Dark Photon, CMB and radio data in our inhomogeneous universe

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Kavli IPMU | APEC seminar
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In collaboration with H. Liu, S. Mishra-Sharma, J. Ruderman
arXiv 2004.06733 (PRD)
arXiv 2002.05165 (PRL)

and Maxim Pospelov (arXiv 21XX.XXXX to appear)

TEL AVIV UNIVERSITY

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WEIZMANN INSTITUTE OF SCIENCE

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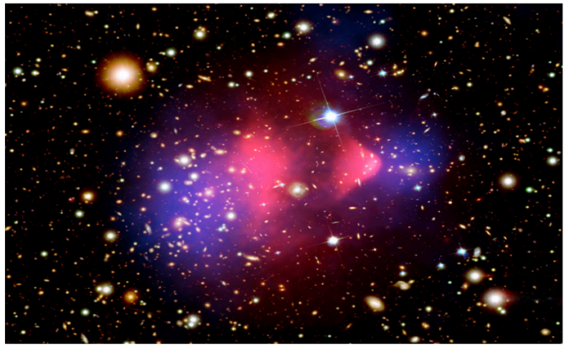
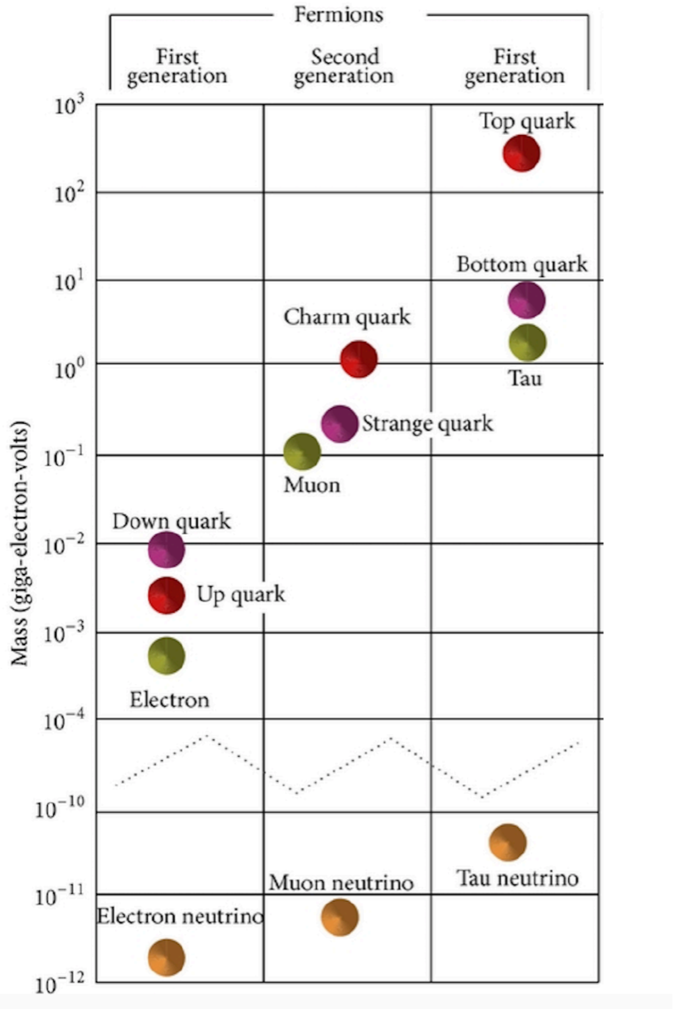
https://github.com/andrea0292/ -------- https://github.com/smsharma ------------ https://github.com/hongwanliu

