Feigin–Semikhatov conjecture and its applications.

based on joint work w/ T. Creutzig, N. Genra arXiv:2005.10713
— " — and R. Sato, to appear

Goal of this talk

- determine the level $k \in \mathbb{C}$ when the (simple) principal W-superalg

$$W_k(\mathfrak{sl}_{1|n}) \quad \left(\underset{\exists!\ \text{simple quotient}}{\twoheadleftarrow} \underbrace{W^k(\mathfrak{sl}_{1|n})}_{\text{universal}} \overset{\text{def}}{=} H^{\frac{\infty}{2}+0}_{f_{prin}}(V^k(\mathfrak{sl}_{1|n})) \right)$$

$$f_{prin} = \begin{pmatrix} 0 & \\ & \begin{matrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{matrix} \end{pmatrix} \in \mathrm{Mat}(\mathbb{C}^{1|n})$$

defines a rational SCFT

(rational: C_2-cofinite (Huang), s.s. braided tensor supercategory)

- understand the repr. theory at these levels. $\{\mathrm{Irr}(\text{—}),\ K(\text{—})\}$

Main theorem

(1) $k = k(r) = -(n-1) + \dfrac{n-1}{n+r}$ $\quad (r \in \mathbb{Z}_{\geq 3},\ \gcd(n-1, n+r) = 1)$

($h^\vee(\mathfrak{sl}_{1|n})$; admissible levels.)

$\Rightarrow W_{k(r)}(\mathfrak{sl}_{1|n})$ defines a rational SCFT.

(2) $$K(W_{k(r)}(\mathfrak{sl}_{1|n})) \simeq \left(K(L_r(\mathfrak{sl}_n)) \underset{\mathbb{Z}[\mathbb{Z}_n]}{\otimes} \mathbb{Z}[\mathbb{Z}_{(n+r)n}] \right)^{\mathbb{Z}_n}$$

$\circlearrowleft \mathbb{Z}_n$

$\pi_{P/Q}(\lambda) + a = 0 \in \mathbb{Z}_n.$

$$\lambda = \sum_{\substack{i=0 \\ \uparrow \\ i \in \mathbb{Z}_n}}^{n-1} n_i \Lambda_i \qquad \Lambda_i \overset{\sigma}{\longmapsto} \Lambda_{i+1}$$

$$\Big\downarrow \pi_{\phi/a}$$

$$i \in \mathbb{Z}_n$$

The case $n=2$ (Adamović around 2000)

$W^k(sl_1|_2) \simeq N=2$ superconformal alg

	Heisenberg	Virasoro	
generators:	$J(z)$,	$L(z)$,	$G^{\pm}(z)$
Δ :	$\underline{\underline{1}}$	2	$\underline{3/2}$
J_0 :	0	0	± 1
	even	even	odd

$$\begin{cases} G^{\pm}(z)\, G^{\pm}(w) \sim 0 \\ G^{+}(z)\, G^{-}(w) \sim \dfrac{\frac{2c(k)}{3}}{(z-w)^3} + \dfrac{2J(w)}{(z-w)^2} + \dfrac{2L(w) + \partial J(w)}{(z-w)} \end{cases}$$

Key Point (duality w/ $V^k(sl_2)$)

- Kazama-Suzuki coset construction

$$W^{k(r)}(sl_1|_{\underset{n}{2}}) \simeq \mathrm{Com}\big(\pi,\ \underline{V^{\cancel{k}}(sl_2) \otimes V_{\mathbb{Z}}}\big)$$

M $\qquad$? $\quad \pi_{\mathbb{Z}} = \langle x_1(z) \rangle$

$$b(z) \longmapsto h(z) \otimes 1 + 1 \otimes 2x_1(z)$$

- Feigin-Semikhatov-Tipunin construction

identify with known alg?

$$\boxed{V^{r}(sl_2)} \simeq \mathrm{Com}\big(\pi,\ W^{k(r)}(sl_1|_{\underset{n}{2}}) \otimes V_{\sqrt{-1}\mathbb{Z}}\big)$$

$$\Longrightarrow \quad W^{k(r)}(\mathfrak{sl}_{1|2})\text{-mod} \underset{\simeq}{\overset{\times}{\rightleftarrows}} V^r(\mathfrak{sl}_2)\text{-mod} \longrightarrow L_r(\mathfrak{sl}_2)\text{-mod} \ (r \in \mathbb{Z}_{\geq 0})$$

