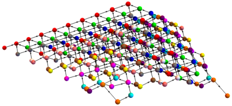

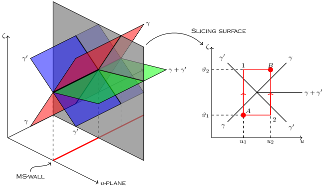

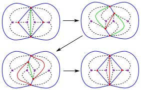

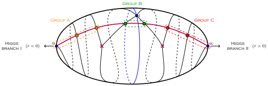

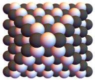

BPS States and Geometry

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Kavli IPMU Postdoc Colloquium Series, October 30, 2020

BPS States

$\mathcal{N}=2$ 4D superalgebra $\Longrightarrow$ Bogomolny-Prasad-Sommerfeld bound

$$\left.\begin{array}{r}\left\{Q^A_\alpha,\bar{Q}_{\dot\beta B}\right\}=2\sigma^\mu_{\alpha\dot\beta}P_\mu\delta^A_B\\ \left\{Q^A_\alpha,Q^B_\beta\right\}=2\epsilon_{\alpha\beta}\epsilon^{AB}\bar{Z}\\ \left\{\bar{Q}_{\dot\alpha A},\bar{Q}_{\dot\beta B}\right\}=-2\epsilon_{\alpha\beta}\epsilon_{AB}Z\end{array}\right\}\Longrightarrow \text{BPS bound: } \boxed{M=\sqrt{P_\mu P^\mu}\geqslant|Z|}$$

P_μ – momentum vector, $Z\in\mathbb{C}$ – central charge

BPS state = short multiplet:

$$\left(\xi^{-1}{Q_1}^1+\xi\bar{Q}^{\dot{1}1}-\xi^{-1}{Q_2}^2-\xi\bar{Q}^{\dot{2}2}\right)|\text{BPS}\rangle=0$$
$$\left(\xi^{-1}{Q_1}^2+\xi\bar{Q}^{\dot{1}2}+\xi^{-1}{Q_2}^1+\xi\bar{Q}^{\dot{2}1}\right)|\text{BPS}\rangle=0$$

For a BPS state we have:

$$\xi^{-2}=-e^{i\arg Z}\qquad M_{\text{BPS}}=|Z|$$

$$\mathcal{H}_{\text{BPS}}\subset\mathcal{H}$$

Seiberg - Witten solution for Super - Yang - Mills

$$\mathcal{L} = \mathrm{Tr}\left[-\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{32\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} + |\nabla_\mu\phi|^2 - \frac{g^4}{2}\left[\phi,\phi^\dagger\right]^2 + \text{fermions}\right]$$

Classical vacua $V(\phi) = \frac{g^4}{2}\mathrm{Tr}\left[\phi,\phi^\dagger\right]^2 = 0 \Longrightarrow$ Moduli space $\langle\phi\rangle \sim \frac{1}{2}a\,\sigma_3$
Gauge group is broken in IR, for example, $SU(2) \Longrightarrow U(1)_{\mathrm{IR}}$.

$$\tau(a) := \frac{\theta(a)}{2\pi} + \frac{4\pi i}{g(a)^2} = \frac{i}{\pi}\log\frac{a^2}{\Lambda^2}\text{“1 − loop”} + \sum_{k=1}^{\infty} c_k\left(\frac{\Lambda}{a}\right)^{4k}\text{“4d inst.”}$$

BPS particles – dyons charged in $U(1)_{\mathrm{IR}}$ $\gamma = (p_{\mathrm{electro}}, q_{\mathrm{magnetic}})$:

$$Z = \frac{1}{g^2}\int\limits_{S^2_{R\to\infty}}\left(i\langle\mathrm{Tr}\,F\phi\rangle - \langle\mathrm{Tr}\,\tilde{F}\phi\rangle\right) = a\cdot p + a_D\cdot q \overset{\text{class}}{\sim} a\cdot p + a\tau_{\mathrm{cl}}\cdot q$$

