Gauge theories on geometric spaces

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Kavli IPMU Colloquium

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Origins from Physics

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Electrodynamics
G=U(1)
Maxwell 1864/ Weyl 1918

Weak/Strong force
G=SU(2)/SU(3)
Yang and Mills 1954

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- $\mathcal{G}(E)$ group of gauge transformations

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† Mirror symmetry and geo Langlands program

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† Infinitely many smooth strs on $\mathbb{R}^4$!

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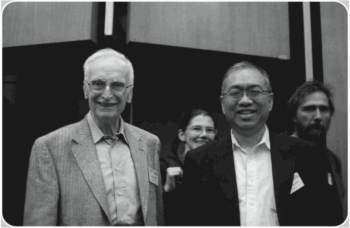

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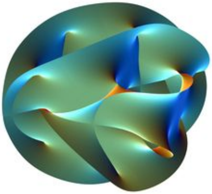

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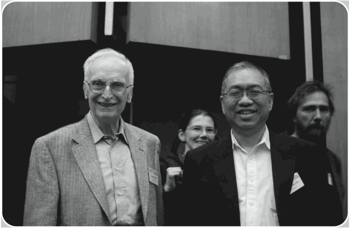

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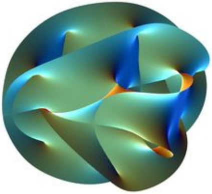

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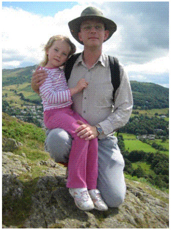
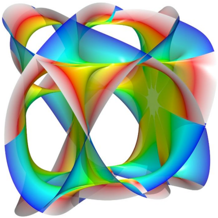


Joyce first constructed examples of $G2$ and $Spin(7)$ manifolds



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For (1), analysis too difficult ! Need alge geo for help

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As we use alge geo method, so can start w/ moduli of sheaves (no bdl inside), which leads to more interesting applications

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In a joint work with M. Kool, we use counting inv of ideal sheaves of pts on $\mathbb{C}^4$ to give a conjectural formula for counting weighted solid partitions.

THE END

Thank you for your attention !