An Introduction to non-Archimedean Geometry

John Welliaveetil

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Another motivation was to define analogues of certain analytic techniques such as power series in a formal algebraic way to use in Number theory.

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However, unlike $\mathbb{R}$ and $\mathbb{C}$, the geometry of the *p*-adics is strange !

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2. We discuss the obstructions to developing a theory of geometry over non-Archimedean fields
3. We introduce Vladimir Berkovich's approach to non-Archimedean geometry which provides analytic spaces with nice topological properties.
4. We describe our recent work that seeks to generalize results of Hrushovski–Loeser concerning the homotopy type of the Berkovich analytifications of quasi-projective varieties.

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Definition (Order of Vanishing at p)

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2. Given $q \in \mathbb{Q}$, we write $q = a/b$ with $a, b \in \mathbb{Z}$ such that $\gcd(a, b) = 1$ and define $\mathrm{ord}_p(a/b) = \mathrm{ord}_p(a) - \mathrm{ord}_p(b)$.

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Definition (p-adic norm)

Given $z \in \mathbb{Q}$, we define

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This property is stronger than the triangle inequality and it is this inequality that *warps* the geometry of the p-adic numbers under the p-adic metric.

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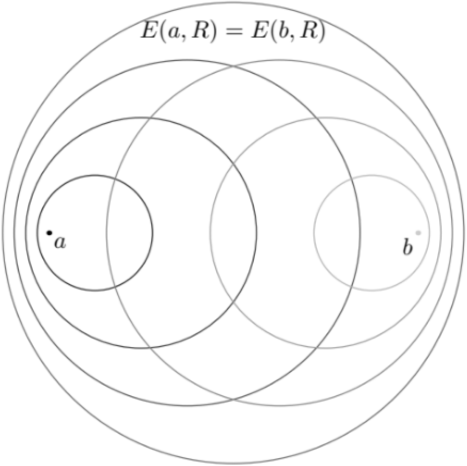

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This makes it hard to draw pictures of simple objects like the closed or open ball in $\mathbb{Q}_p$.

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$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 \ldots$$

where for every i, $a_i \in \mathbb{Q}_p$ and the series converges to $f(x)$ in some open ball around x_0.

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Let $c_1, \ldots, c_m \in \mathbb{Q}_p$. For every $1 \leq i \leq m$ and $z \in B(x_i, 1)^-$, we set

$$f(z) = c_i.$$

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3. Develop a good notion of structure sheaf for such spaces which takes the place of a theory of non-Archimedean analytic functions.

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$$\mathbb{Q}_p\langle T_1, \dots, T_n\rangle := \{ \sum_{\mathbf{i}:=(i_1,\dots,i_n)\in\mathbb{N}^n} a_{\mathbf{i}} T_1^{i_1} \dots T_n^{i_n} | a_{\mathbf{i}} \in \mathbb{Q}_p, a_{\mathbf{i}} \mapsto 0 \text{ when } |\mathbf{i}| \mapsto \infty\}.$$

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Such algebras are called *Tate algebras* and belong to a more general class called *Affinoid algebras*.

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The main idea of Berkovich was to add points to Tate's rigid spaces !

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Given $f \in \mathbb{Q}_p\langle T\rangle$, we can define a function

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The topology on $\mathcal{M}(\mathbb{Q}_p\langle T\rangle)$ is the weakest topology such that such functions are continuous.

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defines a path from a to the *Gauss point* - $\eta_{0,1}$.