$\mathrm{Spec}(J_0) \subset a + \mathbb{Z}$ $\qquad$ $\mathrm{Spec}(h_0) \subset a' + 2\mathbb{Z}$

$$\begin{aligned} \mathrm{Hom}_\pi(\pi_\lambda, M \otimes V_{\mathbb{Z}}) &\longleftarrow\!\shortmid \ M \\ N &\longmapsto \mathrm{Hom}_\pi(\pi_\lambda, N \otimes V_{\sqrt{-1}\mathbb{Z}}) \end{aligned}$$

$$\mathrm{Irr}(\#) \longleftrightarrow \mathrm{Irr}(\#)$$

In the semisimple case $(r \in \mathbb{Z}_{\geq 0})$ $\qquad$ simple current ext (explain later)

$$L_r(\mathfrak{sl}_2) \otimes V_{\sqrt{-2(r+2)}\mathbb{Z}} \overset{\sim}{\hookrightarrow} W_{k(r)}(\mathfrak{sl}_{1|2}) \otimes V_{\sqrt{-1}\mathbb{Z}}$$

$$\oplus\, L_r(r\Lambda_1) \otimes V_{\frac{-(r+2)}{\sqrt{-2(r+2)}} + \sqrt{-2(r+2)}\mathbb{Z}}$$

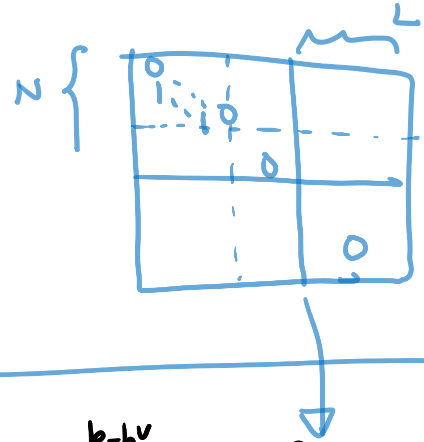
$\Rightarrow$ fusion rules are determined via those of $L_r(\mathfrak{sl}_2)$ & $V_{\sqrt{-2(r+2)}\mathbb{Z}}$

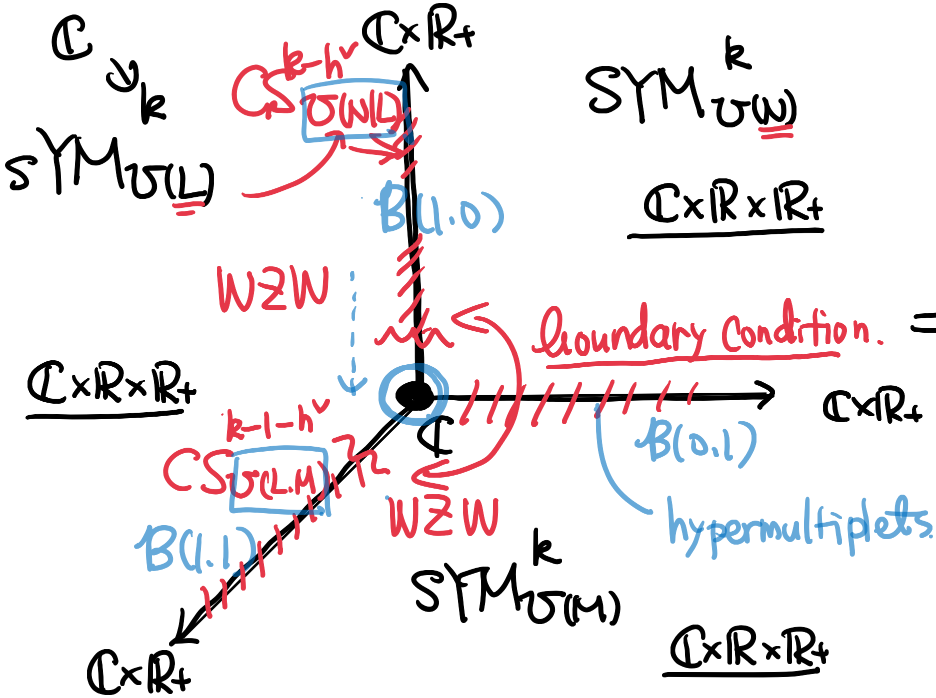
Q How to generalize it to the cases $n \geq 3$?

want to find "$V^r(\mathfrak{sl}_2)$"

Gaiotto–Rapčák conjecture

$$\left\{\begin{array}{l}\text{4d twisted } \mathcal{N}=4 \\ \quad \text{Super Yang-Mills theory}\end{array}\right\} \rightsquigarrow \left\{\underline{\text{vertex algebras}}\right\}$$

