Orbifolds
of
topological quantum field theories

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https://www.carqueville.net/nils/2021-02-08_IPMU.pdf

In a nutshell

A TQFT is a functor

$$\mathcal{Z}\colon \text{Spacetime Caricature} \longrightarrow \text{Algebra}$$

Summary:

– n-dimensional **closed** TQFTs $\Longrightarrow$ **algebras**
– n-dimensional **defect** TQFTs $\Longrightarrow$ **n-categories**
– **orbifolds** $\Longrightarrow$ representation theory in n-categories

Applications for $n \lessapprox 4$:

$n = 2$: Landau–Ginzburg models

$n = 3$: Chern–Simons and Reshetikhin–Turaev theory

$n = 4$: Crane–Yetter and Douglas–Reutter theory

Motivation 1: basic features of quantum physics

- *physical states*: **vector space** V
- *observables*: **linear operators** on V
- *time evolution* of $\Psi \in V$ described by linear map U_t:

$$\mathrm{i}\frac{\partial\Psi}{\partial t} = H\,\Psi \qquad \Psi(t) = U_t\,\Psi(0) \qquad U_t = \mathrm{e}^{-\mathrm{i}Ht}$$

$$U_{t+t'} = U_t \circ U_{t'}$$

Think of **quantum field theory** as a map

Spacetime $\longrightarrow$ Algebra

Motivation 2: group representations

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$$\begin{array}{ccccccc} & & * & \longmapsto & \rho(*) =: V & & \\ \mathrm{End}(*) = G & \ni & g & \longmapsto & \rho(g) & \in & \mathrm{End}(V) \end{array}$$

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Think of **QFT** as a **representation of spacetime on algebra**.

Topological quantum field theory

A **2-dimensional closed TQFT** is a symmetric monoidal functor

$$\mathrm{Bord}_2 \xrightarrow{\mathcal{Z}} \mathrm{Vect}$$

Atiyah 1988

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orient. circles S^1 and surfaces with bdry./diffeom. $\nearrow$ $\nwarrow$ vector spaces and linear maps

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$$\begin{aligned} S^1 \quad &\longmapsto \quad \mathcal{Z}(S^1) =: V \\ S^1 \sqcup \cdots \sqcup S^1 \quad &\longmapsto \quad V \otimes \cdots \otimes V \end{aligned}$$

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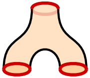

$$\longmapsto \quad \left(V \otimes V \xrightarrow{\mu} V \right)$$

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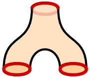

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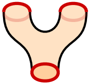

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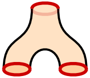

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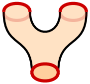

$$\longmapsto \left(V \xrightarrow{\Delta} V \otimes V\right)$$

$$\longmapsto \left(V \xrightarrow{\mathrm{tr}} \Bbbk\right)$$

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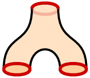
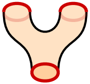


$$\longmapsto \left(V \otimes V \xrightarrow{\mu} V\right)$$



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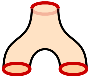
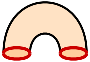
 $\circ$  $=$ 

$$\longmapsto \left(\langle -, - \rangle : V \otimes V \xrightarrow{\mathrm{tr} \circ \mu} \Bbbk\right)$$

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A **2-dimensional closed TQFT** is a symmetric monoidal functor

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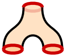
$S^1 \longmapsto V$ (vector space)

 $\longmapsto \big(\mu \colon V \otimes V \longrightarrow V\big)$ (associative multiplication)

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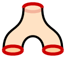 $\longmapsto \big(\mu\colon V \otimes V \longrightarrow V\big)$ (associative multiplication)

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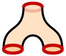 $\longmapsto \big(\mu\colon V \otimes V \longrightarrow V\big)$ (associative multiplication)

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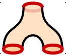
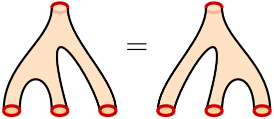
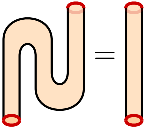
Proof sketch:

Multiplication $\mathcal{Z}\Big($  $\Big)$ associative, pairing $\mathcal{Z}\Big($  $\Big)$ nondegenerate:





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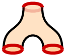 $\longmapsto$ $\big(\mu\colon V \otimes V \longrightarrow V\big)$ (associative multiplication)

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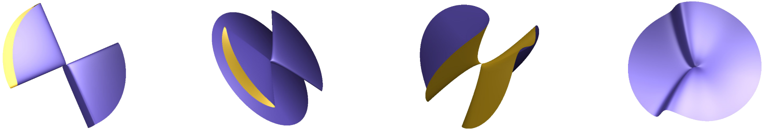
Examples.

- $V = \Bbbk G$ and $\langle g,h\rangle = \delta_{g,h^{-1}}$ for finite abelian group G
- $V = \mathbb{C}[x_1,\dots,x_n]/(\partial_{x_1}W,\dots,\partial_{x_n}W)$ (pairing $\langle -,-\rangle$ from residue theory)



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Options:

- Increase "spacetime" dimension.
- Promote source and target to higher categories.
- Consider other tangential structures.
- Decompose bordisms *without* higher categories as input.

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- Study non-topological QFT...

Defect TQFT

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Davydov/Kong/Runkel 2011

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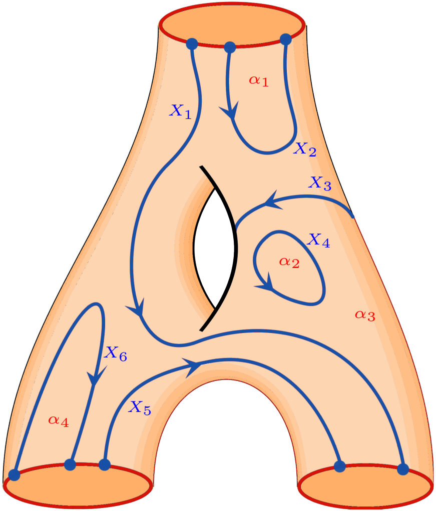

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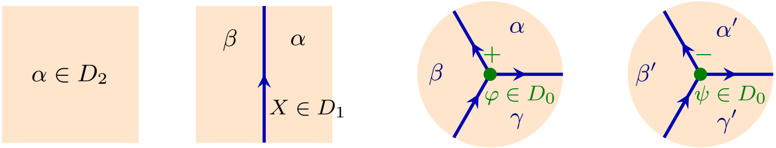
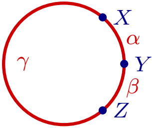
depending on **defect data** $\mathbb{D}$ consisting of:

- set D_2 of bulk theories
- set D_1 of line defects
- set D_0 of junction fields



objects:



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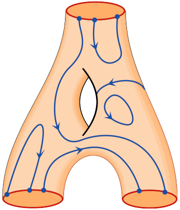


Davydov/Kong/Runkel 2011

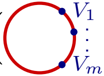
Examples of 2d defect TQFTs

Trivial defect TQFT $\mathcal{Z}^{\mathrm{triv}}$:

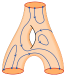
$D_2 := \{\Bbbk\}$

$D_1 := \{\text{finite-dimensional } \Bbbk\text{-vector spaces}\}$

$D_0 := \{\text{linear maps}\}$

$\mathcal{Z}^{\mathrm{triv}}\Bigg($ 

$\Bigg) \overset{\mathrm{def}}{=} V_1 \otimes \cdots \otimes V_m$

$\mathcal{Z}^{\mathrm{triv}}\Bigg($  $\Bigg) \overset{\mathrm{def}}{=}$ (evaluate 0- und 1-strata as string diagrams in Vect)

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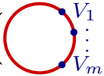
$\mathcal{Z}^{\text{triv}}\Big($ 

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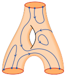
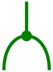
$\mathcal{Z}^{\text{triv}}\Big($  $\Big) \stackrel{\text{def}}{=}$ (evaluate 0- und 1-strata as string diagrams in Vect)

B-twisted sigma models:
Calabi–Yau manifolds and holomorphic vector bundles

Landau–Ginzburg models:
isolated singularities and homological algebra

Δ-separable symmetric Frobenius algebra (over $\Bbbk$)

$A \in \mathrm{Vect}$ with

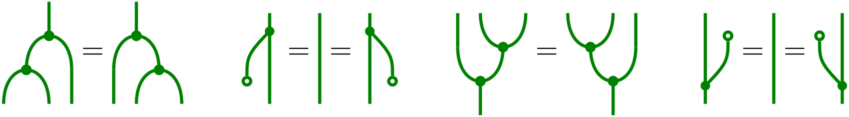
$\mu =$  $: A \otimes A \longrightarrow A$

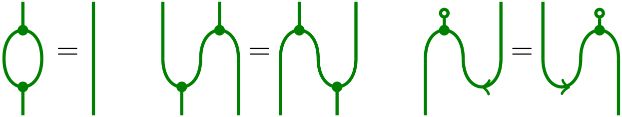
 $: \Bbbk \longrightarrow A$

$\Delta =$  $: A \longrightarrow A \otimes A$

 $: A \longrightarrow \Bbbk$

such that





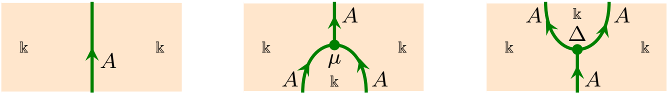
(A need *not* be commutative.)

State sum models

Input: Δ-separable symmetric Frobenius $\Bbbk$-algebra (A, μ, Δ)

(1) Choose oriented **triangulation** t for every bordism Σ in Bord_2

(2) **Decorate Poincaré-dual** graph with $(\Bbbk, A, \mu, \Delta)$:



(3) Obtain $\Sigma^{t,A}$ in $\mathrm{Bord}_2^{\mathrm{def}}(\mathbb{D}^{\mathrm{triv}})$ and define $\boxed{\mathcal{Z}_A^{\mathrm{ss}}(\Sigma) = \mathcal{Z}^{\mathrm{triv}}(\Sigma^{t,A})}$

Fukuma/Hosono/Kawai 1992, Lauda/Pfeiffer 2006

State sum models

Input: Δ-separable symmetric Frobenius $\mathbb{C}$-algebra (A, μ, Δ)

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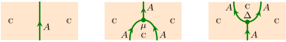

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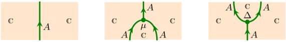
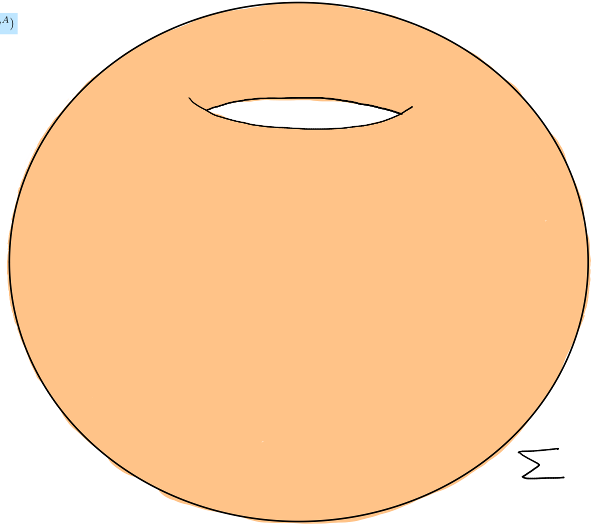


(3) Obtain $\Sigma^{t,A}$ in $\mathrm{Bord}_2^{\mathrm{def}}(\mathbb{D}^{\mathrm{triv}})$ and define $\boxed{\mathcal{Z}_A^{\mathrm{ss}}(\Sigma) = \mathcal{Z}^{\mathrm{triv}}(\Sigma^{t,A})}$



Σ

State sum models

Input: Δ-separable symmetric Frobenius $\mathbb{C}$-algebra (A, μ, Δ)

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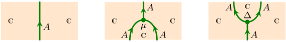

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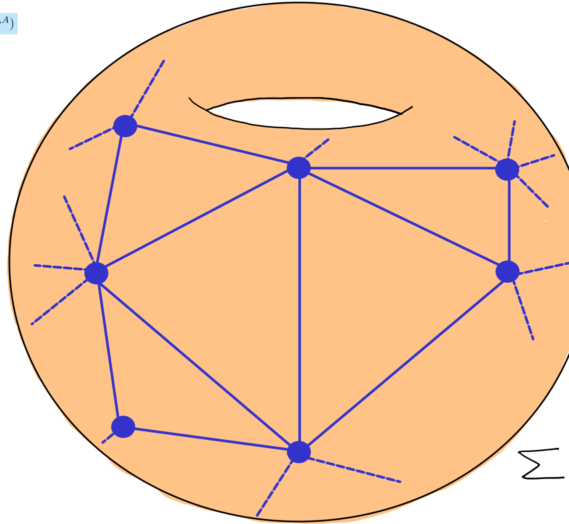

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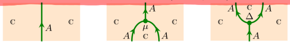

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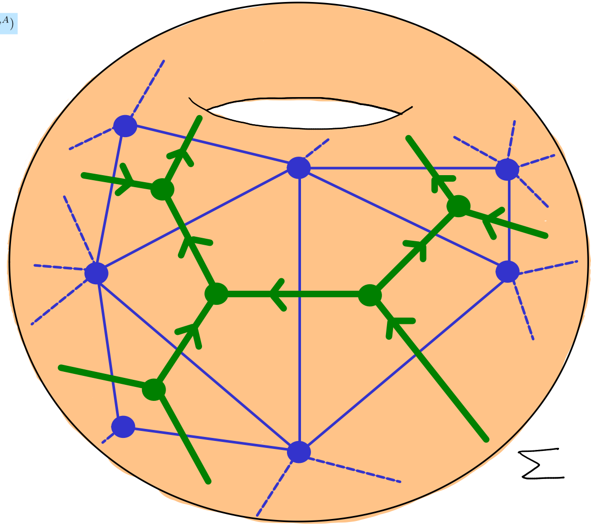

