar field: $S = S_{\rm EH} + S_{\rm GB} + S_{\phi}$,

$$S_{\phi}[g_{\mu\nu}, \phi] = \int d^D x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \partial_{\mu}\phi \, \partial_{\nu}\phi - V(\phi) \right],$$

FLRW ansatz $ds^2 = -dt^2 + a^2 d\boldsymbol{x}^2$, the Friedmann equations in $D \to 4$ limit:

$$3M_p^2 H^2 + 6\alpha H^4 = \frac{1}{2}\dot{\phi}^2 + V(\phi),$$
$$-M_p^2 \Gamma \dot{H} = \frac{1}{2}\dot{\phi}^2 \,,$$

a dimensionless parameter $\Gamma \equiv 1 + \frac{4\alpha H^2}{M_{\rm P}^2}$.

equation of motion: $$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0.$$

Bianchi identity holds.

FLRW space-time

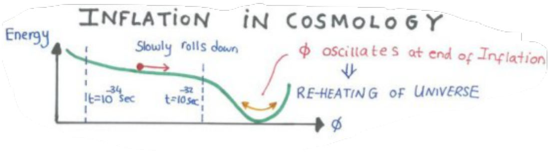
gravitational waves

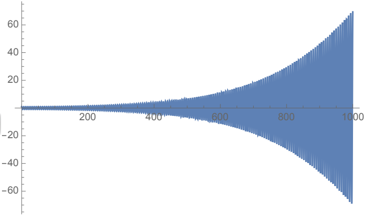
$$g_{ij} = a^2\left(\delta_{ij} + \gamma_{ij}\right),$$

transerve $\partial_i\gamma_{ij} = 0$ and traceless $\gamma_{ii} = 0$.

$$\ddot{\gamma}_{ij} + 3H\left(1 + \frac{8\alpha\dot{H}}{3M_{\rm P}^2\Gamma}\right)\dot{\gamma}_{ij} - c_{\rm s}^2\frac{\partial^2\gamma_{ij}}{a^2} = 0\,,$$

where $c_{\rm s}^2 \equiv 1 + \frac{8\alpha\dot{H}}{M_{\rm P}^2\Gamma}$.





narrow parametric resonance in the gravitational waves sector during reheating epoch (unpublished result).

FLRW space-time

scalar modes of metric perurbations

$$g_{00} = -(1+2\chi)\,, \qquad g_{0i} = \partial_i \beta\,,$$
$$g_{ij} = a^2 e^{2\zeta}(\delta_{ij} + \partial_i \partial_j E)$$

We have to perturb the scalar field as well,

$$\phi(t, \mathbf{x}) = \phi(t) + \delta\phi(t, \mathbf{x}).$$

The space-time diffeomorphism invariance removes $\partial_i \partial_j E$ and $\delta\phi$,

$$\ddot{\zeta} + 3H\left(1 + \frac{\eta}{3} - \frac{8\alpha\epsilon H^2}{3M_{\mathrm{P}}^2\Gamma}\right)\dot{\zeta} - \frac{\partial^2\zeta}{a^2} = 0\,,$$

where $\epsilon \equiv -\dot{H}/H^2$ and $\eta \equiv \dot{\epsilon}/H\epsilon$.

Observational effects?

II.3 Spherical static solution

Spherical static solution

A spherical static ansatz:

$$ds^2 = -e^{2\omega}dt^2 + e^{-2\omega}dr^2 + r^2 d\Omega^2_{D-2},$$

the solution in the limit $D \to 4$,

$$-g_{00} = e^{2\omega} = 1 + \frac{r^2}{32\pi\alpha G}\left(1 \pm \sqrt{1 + \frac{128\pi\alpha G^2 M}{r^3}}\right).$$

At large distances the two branches behave asymptotically as

$$-g_{00} \overset{r\to\infty}{\sim} 1 - \frac{2GM}{r} \quad \text{or} \quad 1 + \frac{r^2}{16\pi\alpha G} + \frac{2GM}{r},$$

The 1st branch recovers the predictions of GR at long distance.

Spherical static solution

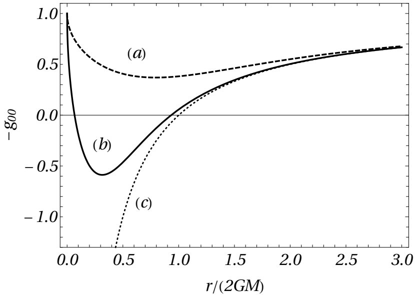


Figure: Radial dependence of gravitational potential $-g_{00}$: (a) $M = M_*/2$; (b) $M = 2M_*$; (c) general relativity;

A critical mass scale

$$M_* = \sqrt{\frac{16\pi\alpha}{G}}\,.$$

Spherical static solution

summary of spherical static solution

- Schwarzschild-like asymptotic behavior at large distance;
- There is a critical mass scale M_*, below which there is no horizon and thus no black hole;
- for $M > M_*$, there are two horizons, the one of a black hole and the one of a white hole;
- resolution of the singularity problem!