We have a lot of evidences for physics beyond the Standard Model

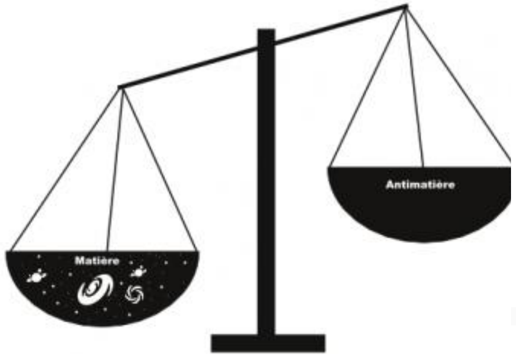
Dark Matter



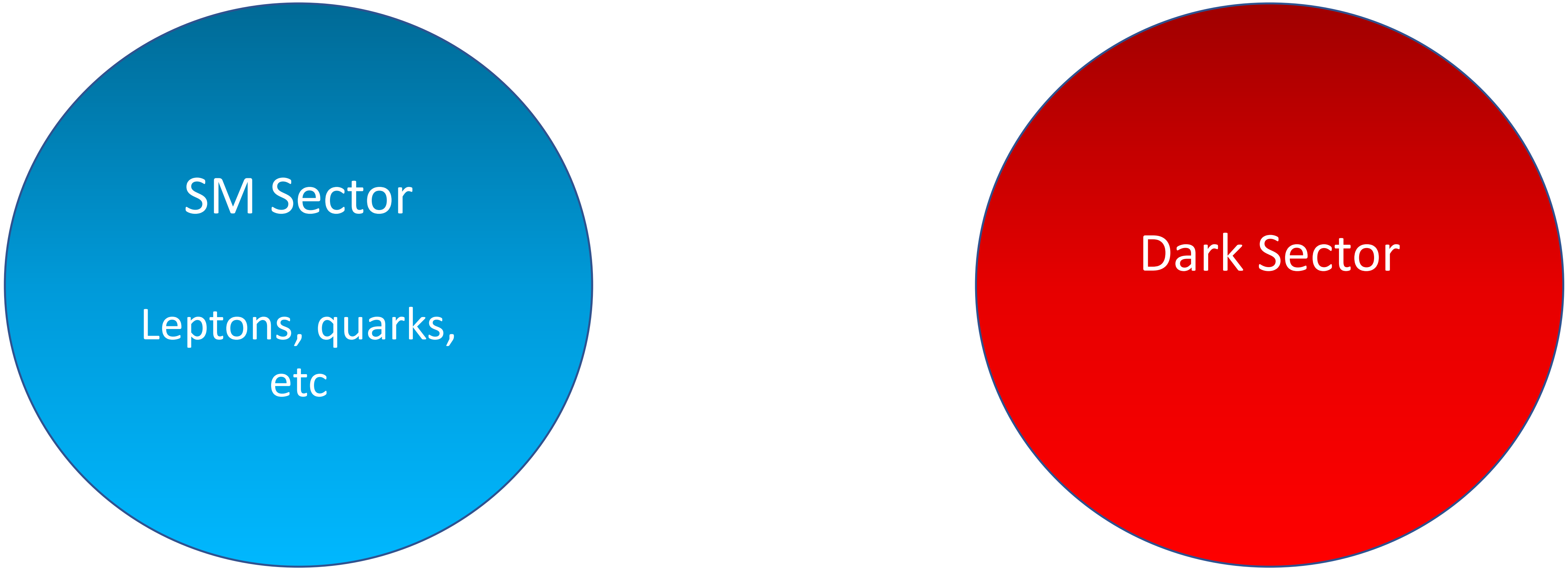
Neutrino Masses



Matter Antimatter asymmetry



Usually what we do is to introduce a Dark Sector, that may or may not be connected with the SM



