Twistors, Integrability and 4d Chern-Simons Theory

David Skinner

DAMTP, Cambridge

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Based on 2011.04638 with Roland Bittleston

Introduction

Costello, Witten & Yamazaki have introduced a beautiful new approach to quantum integrable systems based on a 4d variant of Chern-Simons theory

$$S[A'] = \frac{1}{2\pi i}\int_{\Sigma\times C} \omega \wedge CS(A')$$

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The aim of this talk is to relate this new story to a much older scheme for organising integrable systems, at least at the classical level

The ASDYM Equations

The anti-self-dual Yang-Mills equations on a four-manifold M state

$$F = -\star F$$

- They are conformally invariant
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Other actions for ASDYM break some part of the conformal invariance

Yang's J-matrix

On $\mathbb{R}^4$ with metric $ds^2 = 2(dz\,d\tilde{z} - dw\,d\tilde{w})$ the ASDYM eqns become

$$F_{wz} = 0\,, \qquad F_{\tilde{w}\tilde{z}} = 0\,, \qquad F_{w\tilde{w}} = F_{z\tilde{z}}$$

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- Regarding the first two eqns as flatness conditions, introduce $h, \tilde{h} \in \Omega^0(\mathbb{R}^4, \mathfrak{g})$ by $D_w h = D_z h = 0$ and $D_{\tilde{w}} \tilde{h} = D_{\tilde{z}} \tilde{h} = 0$

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This arises as the eom of the 4d WZW action

$$S[\sigma] = \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{tr}(J \wedge \star J) + \frac{1}{3}\int_{\mathbb{R}^4\times[0,1]} \omega \wedge \mathrm{tr}(\tilde{J} \wedge \tilde{J} \wedge \tilde{J})$$

with $\tilde{J} = -d\tilde{\sigma}\,\tilde{\sigma}^{-1}$ and $\tilde{\sigma}$ any homotopy from σ to 1 [Donaldson;Nair,Schiff; Losev,Moore,Nekrasov,Shatashvili]

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where $\mu = d\tilde{w} \wedge d\tilde{z}$ [Leznov,Mukhtarov; Parkes; Mason,Woodhouse; Siegel]

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- Closely connected to $\mathcal{N} = 2$ heterotic / open strings [Ooguri,Vafa;Berkovits,Vafa]

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Many reductions are possible. For example, reducing by

- a single translation gives Bogomolny equations
- a translation and orthogonal rotation gives the Ernst equation [Penna]
- the Euclidean group on a non-null 2-plane gives Toda theory
- 2-plane and discrete subgroup of rotations gives extended Toda theory
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- It was once hoped that all integrable systems arise as symmetry reductions of ASDYM (though the KP hierarchy in particular does not appear to sit naturally in this framework)

The Lax Connection and Twistor Space

Introducing a Weyl spinor $\pi_{\dot\alpha}$, the ASDYM equations themselves can be written in Lax form as

$$[\pi^{\dot\alpha} D_{\alpha\dot\alpha}, \pi^{\dot\beta} D_{\beta\dot\beta}] = 0 \qquad \text{for all } \pi_{\dot\alpha},$$

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$\pi^{\dot\alpha}\partial_{\alpha\dot\alpha}$ is an anti-holomorphic derivative, so the Lax formulation says an ASDYM connection on $\mathbb{E}^4$ pulls back to give a holomorphic bundle on $\mathbb{PT}$

[Penrose,Ward]

Holomorphic Chern-Simons Theory

Given a (partial) connection $\bar\partial + \mathcal{A}$ on a complex bundle $E \to W$ over a CY 3-fold W, holomorphic Chern-Simons theory has action [Witten]

$$S[\mathcal{A}] = \frac{1}{2\pi i} \int_W \Omega \wedge \mathrm{tr}\left(\mathcal{A}\,\bar\partial\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\right)$$

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The simplest possibilities to consider are

$$\Omega = \frac{D^3 Z}{(A\cdot Z)^2(B\cdot Z)^2} \qquad \text{or} \qquad \Omega = \frac{D^3 Z}{(A\cdot Z)^4}$$

but many other choices are possible

Boundary Conditions

The (3,0)-form $\Omega = D^3Z/(A \cdot Z\, B \cdot Z)^2$ has double poles on two $\mathbb{CP}^2$s in $\mathbb{CP}^3$. Removing their $\mathbb{CP}^1$ intersection leaves us with

$$\mathbb{PT} = (\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{CP}^1)$$

described using homog coordinates $[\pi_{\dot\alpha}] \in \mathbb{CP}^1$ and ω^α on the fibres

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$$2\pi i\,\delta S = \int_{\mathbb{PT}}\Omega\wedge\mathrm{tr}(\delta\mathcal{A}\wedge\mathcal{F}) + \int_{\mathbb{PT}}\bar\partial\Omega\wedge\mathrm{tr}(\delta A\wedge\mathcal{A})$$

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- Similar conditions are imposed on gauge transformations

Reduction to the Donaldson-Nair-Schiff action

The boundary conditions mean that gauge invariant information is contained in

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- Then set $\mathcal{A} = (\mathcal{A}')^{\hat\sigma} = \hat\sigma^{-1}\bar\partial\hat\sigma + \hat\sigma^{-1}\mathcal{A}'\hat\sigma$

