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4d Chern-Simons Theory as a 3d Toda Theory, and a 3d-2d Correspondence

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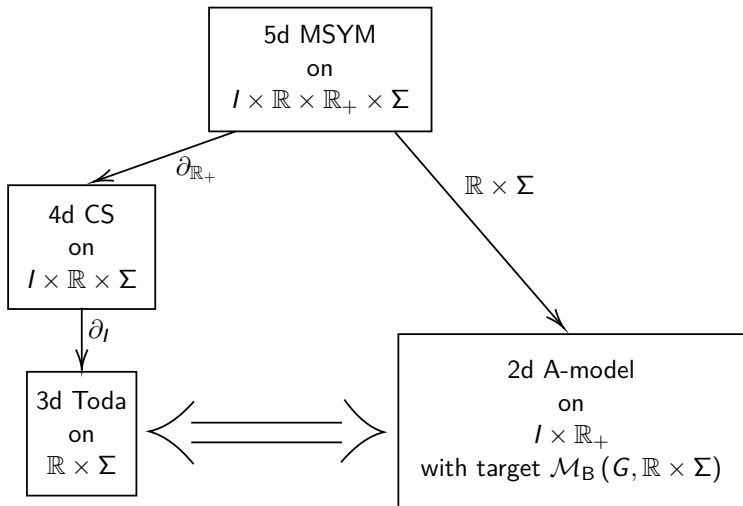
Scope of Presentation

- Introduction/Summary
- Topological-Holomorphic "GL"-twist of 5d MSYM
- 3d Toda Theory
- Open Topological Sigma Model on Bogomolny Moduli Space
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Introduction/Summary

- The AGT correspondence relates four-dimensional $\mathcal{N} = 2$ gauge theory and two-dimensional Toda theory on a Riemann surface, $\tilde{\Sigma}$.

- Among other approaches, it is known that the AGT correspondence can be understood in terms of the relationship between Hitchin's moduli space, $\mathcal{M}(G, \tilde{\Sigma})$, and W-algebra representations.[1]

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1. **nekrasov2010omega**.

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- In this work, we present an analogous relationship, i.e., between the moduli space of solutions to the **Bogomolny equations**, and representations of a novel W-algebra arising from a **three-dimensional analogue of Toda theory**.

- This **3d Toda theory** is a boundary dual of **4d Chern-Simons theory**.

- By embedding 4d CS in partially-twisted five-dimensional $\mathcal{N} = 2$ SYM on a manifold with corners, we shall show that **3d Toda theory is dual to an open topological sigma model on the Bogomolny moduli space**.

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– The equivalence of the $\mathcal{Q}$-cohomology of physical states, will
be shown to imply that **modules of 3d W-algebras** defined
on $\mathbb{R} \times \Sigma$ are modules for the **quantized algebra of certain
holomorphic functions** on $\mathcal{M}_{\mathrm{B}}\left(G, \mathbb{R} \times \Sigma\right)$.

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This talk is based on

- M. Ashwinkumar, K.-S. Png, M.-C. Tan, *4d Chern-Simons Theory as a 3d Toda Theory, and a 3d-2d Correspondence*, To appear

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Topological-Holomorphic "GL"-twist of 5d Maximally Supersymmetric Yang-Mills Theory

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Partial Twist

Five dimensional maximally supersymmetric Yang-Mills involves fields which transform as reps. of $SO_{\mathcal{M}}(5) \times SO_R(5)$:

$$\begin{aligned}
A_M &: (\mathbf{5}, \mathbf{1}) \\
\varphi_{\widehat{M}} &: (\mathbf{1}, \mathbf{5}) \\
\rho_{A\widehat{A}} &: (\mathbf{4}, \mathbf{4})
\end{aligned} \tag{2.2}$$

with the classical action:

$$S = -\frac{1}{g_5^2} \int_{\mathcal{M}} d^5x \, \mathrm{Tr} \left(\frac{1}{4} F_{MN} F^{MN} + \frac{1}{2} D_M \varphi_{\widehat{M}} D^M \varphi^{\widehat{M}} + \frac{1}{4} [\varphi_{\widehat{M}}, \varphi_{\widehat{N}}] [\varphi^{\widehat{M}}, \varphi^{\widehat{N}}] \right.$$
$$\left. + i\rho^{A\widehat{A}} (\Gamma^M)_A{}^B D_M \rho_{B\widehat{A}} + \rho^{A\widehat{A}} (\Gamma^{\widehat{M}})_{\widehat{A}}{}^{\widehat{B}} [\varphi_{\widehat{M}}, \rho_{A\widehat{B}}] \right).$$

We shall choose $G = SU(N)$ as our gauge group in what follows.

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We shall take the 5d worldvolume to be $\mathcal{M} = V \times \Sigma$, whereby we wish to realize a topological twist of the D4-brane worldvolume theory along V.

This amounts to redefining the $SO_V(3)$ rotation group to be the diagonal subgroup

$$SO_V(3)' \subset SO_V(3) \times SO_R(3),$$

where $SO_R(3) \subset SO_R(5)$ rotates $\{\varphi_{\widehat{1}}, \varphi_{\widehat{2}}, \varphi_{\widehat{3}}\}$.

As a result, the scalar fields $\{\varphi_{\widehat{1}}, \varphi_{\widehat{2}}, \varphi_{\widehat{3}}\}$ now transform as the components $\{\varphi_1, \varphi_2, \varphi_3\}$ of a one-form on V.

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The twist results in four supercharges that are scalar along V. We wish to pick a supercharge, $\mathcal{Q}$, **that is scalar along** V, w.r.t. which we shall eventually localize the theory.

We shall choose a certain linear combination of two supercharges to be $\mathcal{Q}$.

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The twisted supersymmetry transformations generated by $\mathcal{Q}$ are

$$
\begin{aligned}
\delta_t A_\alpha &= i\psi_\alpha + it\widetilde{\psi}_\alpha & \delta_t \eta &= t\left(F_{45} + D_\alpha \varphi^\alpha\right) + [\bar{\sigma}, \sigma] \\
\delta_t \varphi_\alpha &= it\psi_\alpha - i\widetilde{\psi}_\alpha & \delta_t \widetilde{\eta} &= -\left(F_{45} + D_\alpha \varphi^\alpha\right) + t[\bar{\sigma}, \sigma] \\
\delta_t A_4 &= i\Upsilon + it\widetilde{\Upsilon} & \delta_t \psi_\alpha &= D_\alpha \sigma + t[\varphi_\alpha, \sigma] \\
\delta_t A_5 &= it\Upsilon - i\widetilde{\Upsilon} & \delta_t \widetilde{\psi}_\alpha &= tD_\alpha \sigma - [\varphi_\alpha, \sigma] \\
\delta_t \sigma &= 0 & \delta_t \Upsilon &= D_4 \sigma + tD_5 \sigma \\
\delta_t \bar{\sigma} &= i\eta + it\widetilde{\eta} & \delta_t \widetilde{\Upsilon} &= tD_4 \sigma - D_5 \sigma
\end{aligned}
\tag{2.3}
$$

$$
\delta_t \chi_\alpha = \frac{1}{2}\left[F_{\alpha 4} + D_5 \varphi_\alpha + \frac{1}{2}\varepsilon_{\alpha\beta\gamma}\left(F^{\beta\gamma} - \left[\varphi^\beta, \varphi^\gamma\right]\right)\right] + \frac{1}{2}t\left[F_{\alpha 5} - D_4 \varphi_\alpha + \varepsilon_{\alpha\beta\gamma}D^\beta \varphi^\gamma\right]
$$