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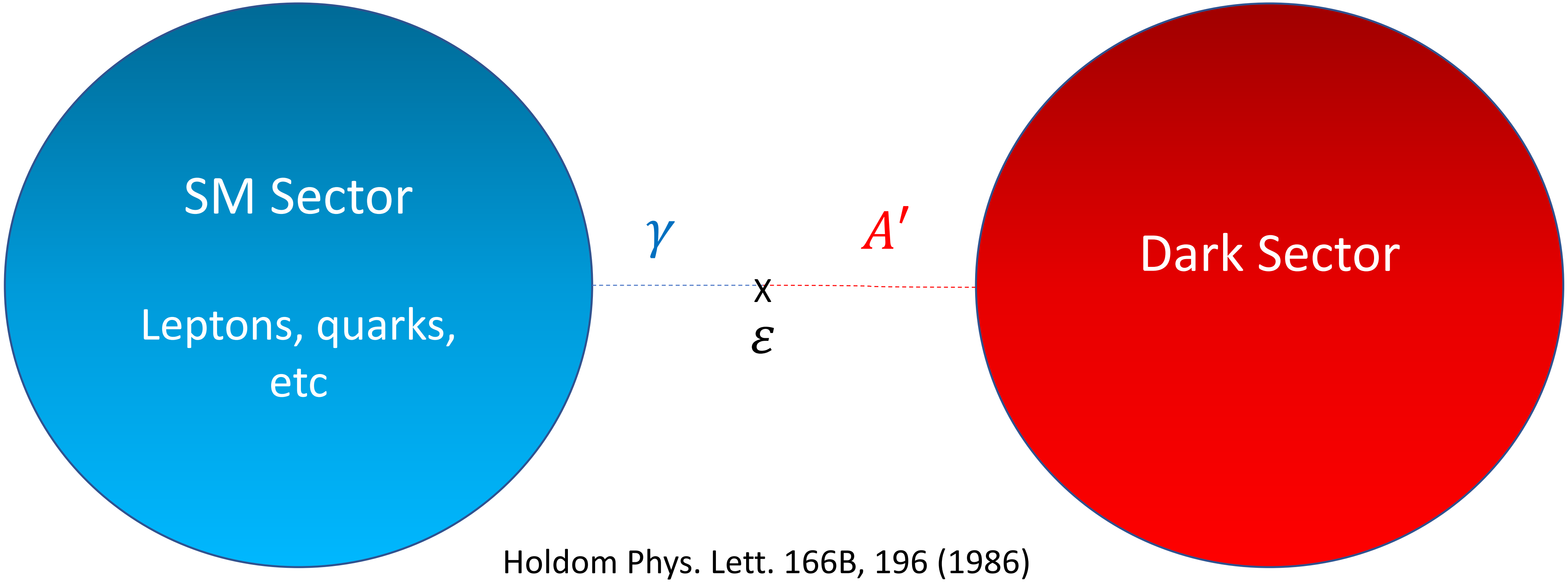
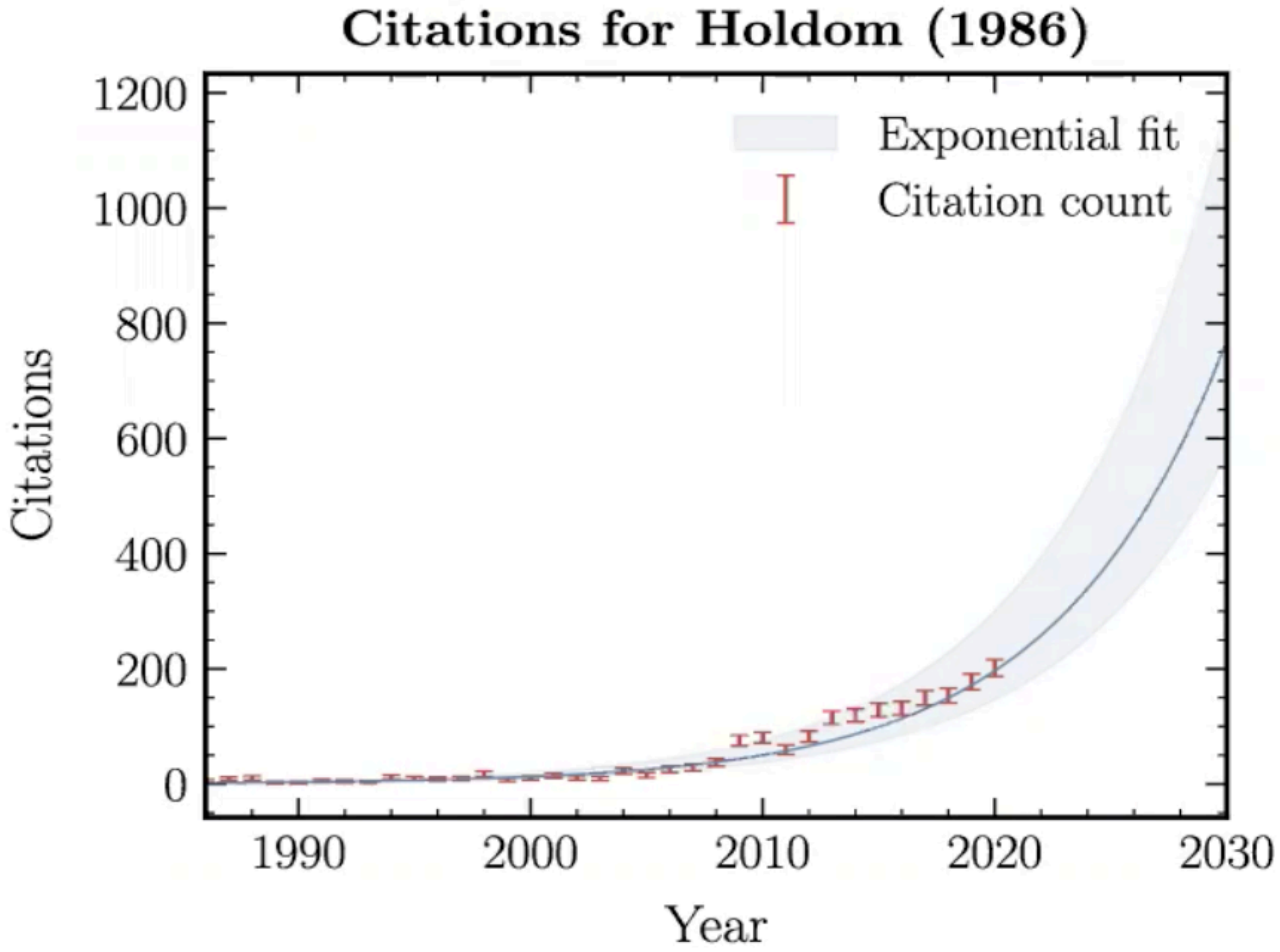