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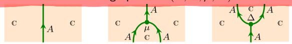

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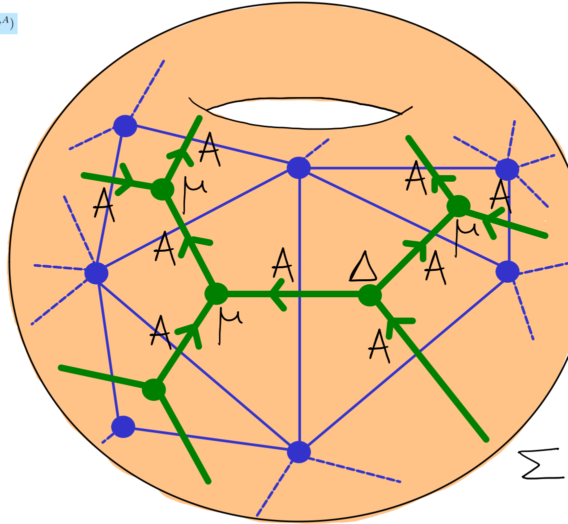

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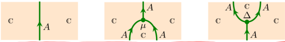

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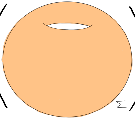
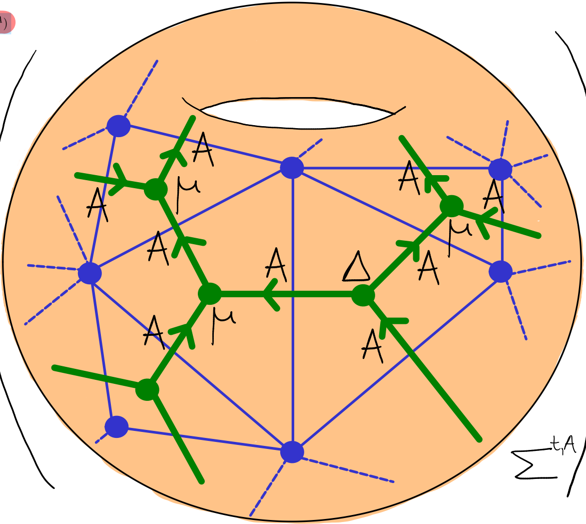
$\mathcal{Z}^{ss}_{A}\Big($  $\Big) \overset{\text{def}}{=} \mathcal{Z}^{triv}\Big($  $\Sigma^{t,A}\Big)$

State sum models

Input: Δ-separable symmetric Frobenius $\Bbbk$-algebra (A, μ, Δ)

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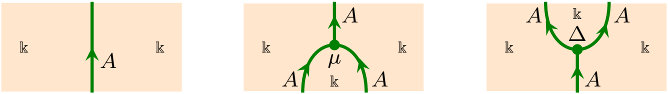

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Fukuma/Hosono/Kawai 1992, Lauda/Pfeiffer 2006

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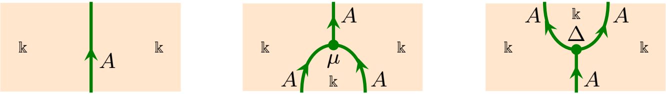

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Theorem. Construction yields TQFT $\mathcal{Z}_A^{\mathrm{ss}} \colon \mathrm{Bord}_2 \longrightarrow \mathrm{Vect}$.

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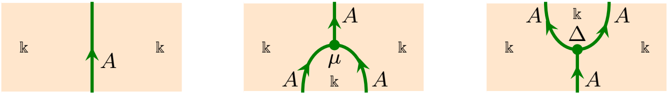

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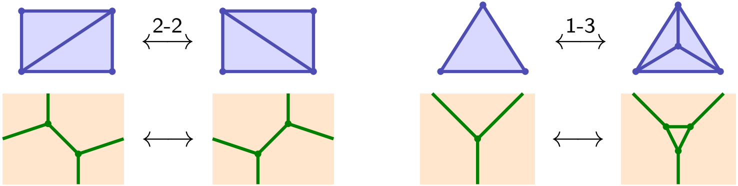
Proof sketch: Defining properties of (A, μ, Δ) encode invariance under **Pachner moves** $\Longrightarrow$ independent of choice of triangulation:



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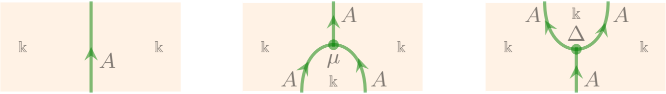

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Theorem. Construction yields TQFT $\mathcal{Z}_A^{\mathrm{ss}} \colon \mathrm{Bord}_2 \longrightarrow \mathrm{Vect}$.

No need to consider only algebras over $\mathbb{k}$!

Orbifolds

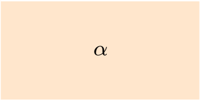
Definition. Let $\mathcal{Z} \colon \mathrm{Bord}_2^{\mathrm{def}}(\mathbb{D}) \longrightarrow \mathrm{Vect}$ be defect TQFT.

Carqueville/Runkel 2012, Fröhlich/Fuchs/Runkel/Schweigert 2009

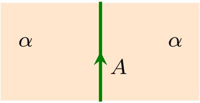
Orbifolds

Definition. Let $\mathcal{Z}\colon \mathrm{Bord}_2^{\mathrm{def}}(\mathbb{D}) \longrightarrow \mathrm{Vect}$ be defect TQFT.

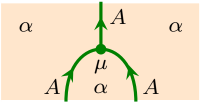
An **orbifold datum** for $\mathcal{Z}$ is $\mathcal{A} \equiv (\alpha, A, \mu, \Delta)$:



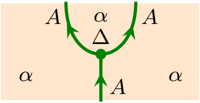
$\alpha \in D_2$



$A \in D_1$

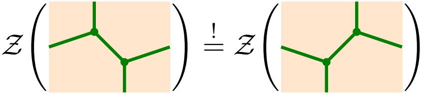


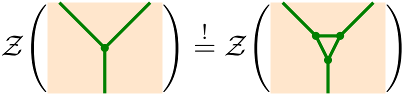
$\mu \in D_0$



$\Delta \in D_0$

such that *Pachner moves become identities* under $\mathcal{Z}$:





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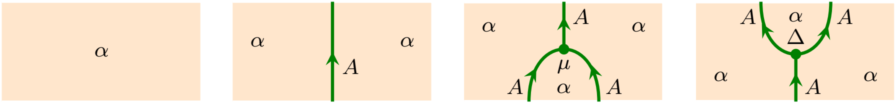

$\alpha \in D_2$ $\qquad$ $A \in D_1$ $\qquad$ $\mu \in D_0$ $\qquad$ $\Delta \in D_0$

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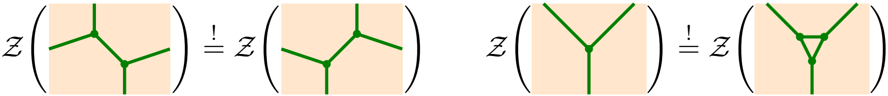

Definition & Theorem.
Triangulation + $\mathcal{A}$-decoration + evaluation with $\mathcal{Z}$ = **$\mathcal{A}$-orbifold TQFT**

$$\mathcal{Z}_{\mathcal{A}}\colon \mathrm{Bord}_2 \longrightarrow \mathrm{Vect}$$

Carqueville/Runkel 2012, Fröhlich/Fuchs/Runkel/Schweigert 2009

Algebraic characterisation

Theorem.
2d defect TQFT $\mathcal{Z} \implies$ pivotal 2-category $\mathcal{B}_{\mathcal{Z}}$

Davydov/Kong/Runkel 2011

Algebraic characterisation

Theorem.