$$
\delta_t \widetilde{\chi}_\alpha = \frac{1}{2}t\left[F_{\alpha 4} + D_5 \varphi_\alpha - \frac{1}{2}\varepsilon_{\alpha\beta\gamma}\left(F^{\beta\gamma} - \left[\varphi^\beta, \varphi^\gamma\right]\right)\right] - \frac{1}{2}\left[F_{\alpha 5} - D_4 \varphi_\alpha - \varepsilon_{\alpha\beta\gamma}D^\beta \varphi^\gamma\right]
$$

i.e., we have $\mathcal{Q} = \mathcal{Q}_L + t\mathcal{Q}_R$, $t \in \mathbb{CP}^1$.[2]

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2. **ashwinkumar2019unifying**.

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Such a twist has also been discussed conceptually by Elliot and Pestun.[3]

The transformations take a form very similar to those of GL-twisted $\mathcal{N} = 4$ SYM, as considered by Kapustin and Witten.[4]

In fact, taking $\Sigma = \mathbb{R} \times S^1$ or T^2, whereby the x^5 direction is S^1, we can dimensionally reduce along the latter to obtain precisely the transformations of Kapustin and Witten via $A_5 \to \varphi_4$, $\chi_\alpha \to \chi_{\alpha 4}^+$, $\widetilde{\chi}_\alpha \to \chi_{\alpha 4}^-$, $\Upsilon \to \psi_4$, $\widetilde{\Upsilon} \to \widetilde{\psi}_4$.

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3. **elliott2019multiplicative**.
4. **kapustin2006electric**.

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In what follows, we shall be interested in $V = I \times \mathbb{R} \times \mathbb{R}_+$ and $\Sigma = \mathbb{CP}^1$, $\mathbb{C}^\times$, or $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. This is a manifold with two corners.

The action of 5d "GL"-twisted theory can then be written as

$$S = \{\mathcal{Q}, \widetilde{V}\} + \frac{\widetilde{\Psi}}{4\pi} \int_{I \times \mathbb{R} \times \mathbb{R}_+ \times \Sigma} dz \wedge \text{Tr}\,(\mathcal{F} \wedge \mathcal{F}). \qquad (2.4)$$

Here, $\widetilde{\Psi}$ is a real parameter (as we have selected $t = -1$ for our purpose).

$\mathcal{F}$ is the curvature of the complex connection involving $\mathcal{A}_\alpha = A_\alpha + i\varphi_\alpha$ (where $\alpha = 1, 2, 3$), and $\mathcal{A}_{\bar{z}} = \frac{1}{2}(A_4 + iA_5)$, where z (i.e., $x^4 + ix^5$) and $\bar{z}$ are local complex coordinates on Σ.

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Boundary conditions

At the origin of $\mathbb{R}_+$ ($x^3 = 0$), we pick NS5-type boundary
conditions to maintain supersymmetry, where we require that

$$\mathcal{Q}(\mathcal{A}_{\widetilde{\alpha}}) = 0, \qquad (2.5)$$

(where $\widetilde{\alpha} = 1, 2$) and

$$\mathcal{Q}(\mathcal{A}_{\overline{z}}) = 0, \qquad (2.6)$$

where $\mathcal{A}_{\widetilde{\alpha}}$ and $\mathcal{A}_{\overline{z}}$ obey Neumann boundary conditions.

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At infinity along $\mathbb{R}_+$, the boundary conditions are taken to be x^3-independent $\mathcal{Q}$-invariant configurations that involve $\mathcal{A}_{\widetilde{\alpha}}$ and $\mathcal{A}_{\overline{z}}$.

At the boundaries of I, we can have Nahm pole-type boundary conditions, defined in terms of the complex gauge fields, where the latter are $\mathcal{Q}$-invariant.