How to draw the Berkovich closed disk

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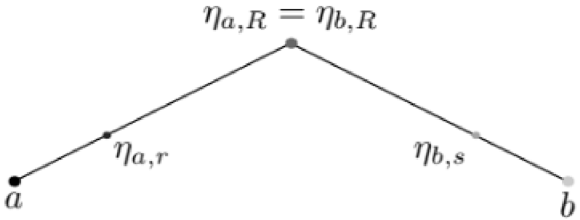
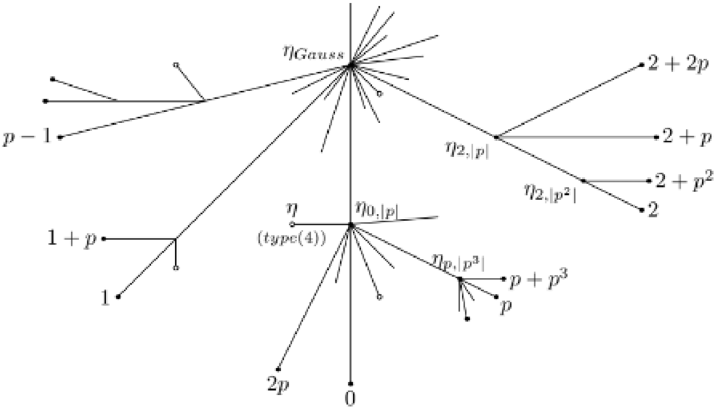
The glueing rule : Let $R := |a - b|_p$.



The full picture



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Let X be a scheme of finite type over the field K. Let X^{an} denote the set of pairs (x, η) where x is a scheme theoretic point of X and η is a rank one valuation on the residue field $K(x)$ that extends the valuation of the field K. The set X^{an} is endowed with a topology whose pre-basic open sets are of the form $\{(x, \eta) \in U^{\mathrm{an}} || f(\eta)| \in W\}$ where U is a Zariski open subset of X, $f \in \mathcal{O}_X(U)$ and W is an open subspace of $\mathbb{R}_{\geq 0}$.

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2. X is separated $\iff$ X^{an} is Hausdorff.
3. X is proper $\iff$ X^{an} is compact.
4. The space X^{an} contains $X(K)$ as a dense subset and the topology induced on $X(K)$ is the valuative topology.

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Theorem (Hrushovski–Loeser)

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Corollary (Hrushovski–Loeser)

Let V be a quasi-projective K-variety. Then V^{an} is locally contractible.

Compatible deformation retractions.

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Furthermore, if e denotes the end point of the interval I then the images of the deformations $\Upsilon := H(e, V^{\mathrm{an}})$ and $\Upsilon' = H'(e, V'^{\mathrm{an}})$ are homeomorphic to finite simplicial complexes.

Generic version of Statement

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Let $\phi\colon V' \to V$ be a morphism between quasi-projective K-varieties whose image is dense. There exists a finite partition $\mathcal{V}$ of V into locally closed sub-varieties such that for every $W \in \mathcal{V}$, there exists a generalized real interval I_W and a pair of deformation retractions

$$H'_W\colon I_W \times {V'_W}^{\mathrm{an}} \to {V'_W}^{\mathrm{an}}$$

and

$$H_W\colon I_W \times V_W^{\mathrm{an}} \to V_W^{\mathrm{an}}$$

which are compatible with respect to the morphism $(\phi_{|V'_W})^{\mathrm{an}}$ and whose images are homeomorphic to finite simplicial complexes.

When the base is a curve

Theorem (JW)

Let S be a smooth connected K-curve and X be a quasi-projective K-variety. Let $\phi\colon X \to S$ be a surjective morphism such that every irreducible component of X dominates S. We assume in addition that the fibres of ϕ are of dimension 1. There exists a pair of deformation retractions

$$H'\colon I \times X^{\mathrm{an}} \to X^{\mathrm{an}}$$

and

$$H\colon I \times S^{\mathrm{an}} \to S^{\mathrm{an}}$$

which are compatible with respect to the morphism ϕ^{an} and whose images are homeomorphic to finite simplicial complexes.

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The key to defining homotopies in this context is to make use of deep continuity criteria for functions defined on $\widehat{V}$. The final homotopy on $\widehat{V}$ is a composition of several homotopies, each of whose continuity can be verified using the results of Hrushovski–Loeser.

Thank you!