Holdom Phys. Lett. 166B, 196 (1986)

We explored the so called vector portal, introducing a kinetic mixing among the SM photon and a **Dark photon** (other portal are possible, such as Higgs portal or neutrinos portal)



Citations for Holdom (1986)
Exponential fit
Citation count
Citations
Year
1200
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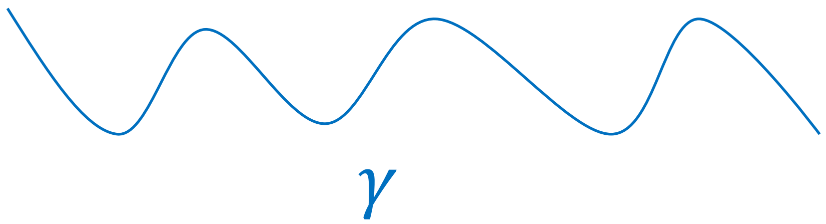
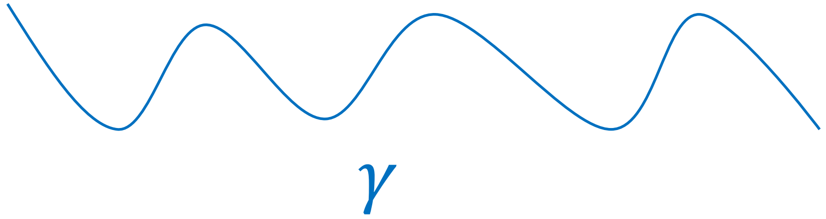
Now some plasma physics