With $x^2 = \tau$ parametrizing $\mathbb{R}$, we first define the convenient coordinates

$$x^{\pm} = \tau \pm i\overline{z}, \qquad (2.7)$$

and

$$\mathcal{A}_{\pm} = \frac{1}{2} \left(\mathcal{A}_\tau \mp i\mathcal{A}_{\overline{z}} \right). \qquad (2.8)$$

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With $x^1 = \sigma$ parametrizing $I = [0, \pi]$, the Nahm pole-type boundary conditions are defined as follows. As $\sigma \to 0$,

$$\mathcal{A} \to \frac{id\sigma}{\sigma} H + \frac{dx^+}{\sigma} T_+. \qquad (2.9)$$

Here, a homomorphism $\rho : \mathfrak{su}(2) \to \mathfrak{su}(N)$ has been chosen such that the image, T_+, of the raising operator of $\mathfrak{su}(2)$ is a maximal length Jordan block, i.e.,

$$T_+ = i \begin{pmatrix} 0 & \mu_1 & 0 & \cdots & 0 \\ 0 & 0 & \mu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \qquad (2.10)$$

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Analogously, as $\sigma \to \pi$,

$$\mathcal{A} \to \frac{id\sigma}{\sigma - \pi} H + \frac{dx^-}{\sigma - \pi} T_-, \qquad (2.11)$$

where

$$T_- = -i \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \nu_1 & 0 & \cdots & 0 & 0 \\ 0 & \nu_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \nu_{N-1} & 0 \end{pmatrix}. \qquad (2.12)$$
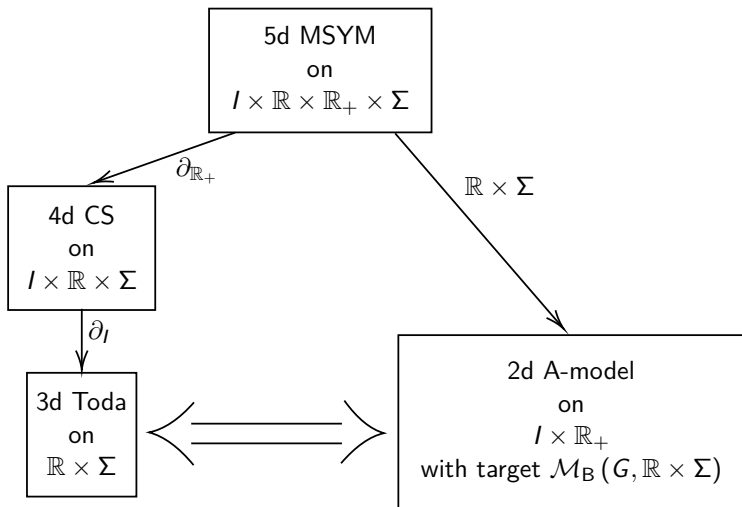
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Localization to 4d Chern-Simons theory

The path integral of this theory localizes to solutions of
$\mathcal{Q}$-invariant configurations, whereby it reduces to the **path
integral of 4d Chern-Simons theory**, i.e.,

$$\int_{\widetilde{\Gamma}} D\mathcal{A} \, \exp\left(\frac{i}{2\pi\hbar} \int_{I \times \mathbb{R} \times \Sigma} dz \wedge \text{Tr}\left(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right) \right), \tag{2.13}$$

where $\widetilde{\Gamma}$ is the integration cycle defined by the $\mathcal{Q}$-invariant
localization equations, and $\frac{i}{\hbar} = \frac{\widetilde{\Psi}}{2}$.

This integration cycle is a **Lefschetz thimble** that ensures the
convergence of the path integral.

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Three-dimensional Toda Theory

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3d Toda Theory from 4d Chern-Simons

A 3d analogue of WZW theory dual to 4d CS theory can be obtained when the the topological directions have a boundary.[5]

Something similar can be done in our current set up, where there are now two boundaries on the ends of I.

Let us focus on the boundary at $\sigma \to 0$, since similar arguments can be applied to the other end of the interval (i.e. $\sigma \to \pi$).

