Approaches to double loop groups

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Introduction

- This talk is about loop groups with a view toward double loop groups.
- We will discuss different approaches to loop groups.
- Double loop groups require a blend of these methods

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Groups

- Recall that a group is a set with an associative multiplication operation and inverses.
  - Captures in mathematics the notion of symmetry.
- We will focus on SL_n, the n-th **special linear group**.

$$SL_n = \{A \in \mathrm{Mat}_{n \times n} \mid \det(A) = 1\}$$

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Simplest example of SL_2

$$\mathbf{SL}_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid ad - bc = 1 \right\}$$

- An algebraic Lie group
  - A manifold defined by (multivariable) polynomial equations.
- For example, we see that SL_2 is 3-dimensional.

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Dynkin diagram

- A special subset of elements of SL_n is the (unipotent) upper triangular matrices

$$\begin{bmatrix} 1 & * & * & \cdots & * \\ & 1 & * & \cdots & * \\ & & \ddots & \cdots & \vdots \\ & & & 1 & * \\ & & & & 1 \end{bmatrix}$$

- Even more special are the ones concentrated on the super diagonal

$$\begin{bmatrix} 1 & * & & & \\ & 1 & * & & \\ & & \ddots & \ddots & \\ & & & 1 & * \\ & & & & 1 \end{bmatrix}$$

- These super diagonal (and the corresponding sub diagonal) matrices generate SL_n.

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- We label these entries by ①, ②, ⋯ (n-1):

$$\begin{bmatrix} 1 & \textcircled{1} & & & \\ & 1 & \textcircled{2} & & \\ & & \ddots & \ddots & \\ & & & 1 & \textcircled{n-1} \\ & & & & 1 \end{bmatrix}$$

- Then organize them into the following **Dynkin diagram**:

$$\textcircled{1} — \textcircled{2} — \cdots — \textcircled{n-1}$$

- Nodes = group generators. Edges = relations.
- One can classify all (simple) Lie groups this way.
  - Gives a uniform combinatorial approach to their representation theory and geometry.

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Loop groups

- Starting with $\mathbf{SL}_n$, we consider the **loop group**

$$\mathcal{L}\mathbf{SL}_n = \{ \text{ functions on a circle with values in } \mathbf{SL}_n \}$$

- $\mathcal{L}\mathbf{SL}_n$ is a group under pointwise multiplication.
- $\mathcal{L}\mathbf{SL}_n$ is infinite dimensional.
  - Each point on the circle has finitely-many degrees of freedom, but the circle has infinitely many points.
- $\mathcal{L}\mathbf{SL}_n$ has been deeply studied from many perspectives
- Our goal is to extend understanding to $\mathcal{L}\mathcal{L}\mathbf{SL}_n$

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Four approaches to loop groups

- Fourier series
- Kac-Moody groups
- p-adic groups
- Coulomb branches of 3d gauge theories

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Fourier series

- Functions on a circle = 2π-periodic functions on a line.
  - We can deal with these functions via Fourier series
  - This is a good way to identify the degrees of freedom
- We can therefore analyse $\mathcal{L}SL_n$ as Fourier series.
- For example: $\mathcal{L}SL_2$ consists of matrices

$$\begin{bmatrix} \sum a_n e^{in\theta} & \sum b_n e^{in\theta} \\ \sum c_n e^{in\theta} & \sum d_n e^{in\theta} \end{bmatrix}$$

  such that

$$(\sum a_n e^{in\theta})(\sum d_n e^{in\theta}) - (\sum b_n e^{in\theta})(\sum c_n e^{in\theta}) = 1$$

- Infinitely many variables, infinitely many equations.