Photons in Vacuum are massless



However medium effects give photon a mass, the usually called plasma mass



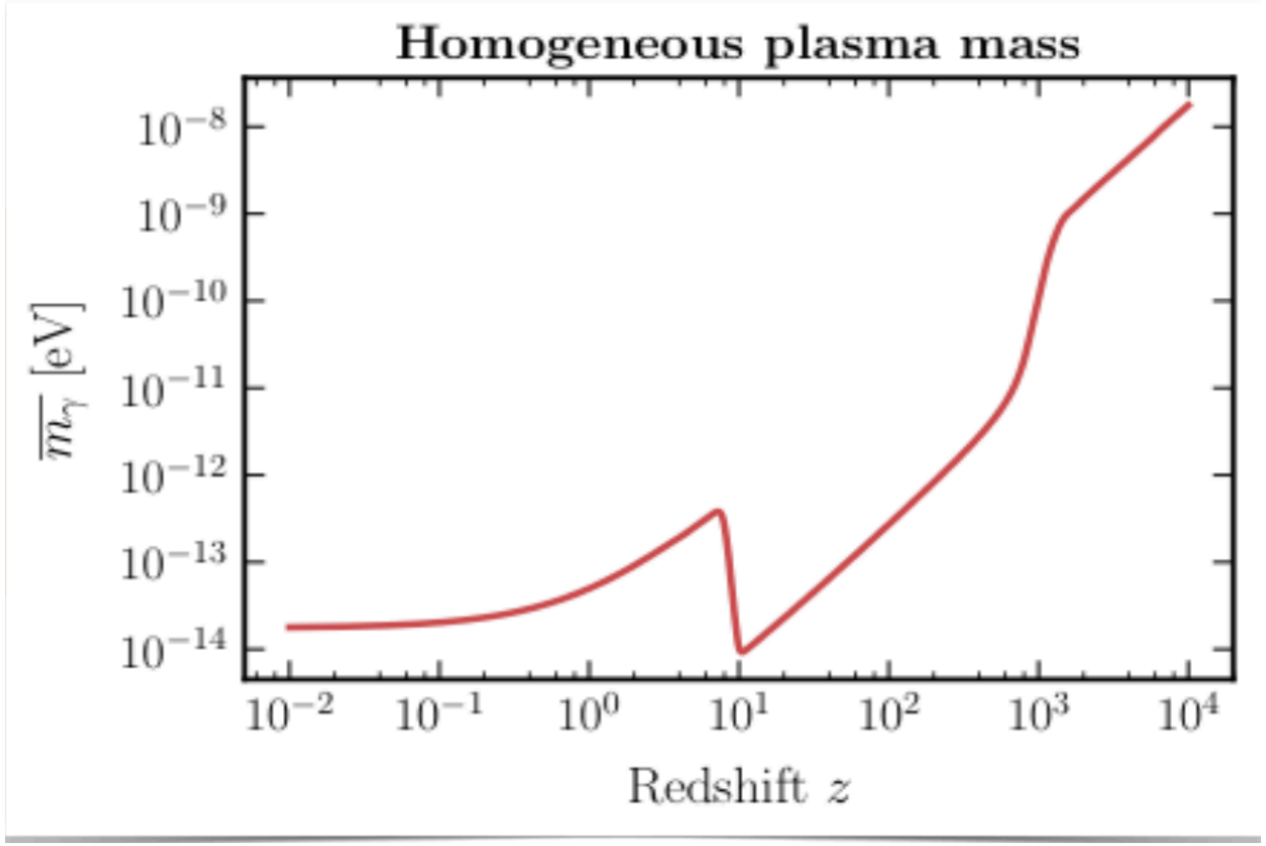
$$m_\gamma^2(z,\vec{x}) \simeq 1.4\times 10^{-21}\,\mathrm{eV}^2\left(\frac{n_\mathrm{e}(z,\vec{x})}{\mathrm{cm}^{-3}}\right)$$