5. **ashwinkumar20203dwzw**.

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Via the boundary condition $\mathcal{A}_- = 0$, the action for $\sigma \to 0$ is that of a 3d analogue of the 2d chiral WZW model

$$
\begin{aligned}
\lim_{\sigma \to 0} S_{\text{4d CS}}[\mathcal{A}] &= S_{\text{3d WZW}}[g] \\
&= \frac{1}{2\pi\hbar} \int_{\mathbb{R} \times \Sigma} dz \wedge dx^+ \wedge dx^- \operatorname{Tr} \left(\partial_+ g\, g^{-1} \partial_- g\, g^{-1} \right) \\
&\quad - \frac{1}{6\pi\hbar} \int_{I \times \mathbb{R} \times \Sigma} dz \wedge \operatorname{Tr} \left(dg\, g^{-1} \wedge dg\, g^{-1} \wedge dg\, g^{-1} \right)
\end{aligned}
$$
(3.2)

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The ensuing conserved currents then take the form

$$J_+ = g^{-1} \partial_+ g \tag{3.3a}$$
$$J_- = \partial_- g \, g^{-1}. \tag{3.3b}$$

Similarly, at the boundary where $\sigma \to \pi$, one finds the currents

$$J'_+ = -g^{-1} \partial_+ g \tag{3.4a}$$
$$J'_- = -\partial_- g \, g^{-1}, \tag{3.4b}$$

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We project the currents onto Cartan directions, as well as positive and negative roots of the (complex) Lie algebra

$$J = J^{-i}R_{-i} + J^{+i}R_{+i} + J^{0i}R_{0i}, \qquad (3.5)$$

where $i = 1, \cdots, N - 1$. Then, the rest of the Nahm pole-type boundary conditions result in current constraints

$$J^{+i}_{+} = \mu_i, \qquad J^{0i}_{+} = 0 \qquad (3.6a)$$

$$J'^{-i}_{-} = -\nu_i \qquad J'^{0i}_{-} = 0 \qquad (3.6b)$$

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Next, we use a Gauss decomposition

$$
g = e^{X_i R_i^+} e^{\phi_i R_i^0} e^{Y_i R_i^-}, \tag{3.7}
$$

which allows the current constraints to be rewritten as

$$
\partial_+ \partial_- \phi_i + 2\mu_i \nu_i e^{C_{ij}\phi^j} = 0 \tag{3.8}
$$

The corresponding action that gives these constraints as an EOM takes the form of a 3d analogue of 2d analytically-continued Toda theory

$$
\boxed{
S_{\text{3d Toda}}[\phi] = \frac{1}{2\pi\hbar} \int_{\mathbb{R} \times \Sigma} dz\, dx^+ dx^- \left(C_{ij} \partial_+ \phi^i \partial_- \phi^j - 4 \sum_i \mu_i \nu_i e^{C_{ij}\phi^j} \right)
}
\tag{3.9}
$$

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3d W-algebras

Writing $\sum_j C_{ij}^{-1} = \gamma_i$, the 3d Toda theory is invariant under the **infinitesimal coordinate transformation** $x^+ \to x^+ + \varepsilon(x^+)$ together with the transformation of ϕ_i as

$$\phi_i(z, x^+, x^-) \to \phi_i(z, x^+, x^-) - \varepsilon \partial_+ \phi_i - \gamma_i \partial_+ \varepsilon. \qquad (3.10)$$

The corresponding Noether current takes the form

$$\Theta = \frac{1}{\hbar} C_{ij} \partial_+ \phi^i \partial_+ \phi^j - \frac{2}{\hbar} C_{ij} \gamma^i \partial_+ \partial_+ \phi^j, \qquad (3.11)$$

where $\partial_- \Theta = 0$.

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We focus on the simplest case where $G = SL(2, \mathbb{C})$, and $[C_{ij}] = 2$.
Take x^- to be the temporal direction, and write $x^+ = \xi$.