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- Doing so we obtain the 4d WZW action [Donaldson;Nair,Schiff] for σ, with α and β related to choice of coords $(w, \tilde{w}, z, \tilde{w})$ above

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- Helpful to note that $\hat{\sigma}$, but not its derivatives along $\mathbb{CP}^1$, is invariant under $U(1)$ rotations of $\mathbb{CP}^1$ around the α,$\hat{\alpha}$ axis

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(Further examples are considered in the paper)

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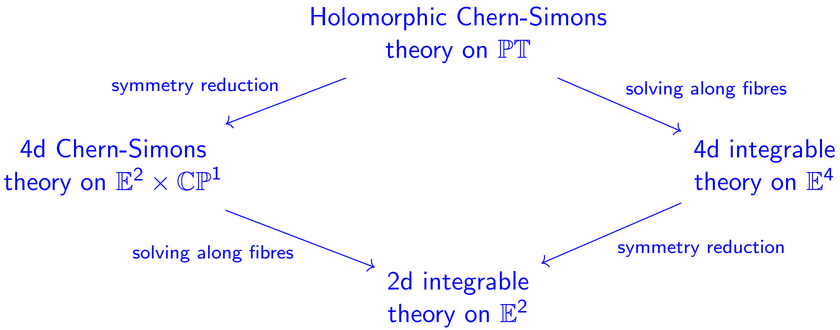
- Costello & Yamazaki show this 4d Chern-Simons theory is equivalent to the 2d PCM model on $\mathbb{E}^2$

The details of which theories are obtained depend on the choice of Ω and choice of symmetry reduction

The relation may be summarized by the following commutative diagram:



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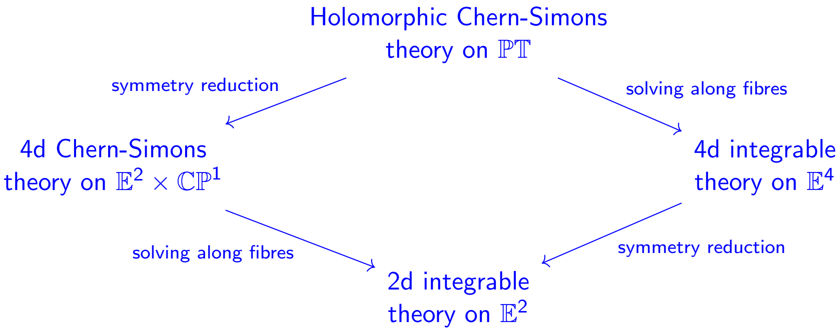

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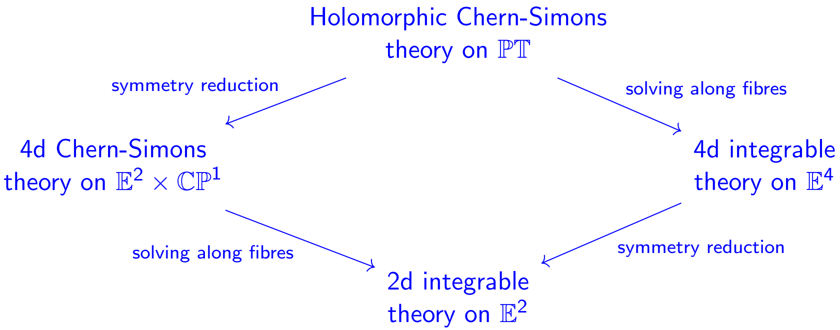

Note that both the twistor theory and 4d Chern-Simons theory include the spectral parameter as part of the geometry. In this sense, they each make integrability manifest.

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- This 5d theory bears the same relation to the 3d Bogomolny theory as Costello-Yamazaki theory does to the 2d WZW model

Anti Self-Dual Gravity [Bittleston,Ma,Sharma,DS] in progress

Deforming the $\mathbb{C}$-str of twistor space deforms the conformal structure on $\mathbb{R}^4$ [Penrose, Atiyah,Hitchin,Singer]. Perturbatively $\bar{\partial} \mapsto \bar{\partial} + V$ for $V \in \Omega^{0,1}(\mathbb{PT}, T_{\mathbb{PT}})$

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- We believe the $4^{\rm th}$-order case gives the action [Plebanski; Ooguri,Vafa]

$$S[\Phi] = \int_{\mathbb{R}^4} \frac{1}{2}\partial\Phi\bar\partial\Phi + \frac{1}{3}\Phi\,\partial\bar\partial\Phi\,\partial\bar\partial\Phi$$

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- For $G \neq SO(8)$ the 4d theory is not anomalous, but no longer comes from a twistor progenitor. Integrability is expected to be broken at the quantum level

Conclusions

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- Further reductions should lead to Toda, sine-Gordon, NLS integrable systems. Some of the systems we obtain are not just symmetry reductions of ASDYM; are there others?

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- Implement the symmetry reduction at the quantum level (perhaps by a partial topological twist?)
- The connection to $\mathcal{N} = 2$ strings seems central and clearly deserves further exploration

Thank You