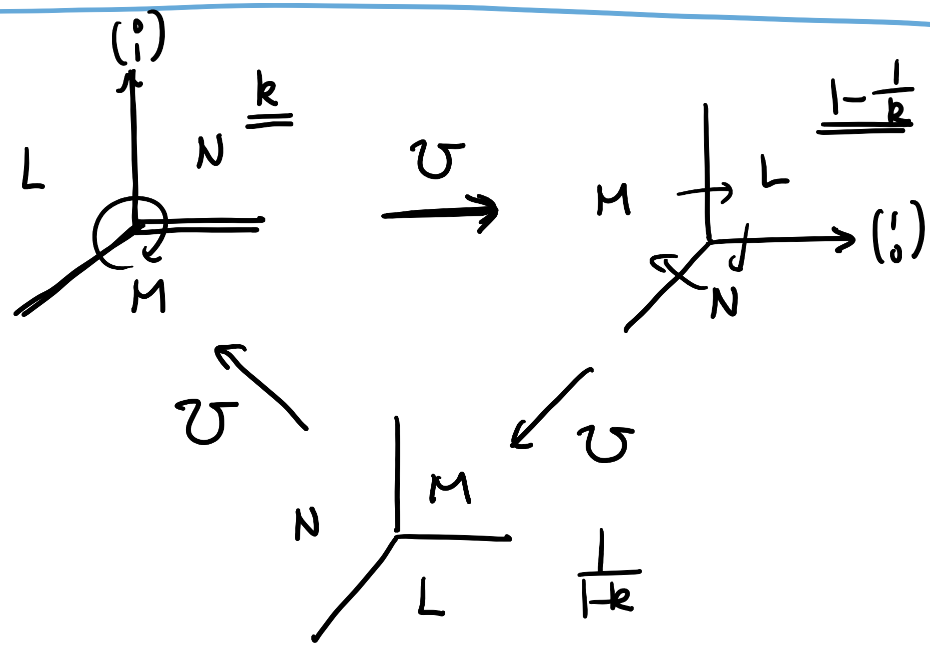




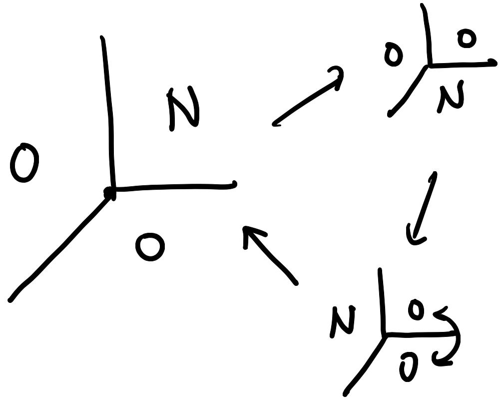
$$Y_{L,M,N}[k] \overset{\mathrm{def}}{=} \begin{cases} \mathrm{Com}\left(V^{k-1-h^\vee}(\mathfrak{gl}_{M|L}),\ W^{k-h^\vee}(\mathfrak{gl}_{N|L}, f_{N-M,1^M|1^L})\right) & (N>M) \\ \mathrm{Com}\left(V^{k-1-h^\vee}(\mathfrak{gl}_{N|L}),\ V^{k-h^\vee}(\mathfrak{gl}_{N|L})\otimes\beta\gamma^N\otimes bc^L\right) & (N=M) \\ \mathrm{Com}\left(V^{-k-h^\vee}(\mathfrak{gl}_{N|L}),\ W^{-k-h^\vee-1}(\mathfrak{gl}_{M|L}, f_{M-N,1^N|1^L})\right) & (N<M) \end{cases}$$

$$\boxed{PSL(2,\mathbb{Z}) \circlearrowright \left\{B(p,q)\right\}_{(p,q)\in\mathbb{Z}^2/\{\pm1\}}} \Longrightarrow$$

$$\sigma = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$



Ex



$\Longrightarrow$

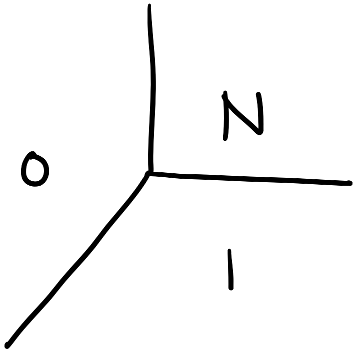
Feigin-Frenkel duality

$$W^{k-N}(\mathfrak{gl}_N) \simeq W^{\boxed{\frac{1}{k}-N}}(\mathfrak{gl}_N)$$

Goddard-Kent-Olive $\simeq$ $\qquad\qquad$ $\simeq$

$$\mathrm{Com}\left(\underline{V^{-\frac{1}{1-k}-N+1}}(\mathfrak{gl}_N),\ \underline{V^{-\frac{1}{1-k}-N}}(\mathfrak{gl}_N)\otimes \underline{\underline{V_{\mathbb{Z}^{\oplus N}}}}\right)$$

Ex (Feigin-Semikhatov conjecture '04)



$\Longrightarrow$

$$\underline{\mathrm{Com}\left(\pi, W^{k-N}(\mathfrak{gl}_N, f_{sub})\right)} \simeq \underline{\mathrm{Com}\left(\pi, W^{\frac{1}{k}-N+1}(\mathfrak{gl}_{N|1})\right)}$$