We use an equivalent gauged 3d WZW model, where Θ takes the
form

$$\Theta(z, \xi) = T(z, \xi) - \partial_\xi J_3(z, \xi), \qquad (3.13)$$

where $T(z, \xi) = -J_a(z, \xi)J^a(z, \xi)$.

Using the symplectic form on phase space, we compute the PB to
obtain a 3d analogue of the classical Virasoro algebra

$$\left[\Theta(z', \xi'), \Theta(z, \xi)\right]_{\text{PB}} = \left(\partial_\xi \Theta(z, \xi)\delta(\xi - \xi') + 2\Theta(z, \xi)\delta'(\xi - \xi') - \frac{1}{2}\delta'''(\xi - \xi')\right)\delta(z - z').$$

$$(3.14)$$

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Via Moyal quantization[6], we then expect (3.14) to be corrected to

$$[\Theta(z',\xi'),\Theta(z,\xi)] = \bigg(\partial_\xi\Theta(z,\xi)\delta(\xi-\xi') + 2\Theta(z,\xi)\delta'(\xi-\xi')$$
$$+ \left(k-\frac{1}{2}\right)\delta'''(\xi-\xi') + k\delta'(\xi-\xi')\bigg)\,\delta(z-z')$$

$$(3.15)$$

where $k \in \mathbb{R}$ describes the quantum deformation of the Poisson bracket expression. This is a **3d analogue of the quantum Virasoro algebra**.

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6. **jorjadze2001poisson**.

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Analogous results for higher spin currents in higher-rank 3d Toda theory can be obtained via a generalized Sugawara construction of W-algebra currents.

We will then obtain 3d W-algebras similar to their 2d counterparts, but with generators having **holomorphic dependence** on Σ, and $\delta(z - z')$ appearing as an overall factor on the RHS.

Choosing x^- instead of x^+ to be the temporal direction leads to **another copy** of the W-algebra, which is x^--dependent.
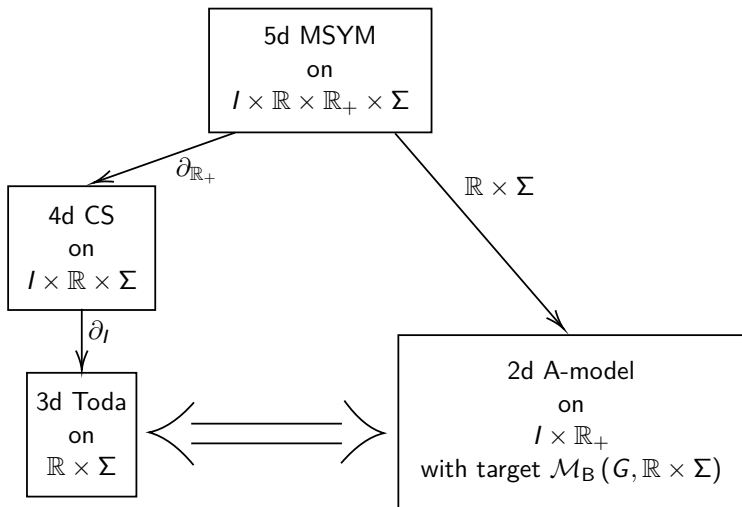
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2d A-model from 5d "GL"-twisted Theory

Using the topological-holomorphic symmetry of the 5d "GL"-twisted theory on $I \times \mathbb{R} \times \mathbb{R}_+ \times \Sigma$, we can scale down the three-manifold $\mathbb{R} \times \Sigma$ while simultaneously blowing up $I \times \mathbb{R}_+$.

To ensure the action remains finite, one requires

$$
\boxed{
\begin{aligned}
F_{\tau\bar{z}} - iD_{\bar{z}}\varphi_\tau &= 0 \\
D_\tau\varphi^\tau - 2iF_{z\bar{z}} &= 0,
\end{aligned}
}
\tag{4.1}
$$

which are the **Bogomolny equations** on $\mathbb{R} \times \Sigma$.