2d defect TQFT $\mathcal{Z} \implies$ pivotal 2-category $\mathcal{B}_{\mathcal{Z}}$

Proof idea:

- objects = closed TQFTs
- 1-morphisms = line defects (= codimension-1 defects)
- 2-morphisms = point defects (= codimension-2 defects)
- adjunctions from orientation reversal

Davydov/Kong/Runkel 2011

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Algebraic characterisation

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Examples.

- **vector spaces**: $\mathrm{Bvect}_{\mathbb{k}}$
  $*$, finite-dimensional $\mathbb{k}$-vector spaces, linear maps
- **algebras over $\mathbb{k}$**
  separable symmetric Frobenius $\mathbb{k}$-algebras, bimodules, intertwiners
- **algebraic geometry**
  Calabi–Yau varieties, Fourier–Mukai kernels, RHom
- **symplectic geometry**
  symplectic manifolds, Lagrangian correspondences, Floer homology
- **Landau–Ginzburg models**
  isolated singularities, matrix factorisations
- **differential graded categories**
  smooth and proper dg categories, dg bimodules, intertwiners
- **categorified quantum groups**
  weights, functors $\mathcal{E}_i, \mathcal{F}_j \ldots$, string diagrams...

Davydov/Kong/Runkel 2011

Algebraic characterisation of orbifolds

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Davydov/Kong/Runkel 2011

Algebraic characterisation of orbifolds

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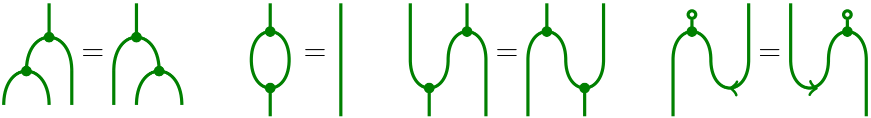
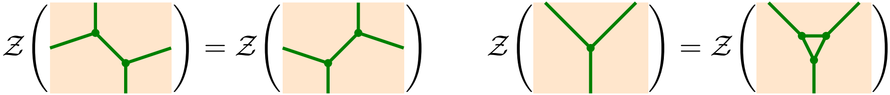
Lemma.

$\{$orbifold data for $\mathcal{Z}\} \cong \{\Delta$-separable symmetric Frobenius algebras in $\mathcal{B}_{\mathcal{Z}}\}$



$$\Longleftrightarrow$$



Davydov/Kong/Runkel 2011

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Examples.

– Δ-separable symmetric Frobenius algebras in BVect
$= \Delta$-separable symmetric Frobenius $\Bbbk$-algebras ☺

Davydov/Kong/Runkel 2011

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  $\implies$ $\boxed{\mathcal{Z}^{\mathrm{ss}}_A = (\mathcal{Z}^{\mathrm{triv}})_A}$ ("*State sum models are orbifolds of the trivial TQFT.*")

Davydov/Kong/Runkel 2011

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$\implies$ $\boxed{\mathcal{Z}^{\mathrm{ss}}_A = (\mathcal{Z}^{\mathrm{triv}})_A}$ (*"State sum models are orbifolds of the trivial TQFT."*)

– A ***G*-action** in $\mathcal{B}_{\mathcal{Z}}$ is 2-functor $\rho\colon \mathrm{B}\underline{G} \longrightarrow \mathcal{B}_{\mathcal{Z}}$.

Lemma. $A_G := \bigoplus_{g \in G} \rho(g)$ is Δ-separable Frobenius algebra in $\mathcal{B}_{\mathcal{Z}}$.

$\implies$ G-orbifolds are orbifolds: $\mathcal{Z}^G = \mathcal{Z}_{A_G}$ $\boxed{\mathcal{C}^G \cong \mathrm{mod}_{\mathcal{C}}(A_G)}$ ☺

Davydov/Kong/Runkel 2011, Fröhlich/Fuchs/Runkel/Schweigert 2009, Brunner/Carqueville/Plencner 2014

Algebraic characterisation of orbifolds

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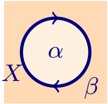
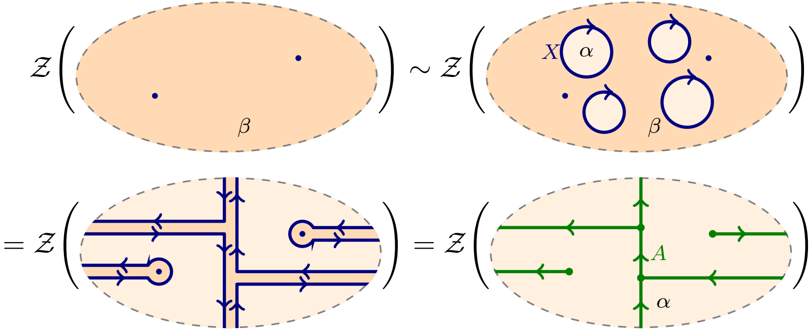
Orbifolds unify gauging of symmetry groups and state sum models.

Davydov/Kong/Runkel 2011, Fröhlich/Fuchs/Runkel/Schweigert 2009, Brunner/Carqueville/Plencner 2014

Orbifold equivalence: main idea

Let $X\colon \alpha \longrightarrow \beta$ be line defect such that  $\neq 0$ in correlators.