$\simeq$ $\qquad\qquad$ $\simeq$

$$\mathrm{Com}\left(\underline{V^{-\frac{1}{1-k}-N+1}(\mathfrak{gl}_N),\ V^{-\frac{1}{1-k}-N+1}(\mathfrak{gl}_{N|1})}\right)$$

ThmA (Creutzig-Linshaw / Creutzig-Genra-N) this level : Heisenberg vertex alg. degenerate

For $(k,\ell) \in \mathbb{C}^2 \setminus \{(-n+\frac{n}{n-1}, -\frac{(n-1)^2}{n})\}$ satisfying

$h^\vee(\mathfrak{sl}_n)$ $\boxed{(k+n)(\ell+n-1)=1,}$ $h^\vee(\mathfrak{sl}_{1|n})$

(1)
$$\mathrm{Com}(\pi, W^k(\mathfrak{sl}_n, f_{sub})) \simeq \mathrm{Com}(\pi, W^\ell(\mathfrak{sl}_{1|n})) \quad -(*)$$

(2) $(*)$ holds when we replace

$$\begin{pmatrix} W^k(\mathfrak{sl}_n, f_{sub}) \\ W^\ell(\mathfrak{sl}_{1|n}) \end{pmatrix} \rightsquigarrow \begin{pmatrix} W_k(\mathfrak{sl}_n, f_{sub}) \\ W_\ell(\mathfrak{sl}_{1|n}) \end{pmatrix}$$

Moreover, we have

ThmB (Creutzig-Genra-N)

For $(k,\ell) \in \mathbb{C}^2$ s.t. $(k+n)(\ell+n-1)=1$,

(1) Kazama-Suzuki type construction

vertex superalg

$$W^\ell(\mathfrak{sl}_{1|n}) \simeq \mathrm{Com}(\pi, W^k(\mathfrak{sl}_n, f_{sub}) \otimes V_{\mathbb{Z}})$$

we can replace the universal W-superalg w/ simple quotients

(2) Feigin-Semikhatov-Tipunin type construction

$$W^k(\mathfrak{sl}_n, f_{sub}) \simeq \mathrm{Com}(\pi, W^\ell(\mathfrak{sl}_{1|n}) \otimes V_{\sqrt{-1}\mathbb{Z}})$$

(sketch of proof of Thm A & B) — use a nice free field realization + screening kernel description at generic level.

Recall the proof of Feigin-Frenkel duality

$$W^k(\mathfrak{g}) \simeq W^{{}^Lk}({}^L\mathfrak{g}) \quad (r(k+h^\vee)({}^Lk+{}^Lh^\vee)=1)$$

r: lacing number.

generic

$$\bigcap_{i=1}^{\mathrm{rk}\,\mathfrak{g}} \mathrm{Ker} \int Y(|{-\alpha_i}\rangle, z)\,dz \simeq \bigcap_{i=1}^{\mathrm{rk}\,{}^L\mathfrak{g}} \mathrm{Ker} \int Y(|{-{}^L\alpha_i}\rangle, z)\,dz$$

root $\Longleftrightarrow$ coroot

$\boxed{\text{Miura map (all levels)}}$ $\longrightarrow$

$$\bigcap \pi^{k+h^\vee}_{\mathfrak{h}(\mathfrak{g})} \simeq \bigcap \pi^{{}^Lk+{}^Lh^\vee}_{\mathfrak{h}({}^L\mathfrak{g})}$$

coroot $\Longleftrightarrow$ root $(\mathfrak{g})$

Ex)

$$W^k(\mathfrak{sl}_2) \simeq W^\ell(\mathfrak{sl}_2) \qquad (k+2)(\ell+2)=1$$

$$W^k(\mathfrak{sl}_2) = \left\langle \frac{1}{4(k+2)} h(z)^2 + \frac{k+1}{2(k+2)}\partial h(z) \right\rangle = \left\langle \frac{1}{4(\ell+2)} {}^Lh(z)^2 + \frac{\ell+1}{2(\ell+2)}\partial\, {}^Lh(z) \right\rangle = W^\ell(\mathfrak{sl}_2)$$

$$\boxed{\pi^{k+2}} \xrightarrow{\sim} \pi^{\ell+2}$$

$$\boxed{h(z)} \longmapsto -\frac{1}{\ell+2}\,{}^Lh(z).$$

Our setting

$$W^k(\mathfrak{sl}_n, f_{sub}) = W^k(\mathfrak{sl}_n, f_{sub}; \boxed{\Gamma}) \hookrightarrow V^{\tau(k)}(\mathfrak{sl}_{n,0}) = V^{\tau(k)}(\mathfrak{sl}_2) \otimes \pi^{k+n}_{\mathfrak{g}^\perp}$$