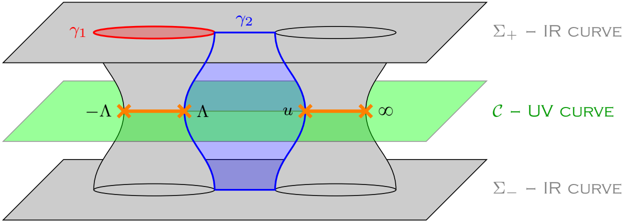
Seiberg-Witten solution:

$$a = \frac{\sqrt{2}}{\pi}\int\limits_{-\Lambda}^{\Lambda}\frac{dz\sqrt{z-u}}{\sqrt{z^2-\Lambda^2}},\quad a_D = \frac{\sqrt{2}}{\pi}\int\limits_{\Lambda}^{u}\frac{dz\sqrt{z-u}}{\sqrt{z^2-\Lambda^2}},\quad u = \langle\mathrm{Tr}\,\phi^2\rangle,\quad \tau = \frac{da_D}{da}$$

M-brane description I

$$a = \oint_{\gamma_1} \lambda, \quad a_D = \oint_{\gamma_2} \lambda, \quad \lambda = \frac{\sqrt{2}}{2\pi} \frac{dz\sqrt{z-u}}{\sqrt{z^2-\Lambda^2}}, \quad z \in \mathcal{C}$$

$$\Sigma : \quad y^2 = (z-u)(z^2-\Lambda^2)$$



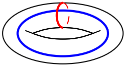
$\gamma_1 = \partial\sigma_1$, $\gamma_2 = \partial\sigma_2$

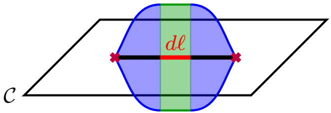
M5-brane:	Σ	$\times$	$\mathbb{R}^{1,3}$
$\dim = 6 =$	2	$+$	4
M2-branes:	σ_i	$\times$	(BPS particle world-line)
$\dim = 3 =$	2	$+$	1

M-BRANE DESCRIPTION II

TWO BRANCHES OF SPECTRAL COVER: $\lambda_{\pm}(z) = \pm \frac{\sqrt{2}}{4\pi} \frac{dz\,\sqrt{z-u}}{\sqrt{z^2-\Lambda^2}}$, $z \in \mathcal{C}$

$$[\partial M2] = p_{\text{electro}} A(\Sigma) + q_{\text{magnetic}} B(\Sigma)$$





$$Z = \oint_{\partial M2} \lambda = \int_\ell \lambda_+ - \int_\ell \lambda_- = \int_\ell (\lambda_+ - \lambda_-)$$

BPS: $dZ = (\lambda_+ - \lambda_-) = e^{i\varphi_0}|dZ|$

$\ell:\quad z(s) = x(s) + iy(s)$

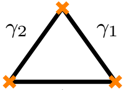
$$\frac{dy}{dx} = -\frac{\mathrm{Im}\, e^{-i\varphi_0}(\lambda_+ - \lambda_-)}{\mathrm{Re}\, e^{-i\varphi_0}(\lambda_+ - \lambda_-)}$$

SIMPLE EXAMPLE: AD_3 SPECTRAL COVER:

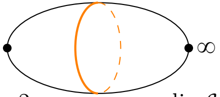
$$\Sigma_u:\ \lambda^2 + \left(z^3 - 3z + u\right) dz^2 = 0, \qquad u = \frac{1}{2}\mathrm{Tr}\,\langle\phi^2\rangle, \qquad \Delta = 27(4 - u^2)$$

$u \to 0$

$u \to \infty$

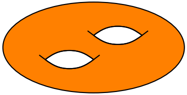


MS WALL $(Z_1||Z_2)$



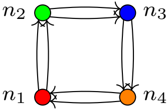
$\dim \mathcal{H}_{\text{BPS}} = 2$ $\qquad$ $\dim \mathcal{H}_{\text{BPS}} = 3$

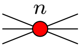
Quiver Quantum Mechanics



Brane physics

Quiver quantum mechanics





– Gauge vector multiplet $U(n)$

– Chiral (Higgs) multiplet $U(n) \times \overline{U(m)}$

$\mathcal{N} = 1$ 4d SYM (Quiver) $\xrightarrow{\text{dim. red.}}$ $\mathcal{N} = 4$ SQM (Quiver)