Then with $A := X^\dagger \otimes X\colon \alpha \longrightarrow \alpha$ we have:



Theorem. (**orbifold equivalence** $\alpha \sim \beta$)

$$\big(\text{theory } \beta\big) \;\cong\; \big(A\text{-orbifold of theory } \alpha\big)$$

Carqueville/Runkel 2012

Orbifold equivalence

Orbifold completion of pivotal 2-category $\mathcal{B}$ is pivotal 2-category $\mathcal{B}_{\mathrm{orb}}$:

– *objects*: **Δ-separable symmetric Frobenius algebras** $A \in \mathcal{B}(\alpha, \alpha)$
– *1-morphisms* $(\alpha, A) \longrightarrow (\beta, B)$: B-A-bimodules in $\mathcal{B}(\alpha, \beta)$
– *2-morphisms*: bimodule maps

Lemma. $\mathcal{B} \hookrightarrow \mathcal{B}_{\mathrm{orb}} \cong (\mathcal{B}_{\mathrm{orb}})_{\mathrm{orb}}$

Carqueville/Runkel 2012

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Theorem & Definition. (**Orbifold equivalence** $\alpha \sim \beta$)
If $X \in \mathcal{B}(\alpha, \beta)$ has *invertible* $\dim(X) \in \mathrm{End}(1_\beta)$, then:
– $A := X^\dagger \otimes X$ is *separable* symmetric Frobenius algebra in $\mathcal{B}(\alpha, \alpha)$
– $X : (\alpha, A) \rightleftarrows (\beta, 1_\beta) : X^\dagger$ is adjoint equivalence in $\mathcal{B}_{\mathrm{orb}}$

Remark.
$\mathcal{B}_{\mathrm{orb}}$ as *oriented* gapped condensation of topological phases of matter

Carqueville/Runkel 2012

Orbifold equivalence

Orbifold completion of 2-category $\mathcal{B}$ is 2-category $\mathcal{B}_{\mathrm{eq}}$:
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– *2-morphisms*: bimodule maps

Lemma. $\mathcal{B} \hookrightarrow \mathcal{B}_{\mathrm{orb}} \cong (\mathcal{B}_{\mathrm{orb}})_{\mathrm{orb}}$, $\quad \mathcal{B} \hookrightarrow \mathcal{B}_{\mathrm{eq}} \cong (\mathcal{B}_{\mathrm{eq}})_{\mathrm{eq}}$

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If $X \in \mathcal{B}(\alpha, \beta)$ has *invertible* $\dim(X) \in \mathrm{End}(1_\beta)$, then:
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Remark.
$\mathcal{B}_{\mathrm{orb}}$ as *oriented* gapped condensation of topological phases of matter
$\mathcal{B}_{\mathrm{eq}} =$ "condensation completion"

Carqueville/Runkel 2012, Gaiotto/Johnson-Freyd 2019

Orbifolds of Landau–Ginzburg models

Theorem. There is a (graded) pivotal 2-category $\mathcal{LG}$ with:

– objects = isolated singularities $W \in \mathbb{C}[x_1, \dots, x_n]$

– $\mathcal{LG}(W, V)$ = homotopy category of matrix factorisations $\mathcal{D}$ of $V - W$

Carqueville/Murfet 2012

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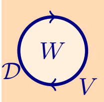
– 

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Carqueville/Murfet 2012

Orbifolds of Landau–Ginzburg models

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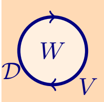
– 

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– symmetric monoidal pivotal 2-category under very good control!

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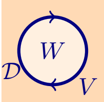
– 

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– CFT/LG correspondence

– CY/LG correspondence

– derived geometry & representation theory

– homological knot invariants

– surface defects in Rozansky–Witten models

Carqueville/Murfet 2012, Buchweitz 1986, Khovanov/Rozansky 2004, Orlov 2005, Kajiura/Saito/Takahashi 2005, ...

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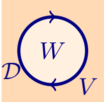
– 

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Theorem. (**Orbifold equivalences** in $\mathcal{LG}$)

$$\begin{array}{rcll} x^k + xy^2 & \sim & u^{2k} + v^2 & \qquad (\mathrm{D}_{k+1} \sim \mathrm{A}_{2k-1}) \\ x^3 + y^4 & \sim & u^{12} + v^2 & \qquad (\mathrm{E}_6 \sim \mathrm{A}_{11}) \\ x^3 + xy^3 & \sim & u^{18} + v^2 & \qquad (\mathrm{E}_7 \sim \mathrm{A}_{17}) \\ x^3 + y^5 & \sim & u^{30} + v^2 & \qquad (\mathrm{E}_8 \sim \mathrm{A}_{29}) \end{array}$$

Carqueville/Murfet 2012, Carqueville/Runkel 2012, Carqueville/Ros Camacho/Runkel 2013

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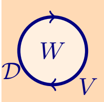
– 

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x^5y + y^3 &\sim u^3v + v^5 && (\mathrm{E}_{13} \sim \mathrm{Z}_{11}) \\
x^6 + xy^3 + z^2 &\sim vw^3 + v^3 + u^2w && (\mathrm{Z}_{13} \sim \mathrm{Q}_{11})
\end{aligned}
$$

Carqueville/Murfet 2012, Carqueville/Runkel 2012, Carqueville/Ros Camacho/Runkel 2013, Recknagel/Weinreb 2017

Aside: Non-semisimple fully extended TQFTs

Theorem.
For every $W \in \mathcal{LG}$, the associated Landau–Ginzburg model $\mathrm{Bord}_2 \longrightarrow \mathrm{Vect}$ can be lifted to a **fully extended TQFT**

$$\begin{aligned} \mathrm{Bord}^{\mathrm{fr}}_{2,1,0} &\longrightarrow \mathcal{LG} \\ \mathrm{pt}_+ &\longmapsto W \\ S^1_1 &\longmapsto \mathbb{C}[x_1,\dots,x_n]/(\partial_{x_1}W,\dots,\partial_{x_n}W) \end{aligned}$$

Remarks.

– Jacobi algebra $\mathbb{C}[x_1,\dots,x_n]/(\partial_{x_1}W,\dots,\partial_{x_n}W)$ is **non-semisimple**.

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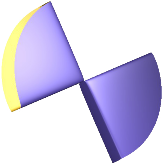
Remarks.

- Jacobi algebra $\mathbb{C}[x_1, \dots, x_n]/(\partial_{x_1} W, \dots, \partial_{x_n} W)$ is **non-semisimple**.
- Get **oriented TQFT** from $\mathrm{SO}(2)$-homotopy fixed points, i. e. trivialisations of Serre automorphism $S_W = 1_W[n]$.
- Get **r-spin TQFTs** in $\mathcal{LG}$ and $\mathcal{LG}_{\mathrm{eq}}$.

Carqueville/Montiel Montoya 2018, Hesse/Valentino 2017, Carqueville/Szegedy 2021

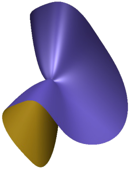
Summary so far

A_{11}

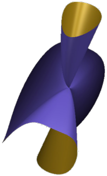


$\sim$

E_6

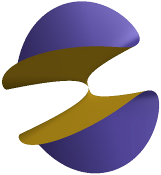


S_{11}



$\sim$

W_{13}



$$\mathcal{C}^G \;\cong\; \mathrm{mod}\left(\mathcal{C} \xrightarrow{A_G \otimes (-)} \mathcal{C}\right)$$

$$\mathcal{Z}^{\mathrm{ss}}_A \;=\; \left(\mathcal{Z}^{\mathrm{triv}}\right)_A$$

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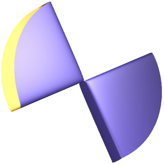
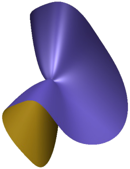
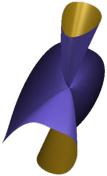
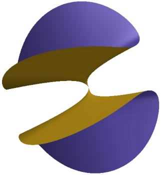
A_{11}  $\sim$ E_6 

S_{11}  $\sim$ W_{13} 

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2d orbifolds

- encode triangulation invariance in algebraic structure
- representation theory of algebras in 2-categories
- unify gauging of symmetry groups and state sum models
- new relations in Landau–Ginzburg models, algebra and geometry

The **orbifold construction** *can be generalised to n-dimensional defect TQFTs*

$$\mathcal{Z}\colon \operatorname{Bord}_n^{\mathrm{def}}(\mathbb{D}) \longrightarrow \operatorname{Vect}$$

in any dimension $n \geqslant 1$.

