A Quantum Toroidal Categorification On Hilbert Schemes

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- It generalized the Heisenberg algebra action on the cohomology by Nakajima and Grojnowski.
- In this talk, we will categorify the above $U_{q_1,q_2}(\ddot{gl}_1)$ action.

Hilbert Schemes and Cohomology

Given a quasi-projective smooth surface S over $k = \mathbb{C}$, we consider $S^{[n]}$ the Hilbert scheme of n points on S, and let

$$\mathcal{M} := \bigsqcup_{n=0}^{\infty} S^{[n]}$$

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Theorem (Nakajima,Grojnowski)

The homology group $H_*(M)$ is a irreducible highest weight represenation as a representation of the Heisenberg superalgebra associated with $H^*(X)$, where the highest weight vector is the generator of $H_0(X^{[0]}) \cong \mathbb{Q}$.

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Quantum Toroidal Algebra $U_{q_1,q_2}(\ddot{gl}_1)$

- $U_{q_1,q_2}(\ddot{gl}_1)$ is an affinization of the q-Heisenberg algebra.
- The study of $U_{q_1,q_2}(\ddot{gl}_1)$ started from many different origins in algebraic geometry, representation theory and mathematical physics.
- Let $\mathbb{K} = \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]^{Sym}_{([1],[2],[3],\cdots)}$ where "Sym" means symmetric in q_1 and q_2 and let $q = q_1 q_2$. Then $U_{q_1,q_2}(\ddot{gl}_1)$ is the $\mathbb{K}$-algebra with generators

$$\{E_k, F_k, H_l^{\pm}\}_{k\in\mathbb{Z}, l\in\mathbb{N}}$$

modulo the following relations:

Quantum Toroidal Algebra $U_{q_1,q_2}(\ddot{gl}_1)$

$$(z - wq_1)(z - wq_2)(z - \frac{w}{q})E(z)E(w) = \\ = (z - \frac{w}{q_1})(z - \frac{w}{q_2})(z - wq)E(w)E(z) \quad (1)$$

$$(z - wq_1)(z - wq_2)(z - \frac{w}{q})E(z)H^{\pm}(w) = \\ = (z - \frac{w}{q_1})(z - \frac{w}{q_2})(z - wq)H^{\pm}(w)E(z) \quad (2)$$

$$[[E_{k+1}, E_{k-1}], E_k] = 0 \quad \forall k \in \mathbb{Z} \quad (3)$$

together with the opposite relations for $F(z)$ instead of $E(z)$, and:

$$[E(z), F(w)] = \delta(\frac{z}{w})(1 - q_1)(1 - q_2)(\frac{H^+(z) - H^-(w)}{1 - q}) \quad (4)$$

where

Hilbert Schemes and Grothendieck Groups

- Let $S^{[n,n+1]}$ be the nested Hilbert scheme parameterized by

$$\{(\mathcal{I}_n, \mathcal{I}_{n+1}, x) \in S^{[n]} \times S^{[n+1]} \times S | \mathcal{I}_{n+1} \subset \mathcal{I}_n, \mathcal{I}_n/\mathcal{I}_{n+1} = k_x\}.$$

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- There is a unique tautological line bundle $\mathcal{L}$ on $S^{[n,n+1]}$ whose fiber over a closed point is $\mathcal{I}_n/\mathcal{I}_{n+1}$.
- $$e_k := [\mathcal{L}^k \mathcal{O}_{S^{[n,n+1]}}], \quad f_k = [\mathcal{L}^{k-1} \mathcal{O}_{S^{[n,n+1]}}]$$

  could be regarded as operators: $K(\mathcal{M}) \to K(\mathcal{M} \times S)$ through the K-theoretic correspondences.

Hilbert Schemes and Grothendieck Groups

Theorem (Schiffmann-Vasserot, Feigin-Tsymbaliuk)

- *There exists* $h_m \in K(\mathcal{M} \times S)$ *which is a combination of symmetric product and wedge product of the universal ideal sheaf on* $\mathcal{M} \times S$ *such that*

$$[e_k, f_l] = \Delta_* \left(\frac{h_{k+l}}{1-q} \right)$$

where $\Delta : \mathcal{M} \times S \to \mathcal{M} \times \mathcal{M} \times S \times S$ *is the diagonal embedding and* $q = [\omega_S]$ *is the canonical line bundle of* S.