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2d A-model from 5d "GL"-twisted Theory
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This results in a **sigma model** on $I \times \mathbb{R}_+$, with target the moduli space, $\mathcal{M}_{\mathrm{B}}(G, \mathbb{R} \times \Sigma)$, of solutions to these equations, with metric

$$\widetilde{G} = \frac{1}{g_5^2} \int_{\mathbb{R} \times \Sigma} d^3 x \, \mathrm{Tr} \left(\delta A^p \otimes \delta A_p + \delta \varphi^\tau \otimes \delta \varphi_\tau \right), \qquad (4.2)$$

where A_p is the gauge field defined along $\mathbb{R} \times \Sigma$.

To be precise, we obtain an **A-model** in symplectic structure

$$\omega_K^{\mathrm{B}} \propto \int_{\mathbb{R} \times \Sigma} d^3 x \, \mathrm{Tr}(\delta \varphi_\tau \wedge \delta A_4 - \delta A_5 \wedge \delta A_\tau) \qquad (4.3)$$

(the pullback of ω_K^{B} to $I \times \mathbb{R}_+$ arises from non-$\mathcal{Q}$-exact terms in the 5d action).

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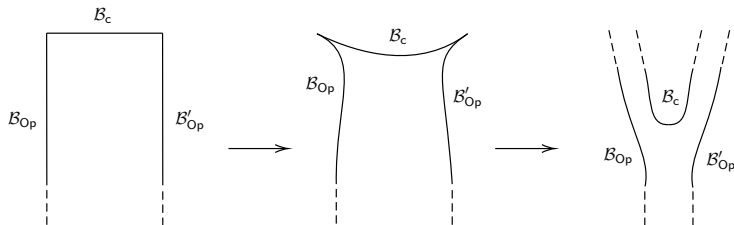**Branes**
Physical States

Branes

The NS5-type boundary condition (b.c.) (of the 5d gauge theory) gives rise to a space-filling **coisotropic** A-brane, $\mathcal{B}_c$, in $\mathcal{M}_B (G, \mathbb{R} \times \Sigma)$. Defining complex/symplectic structures analogous to those on Hitchin moduli space, this is a (B, A, A) brane.

The Nahm pole-type b.c. of the 5d gauge theory give rise to **Lagrangian** branes $\mathcal{B}_{Op}$ and $\mathcal{B}'_{Op}$ in $\mathcal{M}_B (G, \mathbb{R} \times \Sigma)$. These are analogues of branes of opers in Hitchin moduli space, and are both (A, B, A) branes.

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To compute physical states of the A-model we perform the following topological deformation

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If we were to compute the physical states of $(\mathcal{B}_{\mathrm{Op}}, \mathcal{B}'_{\mathrm{Op}})$ strings, we will only find one state, as $\mathcal{B}_{\mathrm{Op}}$ and $\mathcal{B}'_{\mathrm{Op}}$ only intersect at one point in $\mathcal{M}_{\mathrm{B}}(G, \mathbb{R} \times \Sigma)$.

To find more states, we have to consider the $(\mathcal{B}_{\mathrm{c}}, \mathcal{B}_{\mathrm{Op}})$ and $(\mathcal{B}_{\mathrm{c}}, \mathcal{B}'_{\mathrm{Op}})$ strings that arise from the corners on $I \times \mathbb{R}_+$. Their physical states can be computed to be the following spaces of J-holomorphic sections:

$$
\begin{aligned}
\mathcal{H}_{(\mathcal{B}_{\mathrm{c}}, \mathcal{B}_{\mathrm{Op}})} &= H^0(\mathcal{B}_{\mathrm{Op}}, K_{\mathcal{B}_{\mathrm{Op}}}^{1/2}) \\
\mathcal{H}_{(\mathcal{B}_{\mathrm{c}}, \mathcal{B}'_{\mathrm{Op}})} &= H^0(\mathcal{B}'_{\mathrm{Op}}, K_{\mathcal{B}'_{\mathrm{Op}}}^{1/2}).
\end{aligned}
\tag{4.4}
$$