C/Meusburger/Schaumann 2016, C/Runkel/Schaumann 2017–2018, C/Mulevičius/Runkel/Schaumann/Scherl 2021

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n-dimensional orbifolds

- **triangulation invariance $\Longrightarrow$ algebraic structures**
  - $n = 2$: Frobenius algebras in 2-categories
  - $n = 3$: spherical fusion categories in 3-categories
- representation theory internal to n-categories

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- **triangulation invariance $\Longrightarrow$ algebraic structures**
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- **Applications**:
  - unify gauging of symmetry groups and state sum models
  - lift Reshetikhin–Turaev theory to defect TQFT
  - Reshetikhin–Turaev theories close under orbifolds
  - models for topological quantum computation

C/Meusburger/Schaumann 2016, C/Runkel/Schaumann 2017–2018, C/Mulevičius/Runkel/Schaumann/Scherl 2021

n-dimensional defect TQFTs

An **n-dimensional defect TQFT** is a symmetric monoidal functor

$$\mathcal{Z}\colon \operatorname{Bord}_n^{\mathrm{def}}(\mathbb{D}) \longrightarrow \operatorname{Vect}$$

that depends on **defect data** $\mathbb{D}$, consisting of:

- sets D_j, whose elements decorate j-strata of bordisms
- rules how strata are allowed to meet (defined recursively via cones and cylinders)

Carqueville/Meusburger/Schaumann 2016, Carqueville/Runkel/Schaumann 2017–18, Kapustin/Rozansky/Saulina 2009 + wip

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Examples of 3d defect TQFTs.

- quantum **Chern–Simons theory** ($\subset$ Reshetikhin–Turaev theory $\mathcal{Z}^{\mathcal{C}}$)
  - $D_3 = \{$gauge group$\}$ (more generally: modular tensor category $\mathcal{C}$)
  - $D_2 = \{\Delta$-separable symmetric Frobenius algebras in $\mathcal{C}\}$
  - $D_1 = \{$cyclic modules$\} \supset \{$Wilson line labels$\}$
- **Rozansky–Witten theory** (conjecturally)
  - $D_3 = \{$holomorphic symplectic manifolds$\}$
  - $D_2 = \{$ "generalised Landau–Ginzburg models" $\}$
  - $D_1 = \{$ "fibred matrix factorisations" $\}$

Carqueville/Meusburger/Schaumann 2016, Carqueville/Runkel/Schaumann 2017–18, Kapustin/Rozansky/Saulina 2009 + wip

Reshetikhin–Turaev theory with defects

Theorem.
For modular tensor category $\mathcal{C}$, there is a **defect TQFT $\mathcal{Z}^{\mathcal{C}}$** with

$D_3 = \{\mathcal{C}\}$

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that **lifts Reshetikhin–Turaev theory $\mathcal{Z}^{\mathcal{C},\mathrm{RT}}$**.

Kapustin/Saulina 2009, Carqueville/Runkel/Schaumann 2017

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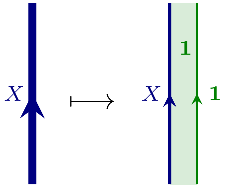
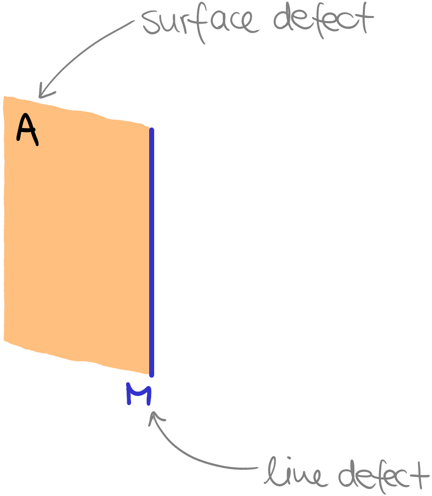
Proof idea:

– replace A-decorated 2-strata by trivalent network of A-ribbons
– evaluate with $\mathcal{Z}^{\mathcal{C},\mathrm{RT}}$
– model X-ribbons by 1- and 2-strata:



Kapustin/Saulina 2009, Carqueville/Runkel/Schaumann 2017

Reshetikhin–Turaev theory with surface defects



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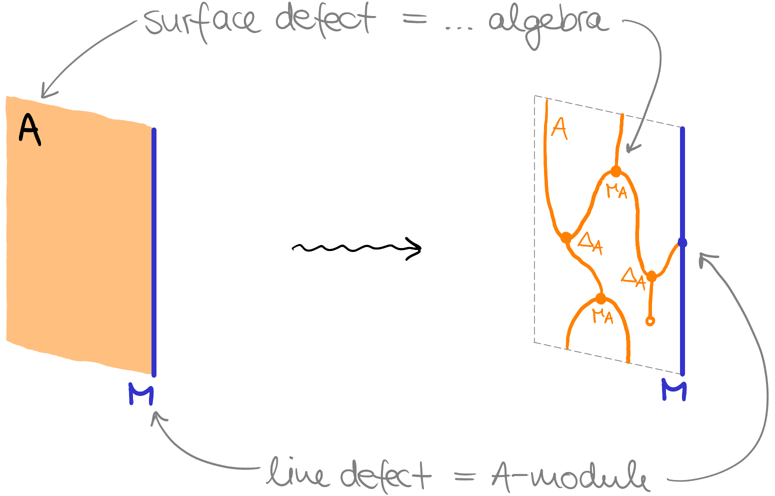

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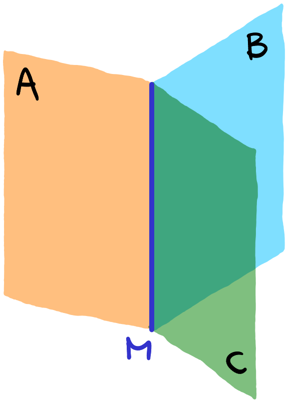