Generally in media the dispersion relations are generally modified by the interactions with the background.

$$\omega^2 = k^2 + \omega_\mathrm{P}^2\left(1+\frac{k^2}{\omega^2}\frac{T}{m_e}\right) \quad \text{Transverse,}$$

$$\omega^2 = \omega_\mathrm{P}^2\left(1+3\,\frac{k^2}{\omega^2}\frac{T}{m_e}\right) \quad \text{Longitudinal.}$$

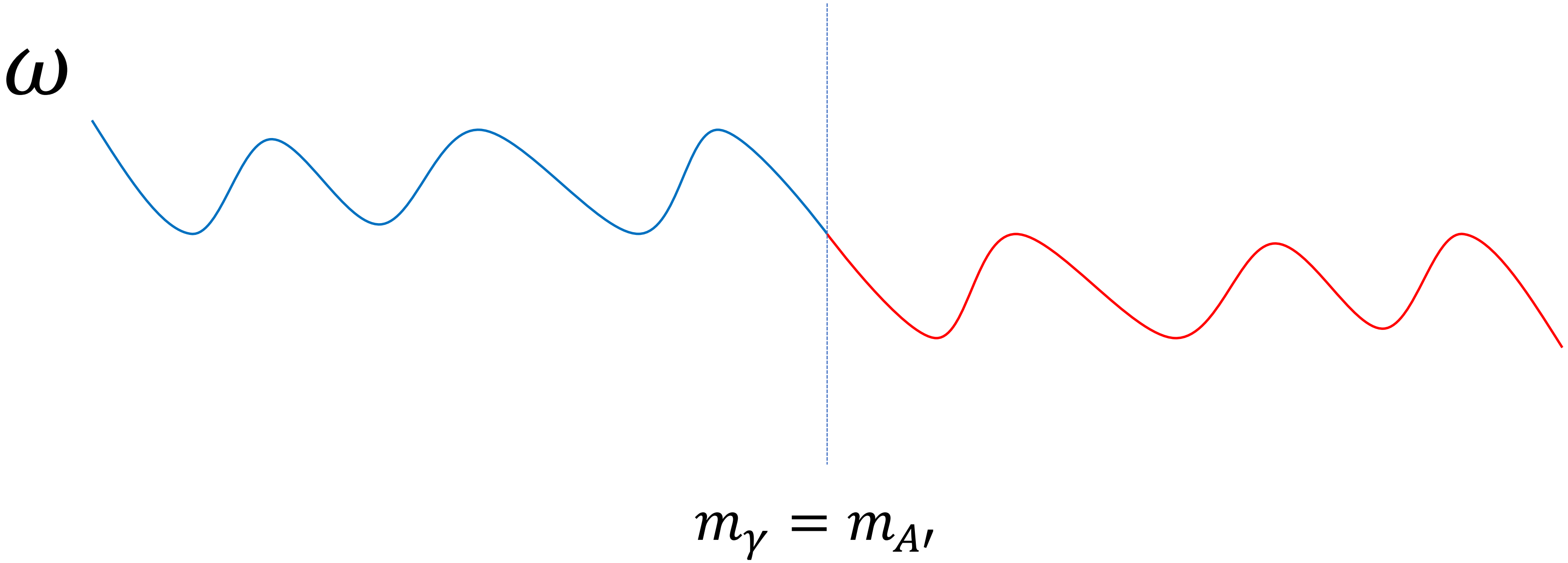
Homogeneous Plasma Mass



In the assumption of homogeneous medium, the photon mass after recombination varies between $10^{-9} and\ 10^{-14} eV$