Summary of cosmological solution

- Fredmann equation is modified;
- scalar mode and tensor mode are modified.

III. challenges that we are facing

pro and cons of modifying action principle

- Pro: extend the way we define a theory, and Lovelock theorem can be bypassed in the more general framework;
- Cons: unable to count number of d.o.f with Hamiltonian analysis.

What theory do we get
if we recover the standard action principle?

Lovelock theorem

the General relativity (with a c.c) is unique if we assume:

1. 4 dimensional space-time;
2. metricity, metric only $\rightarrow$ probably violated;
3. space-time diffeomorphism invariance.
4. 2nd order equation of motion;

challenges that we are facing

17:13 Fri Apr 17 arxiv.org 27%

Is there a novel Einstein-Gauss-Bonnet theory in four dimensions?

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No! We show that the field equations of Einstein-Gauss-Bonnet theory defined in generic $D > 4$ dimensions split into two parts one of which always remains higher dimensional, and hence the theory does not have a non-trivial limit to $D = 4$. Therefore, the recently introduced four-dimensional, novel, Einstein-Gauss-Bonnet theory does not admit an *intrinsically* four-dimensional definition as such it does not exist in four dimensions. The solutions (the spacetime, the metric) always remain $D > 4$ dimensional. As there is no canonical choice of 4 spacetime dimensions out of D dimensions for generic metrics, the theory is not well defined in four dimensions.

main conclusion:

no pure metric theory in $D = 4$.

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IS THE SEQUENCE OF EARTHQUAKES IN SOUTHERN CALIFORNIA, WITH AFTERSHOCKS REMOVED, POISSONIAN?

BY J. K. GARDNER and L. KNOPOFF

ABSTRACT

Yes.

Early attempts to determine whether or not earthquake events were Poisson-distributed led to the conclusion that main sequence events were significantly non-Poissonian, whether worldwide catalogs (Aki, 1956), or local catalogs (Knopoff, 1964) were considered. Such a conclusion is of importance in earthquake prediction; a demonstration of nonrandomness in earthquake catalogs gives rise to the hope that a basis for prediction of occurrence times can be found by decoding the sequential message in the list of events. Conversely, if earthquakes are random events in time, no hope exists for finding order in earthquake catalogs and, hence, attempts to predict times of occurrence based on studies of catalog sequences are very likely doomed. Nevertheless, a result which shows that earthquake sequences are Poissonian can be of use in estimating seismic risk (Molchan, *et al.*, 1970) by virtue of the simple statistical models which result; we assume that more complicated statistical models will make the analytical procedures more complicated, although such analyses have not been carried out in detail.

Aftershocks are earthquake events which are causally connected with a parent event

challenges that we are facing

dimensional reduction

Einstein-Gauss-Bonnet gravity with D-dimensional metric

$$ds_D^2 = ds_4^2 + e^{2\phi} d\Sigma_{D-4}^2 \,,$$

maximally symmetric internal space,

$$R_{abcd} = \lambda(g_{ac}g_{bd} - g_{ad}g_{bc}) \,.$$

rescale the coupling constant $\alpha \to \alpha/(D-4)$, and then taking the limit $D \to 4$, we get the $4D$ reduced action

$$\begin{aligned} S_{EGB} \;\; = \;\; & \int d^4x\sqrt{-g}\Big[R - 2\Lambda_0 + \alpha\Big(\phi\,\mathrm{GB} + 4G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2\lambda R e^{-2\phi} \\ & -4(\partial\phi)^2\Box\phi + 2(\partial\phi)^4 - 12\lambda(\partial\phi)^2 e^{-2\phi} - 6\lambda^2 e^{-4\phi}\Big)\Big] \end{aligned}$$

Ref: ArXiv:2003.11552, by H. Lu, Y. Pang, ArXiv:2003.12771 by T. Kobayashi the same 4D action (with $\lambda = 0$) is obtained by taking the difference between

$$\lim_{D\to4}\frac{1}{D-4}\int d^Dx\left\{\sqrt{-g}\,\mathcal{G}\,[g_{\mu\nu}]-\sqrt{-\widetilde{g}}\,\mathcal{G}\,[\widetilde{g}_{\mu\nu}]\right\},$$

where

$$\widetilde{g}_{\mu\nu}=e^{2\phi}g_{\mu\nu}$$

The resulting theory is equivalent to the original one, only if

$$\lim_{D\to4}\frac{1}{D-4}\int d^Dx\sqrt{-\widetilde{g}}\,\mathcal{G}\,[\widetilde{g}_{\mu\nu}]$$