$\mathfrak{sl}_{n-1}$ $\in$

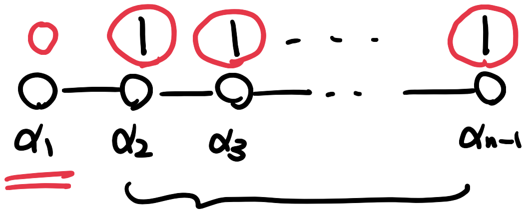
Miura map

0	0	1	2
0	0	1	2
-1	-1	0	1
-1	-2	-1	0

$(\mathfrak{sl}_4)$

$\Gamma : \mathfrak{sl}_n = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{sl}_{n,j}$

(good grading for f_{sub})



$\sum f_{\alpha_i} = f_{sub}$

$\hookrightarrow \beta\gamma \otimes \pi^{k+n}_{\mathfrak{g}}$

Wakimoto repr. of $V^{\bullet}(\mathfrak{sl}_2)$

$\mathbb{P}^1$ $\cup$ $(\mathfrak{sl}_2 \to \mathrm{Vect}(\mathbb{C}))$

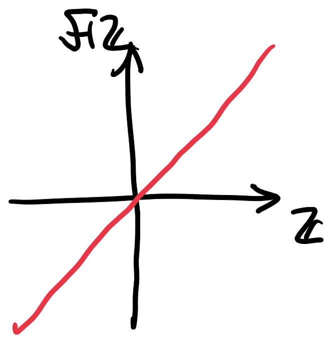
full Heisenberg

$\hookrightarrow \hat{V}_{\mathbb{Z}(1+\sqrt{-1})} \otimes \pi^{k+n}_{\mathfrak{g}}$

bosonization of $\beta\gamma$

(Friedan-Martine-Shenker)

$\subset V_{\mathbb{Z} \oplus \sqrt{-1}\mathbb{Z}}$



$\Rightarrow$ $\forall k \in \mathbb{C}$, $W^k(\mathfrak{sl}_n, f_{sub}) \hookrightarrow \hat{V}_{\mathbb{Z}(1+\sqrt{-1})} \otimes \pi^{k+h^\vee}_{\mathfrak{g}}$

k: generic $\Rightarrow W^k(\mathfrak{sl}_n, f_{sub}) \simeq \bigcap_{i=0}^{n-1} \mathrm{Ker}_{\hat{V}_{\mathbb{Z}(1+\sqrt{-1})} \otimes \pi^{k+h^\vee}_{\mathfrak{g}}} \int S_i^1(z)\,dz.$

Miura	:	$i = 2 \cdots n-1$
Wakimoto	:	$i = 1$
$\beta\gamma$	:	$i = 0$

Similarly,

Miura map

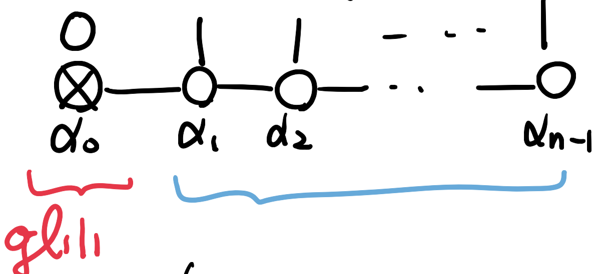
$$W^{\ell}(\mathfrak{sl}_{1|n}) = W^{\ell}(\mathfrak{sl}_{1|n}, f_{prin}; \Gamma) \hookrightarrow V^{\tau(\ell)}(\mathfrak{sl}_{1|n,0}) = V^{\tau(\ell)}(\mathfrak{gl}_{1|1}) \otimes \pi^{\ell+n-1}_{\mathfrak{g}^{\perp}}$$

good grading

$$\begin{array}{|c|c|c|c|} \hline 0 & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 \\ \hline -1 & -1 & 0 & 1 \\ \hline -2 & -2 & -1 & 0 \\ \hline \end{array}$$

$\mathfrak{sl}_{1|3}$

$$\Gamma : \mathfrak{sl}_{1|n} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{sl}_{1|n,j}$$



Wakimoto repr of $V^{\cdot}(\mathfrak{gl}_{1|1})$

$$\hookrightarrow bc \otimes \pi^{\ell+n-1}_{\mathfrak{g}} = V_{\mathbb{Z}} \otimes \pi^{\ell+n-1}_{\mathfrak{g}}$$

super analog of $\beta\gamma \leftrightarrow \Gamma(\mathbb{C}^{1}, \mathcal{D})$

$bc \leftrightarrow \Gamma(\mathbb{C}^{0|1}, \mathcal{D})$

$$\Longrightarrow \begin{cases} \forall \ell \in \mathbb{C}, \quad W^{\ell}(\mathfrak{sl}_{1|n}) \hookrightarrow V_{\mathbb{Z}} \otimes \pi^{\ell+n-1}_{\mathfrak{g}} \\ \ell : \text{generic} \Longrightarrow W^{\ell}(\mathfrak{sl}_{1|n}) \simeq \bigcap_{i=0}^{n-1} \mathrm{Ker}_{V_{\mathbb{Z}} \otimes \pi^{\ell+n-1}_{\mathfrak{g}}} \int S_i^{2}(z)\, dz \end{cases}$$