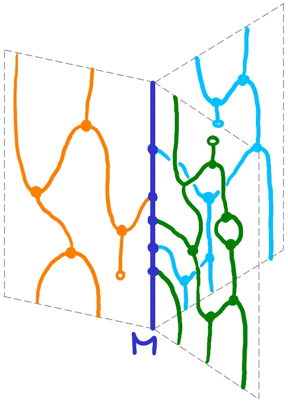

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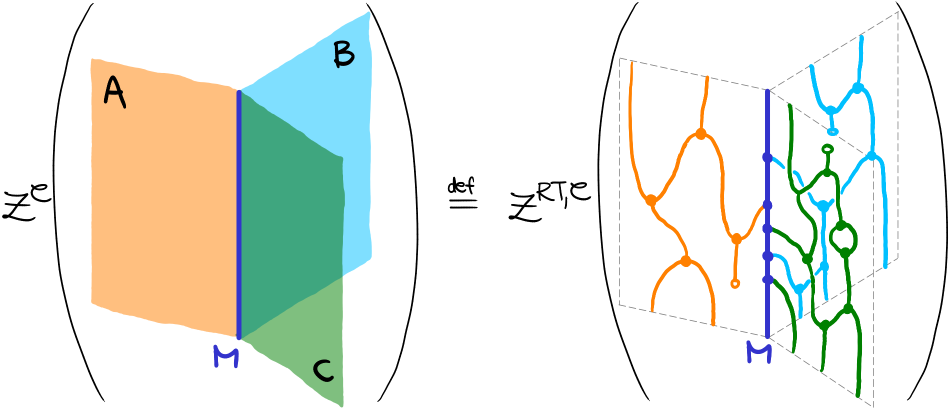

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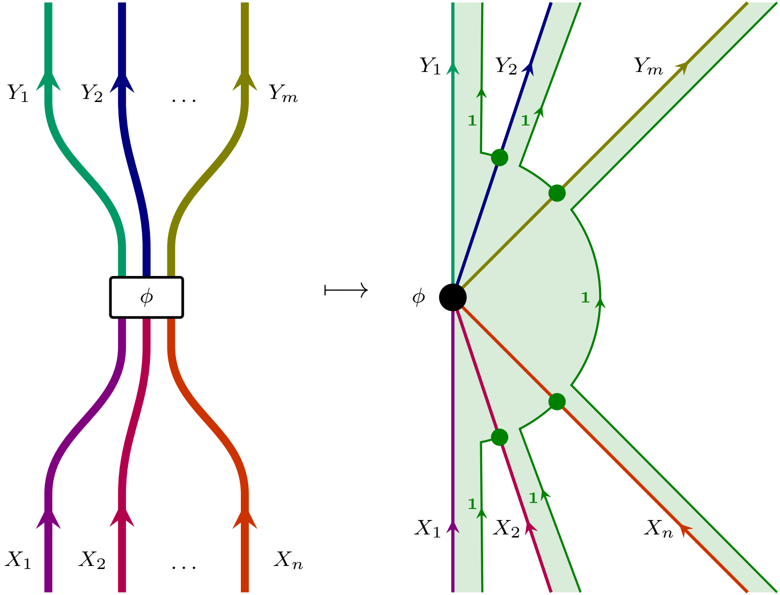
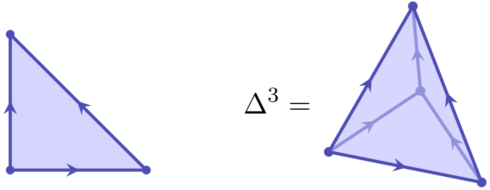


Triangulations

standard n-simplex $\Delta^n := \left\{ \sum_{i=1}^{n+1} t_i e_i \;\middle|\; t_i \geqslant 0\,,\; \sum_{i=1}^{n+1} t_i = 1 \right\} \subset \mathbb{R}^{n+1}$

$\Delta^2 =$ $\Delta^3 =$



simplicial complex C is collection of simplices such that

- all faces of all $\sigma \in C$ are also in C
- $\sigma, \sigma' \in C \quad \Longrightarrow \quad \sigma \cap \sigma' = \varnothing$ or $\sigma \cap \sigma' =$ face

triangulation of manifold M is simplicial complex C with homeomorphism $\varphi\colon |C| \xrightarrow{\cong} M$

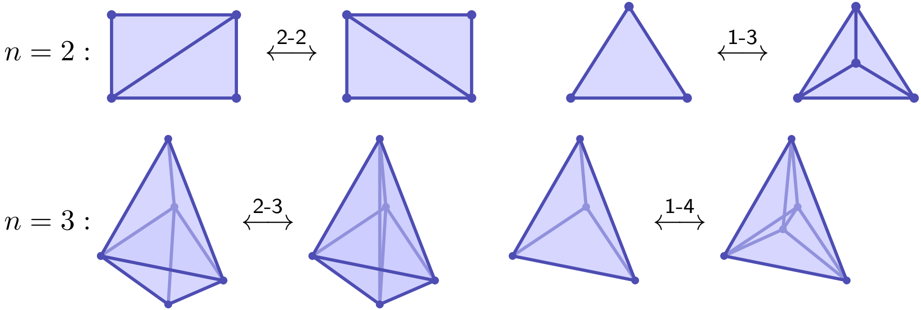
(details for smooth, oriented, . . .)

Pachner moves

Let $\varphi\colon |C| \xrightarrow{\cong} M$ be triangulated n-manifold.
Let $F \subset \partial\Delta^{n+1} \subset C$ be n-dimensional subcomplex.

A **Pachner move** "glues the other side of $\partial\Delta^{n+1}$ into M":

$$M \longmapsto \left|\partial\Delta^{n+1} \setminus \mathring{F}\right| \cup_{\varphi|_{|\partial F|}} \big(M \setminus \varphi(|F|)\big)$$



Theorem. If triangulated PL manifolds are PL isomorphic, then there exists a finite sequence of Pachner moves between them.

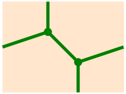
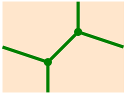
Pachner 1991

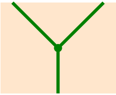
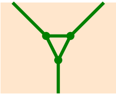
Orbifolds in any dimension n

An **orbifold datum** $\mathcal{A}$ for $\mathcal{Z}\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \longrightarrow \mathrm{Vect}$ consists of

- $\mathcal{A}_j \in D_j$ for all $j \in \{1, \dots, n\}$,
- $\mathcal{A}_0^+, \mathcal{A}_0^- \in D_0$,
- such that "Pachner moves become identities"
  - **compatibility**:
    $\mathcal{A}_j$ is allowed decoration of $(n-j)$-simplices dual to j-strata
  - **triangulation invariance**:
    Let B, B' be $\mathcal{A}$-decorated n-balls dual to two sides of a Pachner move.
    Then: $\mathcal{Z}(B) = \mathcal{Z}(B')$.