Resonant Oscillations

T.-K. Kuo & Pantaleone Rev. Mod. Phys. 61 (1989) 937 Mirizzi, Redondo & Sigl 0901.0014 Caputo, HL, Mishra-Sharma & Ruderman 2004.06733



$$P_{\gamma\to A'} \simeq \sum_i \frac{\pi m_{A'}^2 \epsilon^2}{\omega(t_i)} \left| \frac{\mathrm{d}\ln m_\gamma^2(t)}{\mathrm{d}t} \right|^{-1}_{t=t_i},$$

Landau-Zener approximation

Resonant Oscillations

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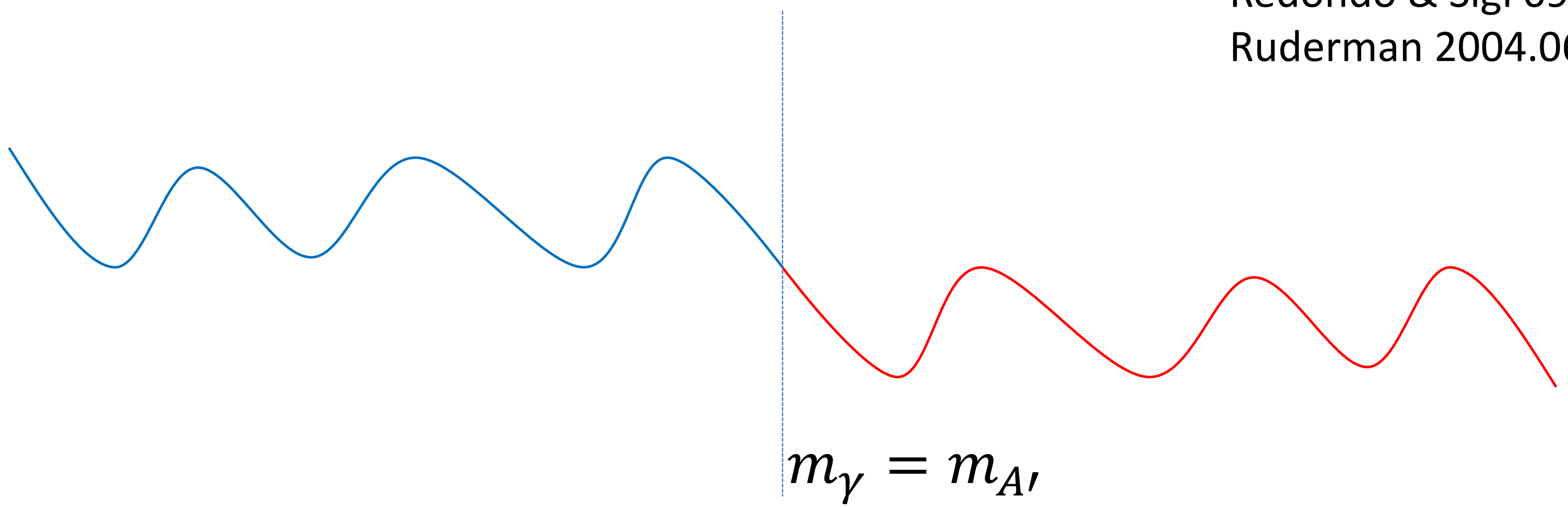