Now,

generic level

$$\mathrm{Com}(\pi, W^k(\mathfrak{sl}_n, f_{sub})) \simeq \bigcap_{i=0}^{n-1} \mathrm{Ker} \int S_i^1(z)\,dz \subset \mathrm{Com}(\pi, \pi_{\mathbb{Z}} \otimes \pi_{\mathbb{Z}\sqrt{-1}} \otimes \pi_{\mathfrak{g}}^{k+n})$$

$\mathrm{rk} = (1+1+(n-1))-1 = n$

$\mathrm{Com}(\pi, \hat{V}_{\mathbb{Z}(1+\sqrt{-1})} \otimes \pi_{\mathfrak{g}}^{k+n})$

$\wr S$ Feigin-Frenkel $\wr S$ $\wr S$

$$\mathrm{Com}(\pi, W^\ell(\mathfrak{sl}_{1|n})) \simeq \bigcap_{i=0}^{n-1} \mathrm{Ker} \int S_i^2(z)\,dz \subset \mathrm{Com}(\pi, \pi_{\mathbb{Z}} \otimes \pi_{\mathfrak{g}}^{\ell+n-1})$$

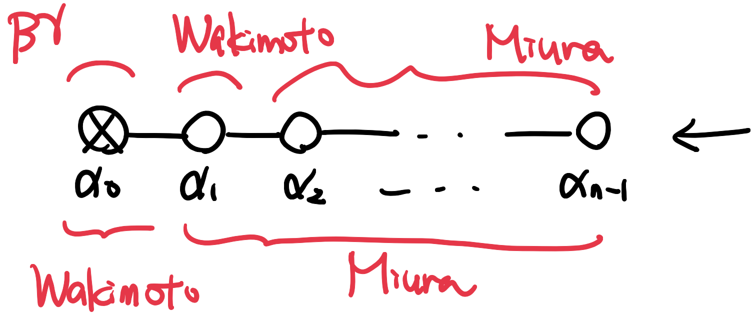
generic level

$\mathrm{Com}(\pi, \boxed{V_{\mathbb{Z}} \otimes \pi_{\mathfrak{g}}^{\ell+n-1}})$

$\mathrm{rk} = (1+n)-1 = n$

π : simple

$\mathfrak{sl}_n$



$\mathfrak{sl}_{1|n}$

//

Applications
$\left(\begin{array}{l} \text{Repr. theory of } \underline{W_k(\mathfrak{sl}_n, f_{sub})} \\ \Rightarrow \quad '' \quad \text{of } \underline{W_\ell(\mathfrak{sl}_{1|n})} \end{array} \right)$

Fact (1) [Arakawa-van Ekeren]

admissible $-n+\frac{q}{p}$, p ~ $n-1$

$(f_{sub}, n-1)$: exceptional

$$W_{-n+\frac{n+r}{n-1}}(\mathfrak{sl}_n, f_{sub}) \quad (r \in \mathbb{Z}_{\geq 0},\ \gcd(n-1, n+r) = 1) \Leftarrow$$

defines a rational CFT

- $\mathrm{Irr}(W_{-n+\frac{n+r}{n-1}}(\mathfrak{sl}_n, f_{sub})) \overset{1:1}{\longleftrightarrow} \mathrm{Irr}(L_r(\mathfrak{sl}_n))$

$$K(W_{-n+\frac{n+r}{n-1}}(\mathfrak{sl}_n, f_{sub})) \simeq K(L_r(\mathfrak{sl}_n)) \quad (\underline{\text{Verlinde formula}})$$

$\boxed{n : \text{even}}$ fusion category.

(2) [Creutzig-Linshaw]

$$\mathrm{Com}(\pi, W_{-n+\frac{n+r}{n-1}}(\mathfrak{sl}_n, f_{sub})) \simeq \boxed{W_{-r+\frac{r+n}{r+1}}(\mathfrak{sl}_r)} \quad (r \geq 3)$$

level-rank duality

$$\mathrm{Com}(\mathrm{RHS}, \ '' \ ) \simeq V_{\sqrt{nr}\mathbb{Z}}$$

$\#=r$

$E(z) \cdots E(z) \in L_r(\mathfrak{sl}_2)$

$F(z) \cdots F(z)$

$$\rightsquigarrow W_{-n+\frac{n+r}{n-1}}(\mathfrak{sl}_n, f_{sub}) \simeq \bigoplus_{a \in \mathbb{Z}_r} \boxed{\mathbb{L}_W(r\Lambda_i)} \otimes V_{\frac{na}{\sqrt{nr}}+\sqrt{nr}\mathbb{Z}} \quad -(*)$$