BPS state = Gauge invariant ground state (Vacuum)

$$V \sim |\text{Gauge} \cdot \text{Higgs}|^2$$

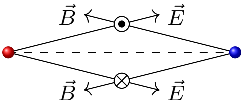
Coulomb branch:	Higgs branch:
$\langle \text{Gauge} \rangle \neq 0$	$\langle \text{Gauge} \rangle = 0$
$\langle \text{Higgs} \rangle = 0$	$\langle \text{Higgs} \rangle \neq 0$

Hall halo

Coulomb branch

$$A_{i=1,2,3} = \begin{pmatrix} x_i^{(1)} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & x_i^{(n)} \end{pmatrix} \longrightarrow \vec{r}_{k=1,\dots,n} = \left(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}\right)$$

Quiver node with $U(n) \longrightarrow n$ "elementary" BPS particles:



DSZ pairing: $\mathcal{J}_{\text{EM}}/\hbar = p_{\color{red}\bullet} q_{\color{blue}\bullet} - p_{\color{blue}\bullet} q_{\color{red}\bullet} = \langle \gamma_{\color{red}\bullet}, \gamma_{\color{blue}\bullet} \rangle = \#({\color{red}\bullet} \to {\color{blue}\bullet}) - \#({\color{blue}\bullet} \to {\color{red}\bullet})$

Back to AD3 theory:

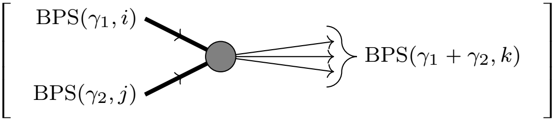
, $\langle \gamma_1, \gamma_2 \rangle = 1$

$$\Psi_{\color{red}\bullet}(r, \vartheta, \varphi) = e^{-\frac{r}{2R_0}} r^{-\frac{1}{2}} (1 - \cos\vartheta)^{-\frac{1}{2}} \begin{pmatrix} 1 - \cos\vartheta \\ -e^{i\varphi} \sin\vartheta \end{pmatrix}$$

$$\langle r \rangle = R_0 = \frac{1}{2} \frac{|Z_1(u) + Z_2(u)|}{\operatorname{Im} Z_1(u) \bar{Z}_2(u)}$$ Diverges on MS wall $Z_1(u) || Z_2(u)$

BPS algebra through scattering

Scattering $\mathcal{S}$-matrix:



$$\left[\ \cdots\ \right](s) = \frac{\mathcal{S}_{ij}^k}{s - M^2_{\gamma_1+\gamma_2}}$$

Define multiplication structure $\mathfrak{m}$:

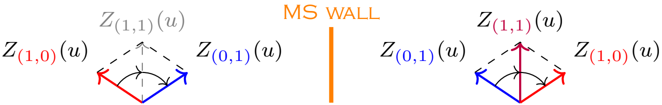
$$\mathfrak{m}: \quad \begin{array}{l} \mathcal{H}_{\mathrm{BPS},\gamma_1} \otimes \mathcal{H}_{\mathrm{BPS},\gamma_2} \longrightarrow \mathcal{H}_{\mathrm{BPS},\gamma_1+\gamma_2} \\ \Psi_i \cdot \Psi_j = \sum_k \mathcal{S}_{ij}^k \Psi_k \end{array}$$

Questions:

1. Does this algebra tell us anything about wall-crossing?
2. Does this algebra resemble anything familiar?

Gas of BPS particles (or Fock spaces) $\Longrightarrow$
$\Longrightarrow$ Kontsevich-Soibelman wall-crossing formula:

$$\mathcal{F}_{(1,0)} \otimes_{\mathfrak{m}} \mathcal{F}_{(0,1)} \cong \mathcal{F}_{(0,1)} \otimes_{\mathfrak{m}} \mathcal{F}_{(1,1)} \otimes_{\mathfrak{m}} \mathcal{F}_{(1,0)}$$

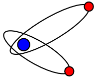


Melting crystals

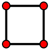
D.G. and Masahito Yamazaki arXiv:2008.07006

Higgs branch

Chiral mass $m_{\mathbb{C}}$: flavour $U(n_f)$ = "freeze" gauge ❄$U(n_f)$❄

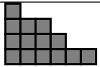
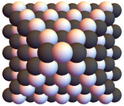
On Coulomb branch we expect $\langle$Gauge$\rangle \sim$ 