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Kac-Moody groups

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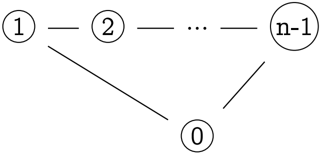

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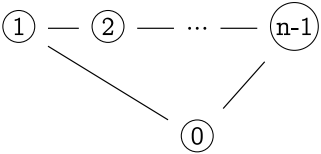

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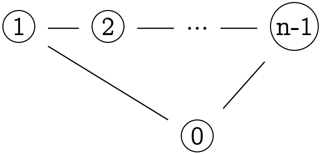

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  - This is a very delicate issue from the Fourier series approach.
  - The Kac-Moody approach gives this for free.
- Unfortunately, there is no Kac-Moody theory for $\mathcal{L}\mathcal{L}\mathbf{SL}_n$.
  - This is main difficulty about $\mathcal{L}\mathcal{L}\mathbf{SL}_n$.

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p-adic groups

- Let p be a prime number
- Analogy between Fourier series and numbers base p.
- $e^{i\theta} \leftrightarrow p$

$$\sum a_n e^{in\theta} \leftrightarrow \sum c_n p^n$$

- Here $c_n \in \{0, 1, \cdots, p-1\}$.
- We write $\mathbb{Q}_p = \{\sum c_n p^n\}$.
- The idea: replace $\mathcal{L}SL_n$ by $SL_n(\mathbb{Q}_p)$
  - Instead of geometry, we do modular arithmetic.
- Aside: you may already know the idea of $e^{i\theta} \leftrightarrow p$.
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Geometry from modular arithmetic

- Let X be an algebraic variety (e.g. an algebraic manifold).
- Then

$$\#\text{ points in } X \text{ modulo } p \sim Cp^{\dim X}$$

- So from arithmetic, we can extract the geometric quantity $\dim X$.
- Similarly we can extract other geometric quantities from arithmetic:
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Some of my results

- Consider $\mathcal{L}SL_n(\mathbb{Q}_p)$ (a p-adic Kac-Moody group)
- M: algebraic algorithm for multiplication in the Iwahori-Hecke algebra for this group
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Coulomb branches

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- There is also

$$\text{(1)} — \text{(2)} — \cdots — \text{(n-1)} \rightsquigarrow \text{quiver} \rightsquigarrow \text{3d } \mathcal{N} = 4 \text{ quiver gauge theory}$$

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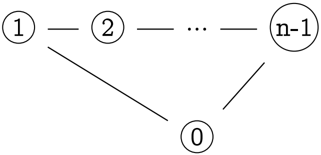
- 

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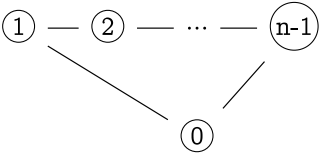
- 

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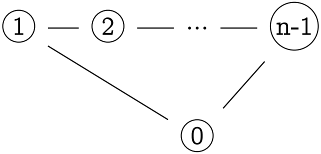
- 

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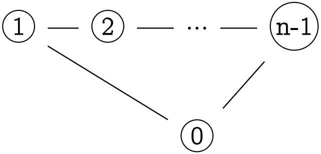
- 

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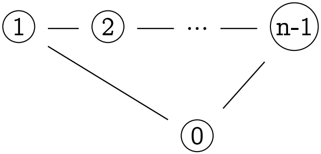
- 

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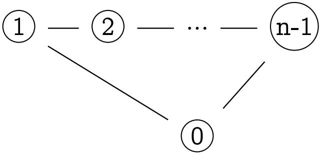
Work just with the Coulomb branch

- Nakajima-Takayama: an explicit model of Coulomb branches in Type A, i.e. associated to



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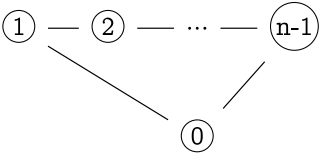

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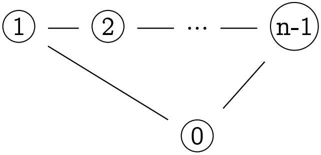

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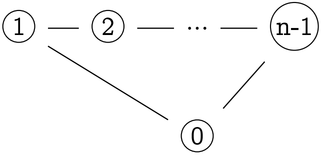

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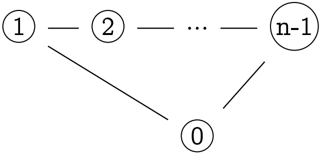

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