$W_{-r+\frac{r+n}{r+1}}(\mathfrak{sl}_r) \otimes V_{\sqrt{nr}\mathbb{Z}}$

(Arakawa - van Ekeren.)

$$\mathcal{K}\left(W_{-r+\frac{r+n}{r+1}}(\mathfrak{sl}_n)\right) \simeq \mathcal{K}\left(L_n(\mathfrak{sl}_r)\right)$$

$$\mathbb{L}_W(\lambda) \longleftrightarrow L_n(\lambda)$$

(*) + Kazama-Suzuki

$\Longrightarrow$

Prop (Creutzig-Genra-N-Sato)

For $r \in \mathbb{Z}_{\geq 3}$, $\gcd(n-1, n+r) = 3$

$$\Longrightarrow \quad W_{-(n-1)+\frac{n-1}{n+r}}(\mathfrak{sl}_{1|n}) \simeq \bigoplus_{a\in\mathbb{Z}_r} \mathbb{L}_W(n\Lambda_a) \otimes V_{\frac{(n+r)a}{\sqrt{(n+r)r}}+\sqrt{(n+r)r}\,\mathbb{Z}} \quad \leftarrow$$

invertible objects (w.r.t. $\boxtimes$) $\leftarrow$ simple current

① Generality of simple current ext.

V: $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded simple C_2-cofinite VOSA

$\Longrightarrow$ V-mod : braided tensor supercategory

(Huang-Lepowsky-Zhang)

not abelian

(parity-inhomogeneous morphisms seldom admits kernel/cokernel)

$\mathcal{C}$: braided tensor supercategory ($\mathbb{1}_{\mathcal{C}}$: simple object) $\leftrightarrow V$

$\mathcal{C} \ni M$: simple current $\overset{\text{def}}{\Longleftrightarrow}$ invertible object

$\{S_g\}_{g\in G} \subset Ob(\mathcal{C})$: group of simple currents

G : finite abelian group : $S_g \boxtimes S_h \simeq S_{gh}$.

$$\Longrightarrow \quad \mathcal{C} = \bigoplus_{\xi \in G^\vee} \mathcal{C}_\xi, \quad \mathcal{C}_\xi \ni M \text{ s.t. } \mathcal{M}_{S_g, M} = \xi(g)\, \mathrm{id}_{S_g \boxtimes M}.$$

$\{S_g\}_{g\in G} \curvearrowright \mathcal{C}$, $\quad G^\vee = \mathrm{Hom}(G, \mathbb{C}^\times)$

$\rightsquigarrow$ $K(\mathcal{C})$: $G^\vee$-graded, $\mathbb{Z}[G]$-alg.

Let $\mathcal{E} = \bigoplus_{g\in G} S_g \in Ob(\mathcal{C})$: commutative alg. object ($\Longleftrightarrow$ vertex alg.)

$$\mathcal{C} \xrightarrow[\text{tensor}]{\boxed{\mathcal{E}\boxtimes ?}} \mathrm{Rep}(\mathcal{E}) = \{\mathcal{E}\text{-module objects in } \mathcal{C}\} \quad (\text{tensor supercat})$$

$\leftarrow$ but twisted modules (vertex alg)

$$\bigoplus_{\xi\in G^\vee} \mathcal{C}_\xi \supset \mathcal{C}_0 \xrightarrow[\mathcal{E}\boxtimes ?]{} \mathrm{Rep}^0(\mathcal{E}) = \{\text{local } \mathcal{E}\text{-module objects}\} \simeq \mathcal{E}\text{-mod}. \quad (\text{braided tensor cat})$$

with $\mathrm{Rep}^0(\mathcal{E}) \subset \mathrm{Rep}(\mathcal{E})$.

Fact $\underline{G} \curvearrowright \mathrm{Ob}(\mathcal{C})$: fixed-point free $\left(S_g \boxtimes M \simeq M \Leftrightarrow g = 0 \text{ for } {}^\forall M \in \mathrm{Ob}(\mathcal{C}) \right)$

$\longrightarrow$ $\mathcal{C}$: semisimple $\Longleftrightarrow$ $\mathrm{Rep}(\mathcal{E})$: semisimple $\left(\Rightarrow \mathrm{Rep}^0(\mathcal{E})\text{: semisimple} \right)$

$$\boxed{K(\mathcal{E})} \simeq \left(\boxed{K(\mathcal{C}) \underset{\mathbb{Z}[G]}{\otimes} \mathbb{Z}} \right)^{G} \quad \text{—} (*)$$

$\mathcal{E} \boxtimes M \simeq \mathcal{E} \boxtimes (S_g \boxtimes M)$

$K(\mathcal{E}) = K(\mathrm{Rep}^0(\mathcal{E}))$

$K(\mathrm{Rep}(\mathcal{E}))$

Cor $W_{-(n-1)+\frac{n-1}{n+r}}(\mathfrak{sl}_{1|n})$ $\left(r \in \mathbb{Z}_{\geq 3},\ \gcd(n-1, n+r) = 1 \right)$

defines a rational SCFT.