is a total derivative, which is unfortunately impossible to be satisfied... Or in other word, its contribution to the equation of motion

$$\widetilde{\mathcal{E}}_{\mu\nu}\equiv\lim_{D\to4}\frac{1}{D-4}\frac{\delta\left(\sqrt{-\widetilde{g}}\,\mathcal{G}\,[\widetilde{g}_{\mu\nu}]\right)}{\delta\widetilde{g}_{\mu\nu}}$$

conformal approach

$$
\begin{aligned}
\widetilde{\mathcal{E}}^{\mu}_{\nu} = \mathcal{E}^{\mu}_{\nu} + \delta^{\mu}_{\nu} \Big[-X^2 + R(X - 2\Box\phi) + 2\Box\phi\Box\phi + 2\nabla^{\alpha}\nabla_{\beta}\phi \left(2\nabla_{\alpha}\phi\nabla^{\beta}\phi - \nabla_{\alpha}\nabla^{\beta} \right. \\
+4R^{\alpha}_{\beta} \left(\nabla^{\beta}\nabla_{\alpha}\phi - \nabla^{\beta}\phi\nabla_{\alpha}\phi \right) \Big] + 4R^{\mu\alpha}{}_{\nu\beta} \left(\nabla_{\alpha}\phi\nabla^{\beta}\phi - \nabla_{\alpha}\nabla^{\beta}\phi \right) \\
+4R^{\mu}{}_{\alpha} \left(\nabla^{\alpha}\phi\nabla_{\nu}\phi - \nabla^{\alpha}\nabla_{\nu}\phi \right) + 4R^{\alpha}_{\nu} \left(\nabla_{\alpha}\phi\nabla^{\mu}\phi - \nabla_{\alpha}\nabla^{\mu}\phi \right) - 2R^{\mu}_{\nu} \left(X - 2\Box \right. \\
+2R \left(\nabla^{\mu}\nabla_{\nu}\phi - \nabla^{\mu}\phi\nabla_{\nu}\phi \right) + 4X\nabla^{\mu}\phi\nabla_{\nu}\phi - 4\nabla^{\mu}\phi\nabla^{\alpha}\phi\nabla_{\nu}\nabla_{\alpha}\phi - 4\nabla^{\mu}\nabla^{\alpha}\phi\nabla_{\nu} \\
+4\nabla^{\mu}\nabla^{\alpha}\phi\nabla_{\nu}\nabla_{\alpha}\phi + 4\Box\phi \left(\nabla^{\mu}\phi\nabla_{\nu}\phi - \nabla^{\mu}\nabla_{\nu}\phi \right).
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{E}^{\mu}_{\nu} &= \frac{1}{D-4} \times 15\, \delta^{\mu}{}_{[\nu} R^{\rho\sigma}{}_{\rho\sigma} R^{\alpha\beta}{}_{\alpha\beta]} \\
&= \frac{1}{D-4} \left(-2R^{\mu\alpha}{}_{\rho\sigma} R^{\rho\sigma}{}_{\nu\alpha} + 4R^{\mu\alpha}{}_{\nu\beta} R^{\beta}{}_{\alpha} + 4R^{\mu}{}_{\alpha} R^{\alpha}{}_{\nu} - 2RR^{\mu}{}_{\nu} + \frac{1}{2}\mathcal{G}\delta^{\mu}_{\nu} \right)
\end{aligned}
$$

unfortunately

these two theories are not equivalent.

IV. Conclusion

conclusion

- Bypass the Lovelock theorem on account of modifying action principle.
- The theory we discovered is written as

$$S[g_{\mu\nu}] = \int d^D x \sqrt{-g} \left[\frac{M_{\rm P}^2}{2} R - \Lambda_0 + \frac{\alpha}{D-4} \mathcal{G} \right],$$

  the 4 dimensional theory is defined in the limit $D \to 4$.
- it satisfies all Lovelock criteria: (1) 4 dimensions; (2) metricity; (3) space-time diffeomorphism; (4) 2nd order e.o.m.
- modified dynamics, singularity resolved;
- generalize to the higher order Lovelock terms, 2003.07068 by A. Casalino et al.
- the standard action principle version is still missing...

Thank you!

A very basic question:

Law of physics $\longrightarrow$ EoM with no more than two time derivatives

$$e.g. \qquad F = m\ddot{x}$$

WHY? Have you ever thought about it?

Ostrogradsky theorem

Consider an action

$$S = \int dt \left[\ddot{q}^2 - V(q)\right]$$

The canonical variables

$$Q_1 = q, \qquad Q_2 = \dot{q},$$

and their conjugate momenta

$$P_1 = \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}, \qquad P_2 = \frac{\partial L}{\partial \ddot{q}}.$$

The Hamiltonian

$$H = \color{red}{P_1 Q_2} + \frac{1}{2}P_1^2 + V(Q_1)$$

unbounded from below.