On Higgs branch we expect $\langle$Gauge$\rangle = 0$

However, we expect $\langle$Gauge$\rangle \sim$  $m_{\mathbb{C}}$

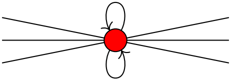
Mathematically $m_{\mathbb{C}}$ resembles equivariant action

For toric Calabi-Yau manifolds we have: BPS states = Crystals

CY 1-fold	1D crystal	
CY 2-fold	2D crystal	
CY 3-fold	3D crystal	

Melting crystal model for DT invariants of CY manifolds

BPS algebra I

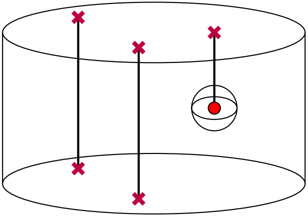


Induced statystics of nodes (particles):

#(selft-loops)	statystics
even	fermionic
odd	bosonic

Algebra generated by:

$$\begin{aligned} \hat{e}_\bullet &: \quad n_\bullet \to n_\bullet + 1 \\ \hat{f}_\bullet &: \quad n_\bullet \to n_\bullet - 1 \\ \hat{\psi}_\bullet &: \quad n_\bullet \to n_\bullet \end{aligned} \qquad \begin{aligned} e_\bullet(z) &= \left[\mathrm{Tr}\,(z-\Phi)^{-1}, \hat{e}_\bullet \right] \\ f_\bullet(z) &= -\left[\mathrm{Tr}\,(z-\Phi)^{-1}, \hat{f}_\bullet \right] \end{aligned}$$



"Monopole operator"
(Hecke modification)

BPS algebra II

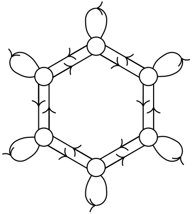
RESULTING ALGEBRA – QUIVER YANGIAN:

$$
\begin{aligned}
[e_{\bullet}(x), f_{\bullet}(y)\} &\sim \delta_{\bullet,\bullet} \frac{\psi_{\bullet}(x) - \psi_{\bullet}(y)}{x-y} , \\
\psi_{\bullet}(x) e_{\bullet}(y) &\simeq \varphi_{\bullet,\bullet}(x-y) e_{\bullet}(y) \psi_{\bullet}(x) , \\
\psi_{\bullet}(x) f_{\bullet}(y) &\simeq [\varphi_{\bullet,\bullet}(x-y)]^{-1} f_{\bullet}(y) \psi_{\bullet}(x) , \\
e_{\bullet}(x) e_{\bullet}(y) &\sim (-1)^{|\bullet||\bullet|} \varphi_{\bullet,\bullet}(x-y) e_{\bullet}(y) e_{\bullet}(x) , \\
f_{\bullet}(x) f_{\bullet}(y) &\sim (-1)^{|\bullet||\bullet|} [\varphi_{\bullet,\bullet}(x-y)]^{-1} f_{\bullet}(y) f_{\bullet}(x) ,
\end{aligned}
$$

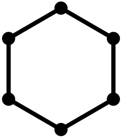
$$
\varphi_{\bullet,\bullet}(z) = (\text{1-LOOP}) = \frac{\prod_{\mathfrak{a} \in \mathrm{arrows}(\bullet \to \bullet)} (z + m_{\mathbb{C},\mathfrak{a}})}{\prod_{\mathfrak{b} \in \mathrm{arrows}(\bullet \to \bullet)} (z - m_{\mathbb{C},\mathfrak{b}})}
$$

$$
\text{BPS ALGEBRA}(xy = z^n w^m) = Y\left(\widehat{\mathfrak{gl}}_{n|m}\right)
$$

QUIVER DIAGRAM $xy = z^6$



DYNKIN DIAGRAM $\widehat{\mathfrak{gl}}_6$



Summary

1. BPS states in supersymmetric field theories give nice model families to study non-perturbative effects
2. There are problems in enumerative geometry whose solutions are given by indices (or others observables) of BPS states
3. BPS states establish a phenomenon of wall-crossing and give rise to BPS algebra
4. In certain cases BPS algebra resemble familiar algebras. This fact can be used as a support for duality relations.

Thank you for your attention!