$W_{-(n-1)+\frac{n-1}{n+r}}(\mathfrak{sl}_{1|n}) \subset W_{-n+\frac{n+r}{n-1}}(\mathfrak{sl}_n, f_i) \otimes V_{\mathbb{Z}}$

Consider

$$\mathcal{E} = \bigoplus_{a \in N/L} S_a \otimes V_{a+L} \in (V \otimes V_L)\text{-mod}.$$

Cor $K(\mathcal{E}) \simeq \left(K(V) \underset{\mathbb{Z}[N/L]}{\otimes} \mathbb{Z}[L'/L] \right)^{N/L}$ — monodromy

$\mathbb{Z}[N/L]$: fusion product

$L' = \{ a \in \mathbb{Q} \otimes L \mid (a|L) \subset \mathbb{Z} \}$

Assume the semisimplicity.

$$V\text{-mod} \hookrightarrow \boxed{\mathrm{Rep}(\mathcal{E})} \simeq \bigoplus_{a\in L'/N'} \mathrm{Rep}^a(\mathcal{E}).$$

$$M \longmapsto \mathcal{E}\boxtimes(M\otimes V_L) = \bigoplus_{a\in N/L} (\mathcal{E}_a\boxtimes M)\otimes V_{a+L}$$

N/L $\qquad$ $V_L\text{-mod}$.

$\mathcal{E}_a = \mathcal{E}\boxtimes(V\otimes V_{a+L})$: simple currents

$\mathcal{E}_a \updownarrow V_{a+L}$

$$\boxed{\mathrm{Rep}^0(\mathcal{E})} \xrightarrow[\simeq]{\mathcal{E}_a\boxtimes ?} \mathrm{Rep}^a(\mathcal{E}) \quad (a\in \cancel{L'/L}\ \boxed{L'/N'})$$

$\mathcal{E}_a\boxtimes ?\ (a\in N/L)$

$$\Rightarrow V\text{-mod} \simeq \mathrm{Rep}(\mathcal{E})^{N'/L}$$

$$\Rightarrow K(V) \simeq \boxed{K(\mathrm{Rep}(\mathcal{E}))}^{N'/L} \simeq \left(\boxed{K(\mathcal{E}) \underset{\mathbb{Z}[N/L]}{\otimes} \mathbb{Z}[L'/L]}\right)^{N'/L}$$

We apply

$$W_{-n+\frac{n+r}{n-1}}(\mathfrak{sl}_n, f_{sub}) \underset{\text{SCE}}{\supset} W_{-r+\frac{r+n}{r+1}}(\mathfrak{sl}_r)\otimes V_{\sqrt{nr}\mathbb{Z}}$$

$$\Rightarrow K(W_{-n+\frac{n+r}{n-1}}(\mathfrak{sl}_n, f_{sub})) \simeq \left(K(W_{-r+\frac{r+n}{r+1}}(\mathfrak{sl}_r)) \underset{\mathbb{Z}[\mathbb{Z}_r]}{\otimes} \mathbb{Z}[\mathbb{Z}_{nr}]\right)^{\mathbb{Z}_r}$$

$$\simeq \left(K(L_n(\mathfrak{sl}_r)) \underset{\mathbb{Z}[\mathbb{Z}_r]}{\otimes} \mathbb{Z}[\mathbb{Z}_{nr}]\right)^{\mathbb{Z}_r} \simeq \boxed{K(L_r(\mathfrak{sl}_n))}$$

Thm (Arakawa-van Ekeren (n: even) / Creutzig-Genra-N-Sato, $r \geq 3$)

$$K(W_{-n+\frac{n+r}{n-1}}(\mathfrak{sl}_n, f_{sub})) \simeq K(L_r(\mathfrak{sl}_n))$$

Next, we apply

$$\boxed{W_{-n+\frac{n+r}{n-1}}(\mathfrak{sl}_n, f_{sub})} \otimes V_{\mathbb{Z}} \supset W_{-(n-1)+\frac{n-1}{n+r}}(\mathfrak{sl}_{1|n}) \otimes V_{\sqrt{(n+r)n}\mathbb{Z}}$$

SCE.

holomorphic $K(V_{\mathbb{Z}}) \simeq \mathbb{Z}[V_{\mathbb{Z}}]$

$\Rightarrow$

Thm (Creutzig-Genra-N-Sato)

$$K(W_{-(n-1)+\frac{n-1}{n+r}}(\mathfrak{sl}_{1|n})) \underset{r \geq 3}{\simeq} \left(K(L_r(\mathfrak{sl}_n)) \underset{\mathbb{Z}[\mathbb{Z}_n]}{\otimes} \mathbb{Z}[\mathbb{Z}_{(n+r)n}] \right)^{\mathbb{Z}_n}.$$

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