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- *e_k, f_k, h_k satisfy the relations in $U_{q_1,q_2}(\ddot{gl}_1)$.*

Some Applications

- It also acts on the Grothendieck group of higher rank stable sheaves, and factors through the deformed W-algebra. It leads to a proof of AGT correspondences for $U(r)$ gauge theory with matter. (Neguț)

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- It also acts on the Grothendieck group of higher rank stable sheaves, and factors through the deformed W-algebra. It leads to a proof of AGT correspondences for $U(r)$ gauge theory with matter. (Neguț)
- It induces a W-algebra action on the Chow groups of Hilbert schemes of points on surfaces which is studied by Li-Qin-Wang in the homology theory, which induced the Beauville conjecture on the Hilbert schemes of $K3$ surfaces (Maulik-Neguț).

Categorification of $U_{q_1,q_2}(\ddot{gl}_1)$

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Categorification of $U_{q_1,q_2}(\ddot{gl}_1)$

- The action of the positive part was also categorified by Neguț
- A monoidal categorification of the postive part is given by Porta-Sala through the Categorified hall algebra.
- We will categorify the commutator of the positive and the negative part:

$$[e_k, f_l] = \Delta_* \frac{h_{k+l}}{1-q}$$

by constructing natural transformations in derived categories explicitly.

The Main Theorem(Y. Zhao)

- For every two integers m and r, there exists natural transformations

$$\begin{cases} f_r e_{m-r} \to e_{m-r} f_r & \text{if } m > 0 \\ e_{m-r} f_r \to f_r e_{m-r} & \text{if } m < 0 \\ f_r e_{-r} = e_r f_{-r} \oplus \mathcal{O}_\Delta[1]. & \end{cases} \tag{6}$$

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- When $m \neq 0$, the cone of above natural transformations has a filtration with associated graded object

$$\begin{cases} \bigoplus_{k=0}^{m-1} R\Delta_*(h^+_{m,k}) & \text{if } m > 0 \\ \bigoplus_{k=m+1}^{0} R\Delta_*(h^-_{m,k}) & \text{if } m < 0 \end{cases}$$

where $h^+_{m,k}, h^-_{m,k} \in D^b(\mathcal{M} \times S)$ are combinations of wedge and symmetric products of universal sheaves on $\mathcal{M} \times S$.

The main theorem

- At the level of Grothendieck groups, we have the formula:

$$(1-[\omega_S])\sum_{k=0}^{m-1}[h^+_{m,k}] = h^+_m \qquad m>0$$

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- The extensions between $h^{\pm}_{m,k}$ are non-trivial and given by a explicit formula (which I will present if there is still time).

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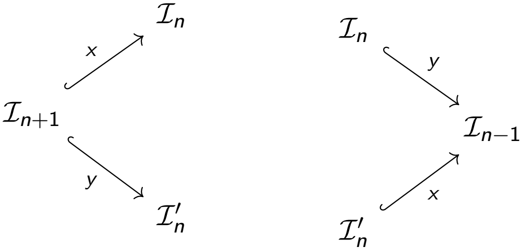
- It is only a categorification in the weak sense. Maps between sheaves and relations between them should be pursued in the future.
- We could also consider the categorification of the action on higher rank stable sheaves with a refinement of the techniques.

Vanishing Theorem

We consider the following triple/quadruple moduli space $\mathfrak{Z}_+, \mathfrak{Z}_-, \mathfrak{Y}$ parameterize diagrams:



Vanishing theorem

$$\begin{array}{ccccc} & & \mathcal{I}_n & & \\ & \overset{x}{\nearrow} & & \overset{y}{\searrow} & \\ \mathcal{I}_{n+1} & & & & \mathcal{I}_{n-1} \\ & \underset{y}{\searrow} & & \underset{x}{\nearrow} & \\ & & \mathcal{I}'_n & & \end{array} \tag{7}$$

respectively, of ideal sheaves where each successive inclusion is colength 1 and supported at the point indicated on the diagrams. We consider line bundles $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}'_1, \mathcal{L}'_2$ over triple/quadruple moduli spaces with fiber $\mathcal{I}_n/\mathcal{I}_{n+1}, \mathcal{I}_{n-1}/\mathcal{I}_n, \mathcal{I}_n/\mathcal{I}'_{n+1}, \mathcal{I}'_{n-1}/\mathcal{I}_n$ respectively.