$n = 2$ is special case:

$\mathcal{Z}\Bigg($  $\Bigg) = \mathcal{Z}\Bigg($  $\Bigg)$

$\mathcal{Z}\Bigg($  $\Bigg) = \mathcal{Z}\Bigg($  $\Bigg)$

Carqueville/Runkel/Schaumann 2017

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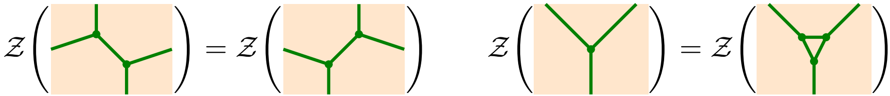
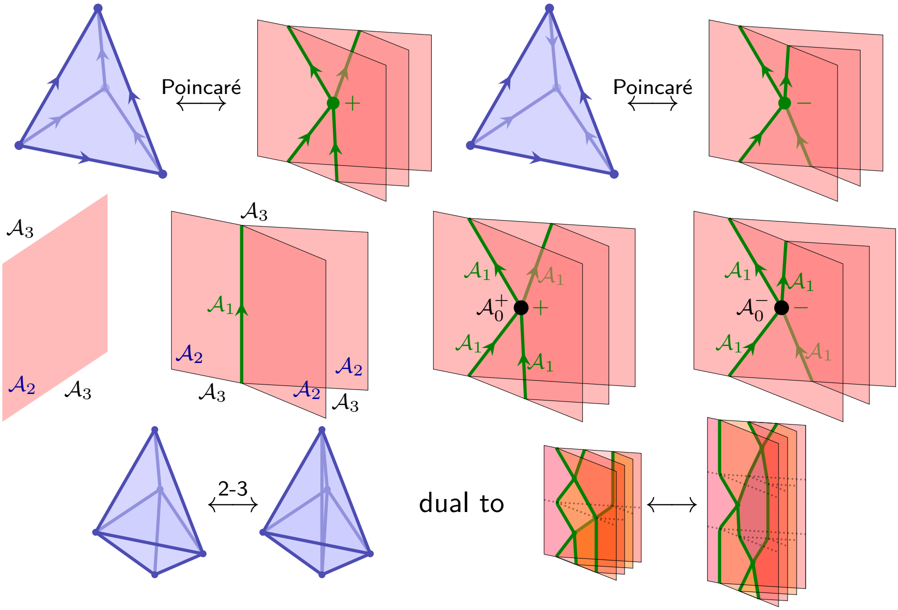


Definition & Theorem.
Triangulation $+$ $\mathcal{A}$-decoration $+$ evaluation with $\mathcal{Z}$ $=$ **$\mathcal{A}$-orbifold TQFT**

$$\mathcal{Z}_{\mathcal{A}}\colon \mathrm{Bord}_n \longrightarrow \mathrm{Vect}$$

Carqueville/Runkel/Schaumann 2017

Orbifold datum $\mathcal{A}$ for $n = 3$



3d orbifolds

Theorem.
3d defect TQFT $\mathcal{Z} \implies$ 3-category $\mathcal{T}_{\mathcal{Z}}$

C/Meusburger/Schaumann 2016, C/Runkel/Schaumann 2017–2018, C/Mulevičius/Runkel/Schaumann/Scherl 2021

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Spherical fusion categories in $\mathcal{T}_{\mathcal{Z}}$ are orbifold data for $\mathcal{Z}$.

C/Meusburger/Schaumann 2016, C/Runkel/Schaumann 2017–2018

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Turaev–Viro–Barrett–Westbury models are orbifolds of $\mathcal{Z}^{\mathrm{vect}}$

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From spherical fusion category $\mathcal{A}$ get orbifold datum

- $\mathcal{A}_3 = *$
- $\mathcal{A}_2 = \mathcal{A}$ (equivalently: $\mathbb{C}^{\#\text{ simples of }\mathcal{A}}$)
- $\mathcal{A}_1 = \otimes \colon \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ (equivalently: fusion rules of $\mathcal{A}$)
- $\mathcal{A}_0^{\pm} =$ associator$^{\pm 1}$ (equivalently: F-matrices of $\mathcal{A}$)

C/Meusburger/Schaumann 2016, C/Runkel/Schaumann 2017–2018

3d orbifolds

Theorem.
3d defect TQFT $\mathcal{Z} \implies$ 3-category $\mathcal{T}_{\mathcal{Z}}$

Theorem.
Spherical fusion categories in $\mathcal{T}_{\mathcal{Z}}$ are orbifold data for $\mathcal{Z}$.

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Theorem.
Orbifolds of Reshetikhin–Turaev theories are Reshetikhin–Turaev theories.

C/Meusburger/Schaumann 2016, C/Runkel/Schaumann 2017–2018, C/Mulevičius/Runkel/Schaumann/Scherl 2021

In a nutshell

A TQFT is a functor

$$\mathcal{Z} : \text{Spacetime Caricature} \longrightarrow \text{Algebra}$$

Summary:

- n-dimensional **closed** TQFTs $\Longrightarrow$ **algebras**
- n-dimensional **defect** TQFTs $\Longrightarrow$ **n-categories**
- **orbifolds** $\Longrightarrow$ representation theory in n-categories

Applications for $n \lessapprox 4$:

$n = 2$: Landau–Ginzburg models

$n = 3$: Chern–Simons and Reshetikhin–Turaev theory

$n = 4$: Crane–Yetter and Douglas–Reutter theory

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  [unify and extend state sum models and symmetry gauging]

Applications for $n \lessapprox 4$:

$n = 2$: Landau–Ginzburg models:

[new dualities; fully extended framed/oriented/spin TQFTs]

$n = 3$: Chern–Simons and Reshetikhin–Turaev theory:

[surface defects; close under orbifolds]

$n = 4$: Crane–Yetter and Douglas–Reutter theory

Application: topological quantum computation

Interpretation of Reshetikhin–Turaev theory $\mathcal{Z}^{\mathcal{C}}$:

- objects u_i in $\mathcal{C}$: anyonic quasiparticles in 2+1 dimensions
- $\mathcal{Z}^{\mathcal{C}}(\Sigma_{u_1,\dots,u_m})$: qubit storage on surface Σ with m anyons
- braiding matrices β_{u_i,u_j}: quantum gates
- $\langle \beta_{u_i,u_j} \rangle$ dense in $\mathrm{U}(N)$ for $N \gg 1$: *universal* quantum computation

Fact. $\mathcal{C} =$ Ising category not universal.
"Gauging" of S_2-symmetry of $\mathcal{C} \boxtimes \mathcal{C}$ is universal!

Conjecture. **Orbifolds of $\mathcal{Z}^{\mathcal{C}}$** construct universal quantum computers with larger qubit storages $\mathcal{Z}^{\mathcal{C}}(\Sigma_{u_1,\dots,u_m})$;
in particular

- $\rho \colon \mathrm{B}S_N \longrightarrow \mathrm{Bimod}_{\mathbb{C}}$ with $\rho(*) = \mathcal{C}^{\boxtimes N}$
- $\mathcal{C}$-$\mathcal{C}'$-bimodules with "invertible quantum bubble"

Kitaev 1997, Freedman/Kitaev/Larsen/Wang 2001, Barkeshli/Jian/Qi 2012, Fuchs/Schweigert 2013