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Moreover, the algebra of $(\mathcal{B}_c, \mathcal{B}_c)$ strings acts on the space of states of $(\mathcal{B}_c, \mathcal{B}_{Op})$ and $(\mathcal{B}_c, \mathcal{B}'_{Op})$ strings, by attaching to the appropriate ends of the latter.
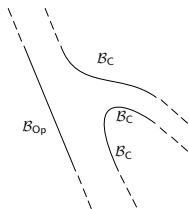
This is just the **quantized algebra** of J-**holomorphic functions** on $\mathcal{M}_B(G, \mathbb{R} \times \Sigma)$.

Therefore the space of states of the $(\mathcal{B}_c, \mathcal{B}_{Op})$ and $(\mathcal{B}_c, \mathcal{B}'_{Op})$ strings are modules for the quantized algebra of J-holomorphic functions on $\mathcal{M}_B(G, \mathbb{R} \times \Sigma)$.

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3d-2d Correspondence

The 5d "GL"-twisted gauge theory on $I \times \mathbb{R} \times \mathbb{R}_+ \times \Sigma$ has given rise to two effective descriptions.

The $\mathcal{Q}$-cohomology of states of the 5d theory ought to remain invariant in reducing to these effective descriptions. This implies a **duality** between the two theories:

3d Toda theory		2d A-model
on	$\Leftrightarrow$	on
$\mathbb{R} \times \Sigma$		$I \times \mathbb{R}_+$
		with target $\mathcal{M}_{\mathrm{B}}(G, \mathbb{R} \times \Sigma)$

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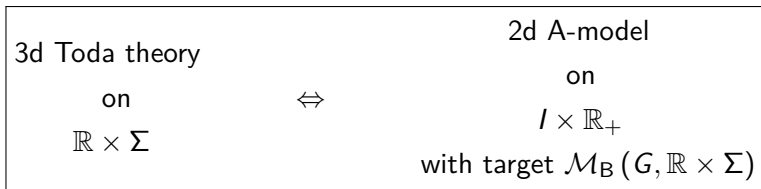
Mathematically, this implies that

Modules of 3d W-algebras
defined on $\quad\Leftrightarrow\quad H^0(\mathcal{B}_{\mathrm{Op}}, K^{1/2}_{\mathcal{B}_{\mathrm{Op}}}) \otimes H^0(\mathcal{B}'_{\mathrm{Op}}, K^{1/2}_{\mathcal{B}'_{\mathrm{Op}}})$
$\mathbb{R} \times \Sigma$

Moreover, we find that

Modules of 3d W-algebras defined on $\mathbb{R} \times \Sigma$ are
modules for the quantized algebra of J-holomorphic functions
on $\mathcal{M}_{\mathrm{B}}(G, \mathbb{R} \times \Sigma)$.

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- We have shown that 3d Toda theory and a 2d A-model on Bogomolny moduli space are dual to each other.

- We have found that modules of 3d W-algebras are modules for quantized algebras of holomorphic functions on the Bogomolny moduli space.

- The crucial ingredient is the fact that the 5d $\mathcal{N} = 2$ SYM theory admits a partial twist that is topological-holomorphic, and analogous to the GL-twist of 4d $\mathcal{N} = 4$ SYM.

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- Future work involves defining a GL-type partial twist (in two directions) for maximally supersymmetric Yang-Mills theory in 6d, whereby one expects to relate **4d W-algebras** with the quantized algebra of holomorphic functions on the **moduli space of instantons**.

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Thank you for your attention!