Vanishing Theorem

The forgetful morphism induces a Cartesian diagram:

$$\begin{array}{ccc}
\mathfrak{Y} & \xrightarrow{\alpha_+} & \mathfrak{Z}_+ \\
\downarrow{\scriptstyle \alpha_-} & \searrow^{\theta} & \downarrow{\scriptstyle \beta_+} \\
\mathfrak{Z}_- & \xrightarrow{\beta_-} & S^{[n]} \times S^{[n]} \times S \times S
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Theorem (Vanishing Theorem)

- $R\alpha_{-*}\mathcal{O}_{\mathfrak{Y}} = \mathcal{O}_{\mathfrak{Z}_-}$
- $R\alpha_{+*}\mathcal{O}_{\mathfrak{Y}} = \mathcal{O}_{W_0}$, *where* W_0 *and* $W_1 = S^{[n,n+1]}$ *are two irreducible components of* $\mathfrak{Z}_+$.

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Given the above two equailities, we are able to compare $e_m f_l$ and $f_l e_m$ through line bundles on $\mathfrak{Y}$.

Some Basic Ideas of the Minimal Model Program

Definition (Discrepancy)

- X be a normal variety that mK_X is Cartier for $m \in \mathbb{Z}_{>0}$

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- Suppose $f : Y \to X$ is a birational morphism from a smooth variety Y.

Remark

In birational geometry, people care more about pairs (X, D), where D is a $\mathbb{Q}$ of $\mathbb{R}$ Cartier divisor. We will not present the definition of pairs for simplicity, but it is essential in part of our proof.

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- Suppose $f : Y \to X$ is a birational morphism from a smooth variety Y.
- There are rational numbers $a(E_i, X)$ such that

$$\mathcal{O}_Y(mK_Y) \cong f^*\mathcal{O}_X(mK_X) \otimes \mathcal{O}_Y(\sum_i ma(E_i, X)E_i).$$

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- $a(E_i, X)$ is called the discrepancy of E_i with respect to X.

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An Overview of the Minimal Model Program

Definition (Classifacation of singularities)

Let X be a normal variety. Assume that mK_X is Cartier for some $m > 0$. We say that X is

$$\begin{cases} \textit{terminal} \\ \textit{canonical} \\ \textit{klt} \\ \textit{plt} \\ \textit{dlt} \\ \textit{lc} \end{cases} \quad \text{if } a(E, X) \text{ is } \begin{cases} > 0, \text{ for every exceptional } E \\ \geq 0, \text{ for every exceptional } E \\ > -1, \text{ for every } E \\ > -1, \text{ for every exceptional } E \\ > -1, \text{ if } center_X E \subset \text{non-snc} X \\ \geq -1. \text{ for every } E \end{cases}$$

Remark

For S_2 and equidimensional schemes, we could also define "semi-lc","semi-dlt".

The Structure Theorem

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- Every irreducible component of a semi-dlt scheme is normal.

By explicit computing the discrepancy, we have the following structure theorem for the geometry of $\mathfrak{Y}, \mathfrak{Z}_-, \mathfrak{Z}_+$

Theorem (Structure Theorem)

- $\mathfrak{Y}$ *is smooth (Neguț);*
- $\mathfrak{Z}_+$ *is semi-dlt and* W_0 *is a canonical singularity (Y. Zhao).*
- $\mathfrak{Z}_-$ *is a canonical singularity (Y. Zhao).*

Review

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Review

- We categorify the commutation of e_k, f_l action on the Grothendieck groups of Hilbert schemes
- For higher rank stable sheaves, $\mathfrak{Z}_+$ is no longer equi-dimensional, and the influence of DAG has to be accounted for.
- It is a toy model of the categorical version "Drinfeld Double of the CoHA", and we expect the generalization to other more complicated settings.