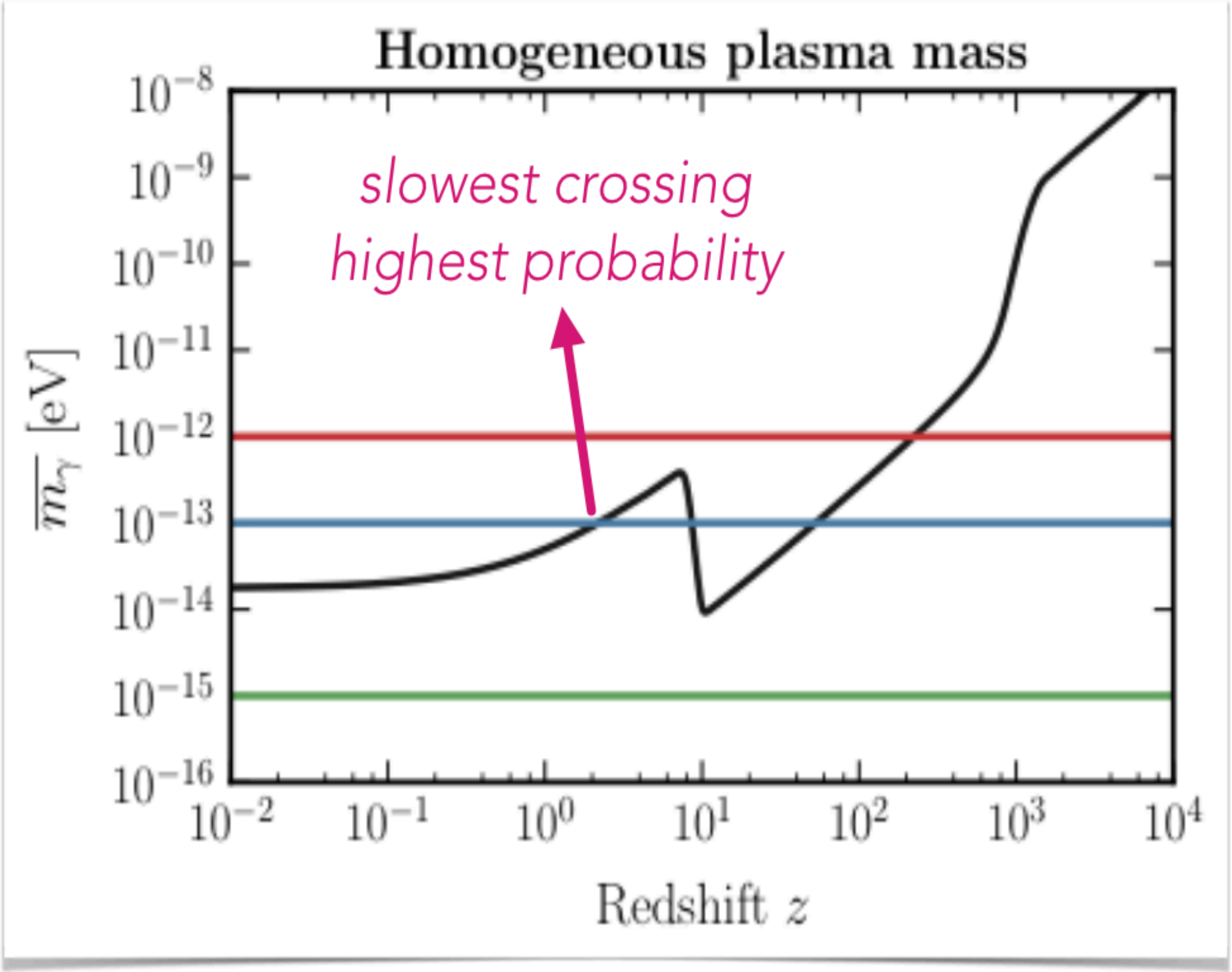
mixing

$(\gamma \to A'$ vacuum oscillation length$)^{-1}$

resonance timescale $\sim H^{-1}$

$$P_{\gamma\to A'} = 2\pi \times \epsilon^2 \times \frac{m_{A'}^2}{2\omega} \times \left| \frac{\mathrm{d}\ln m_\gamma^2}{\mathrm{d}t} \right|^{-1}_{m_\gamma = m_{A'}}$$

You can also have multiple resonances!



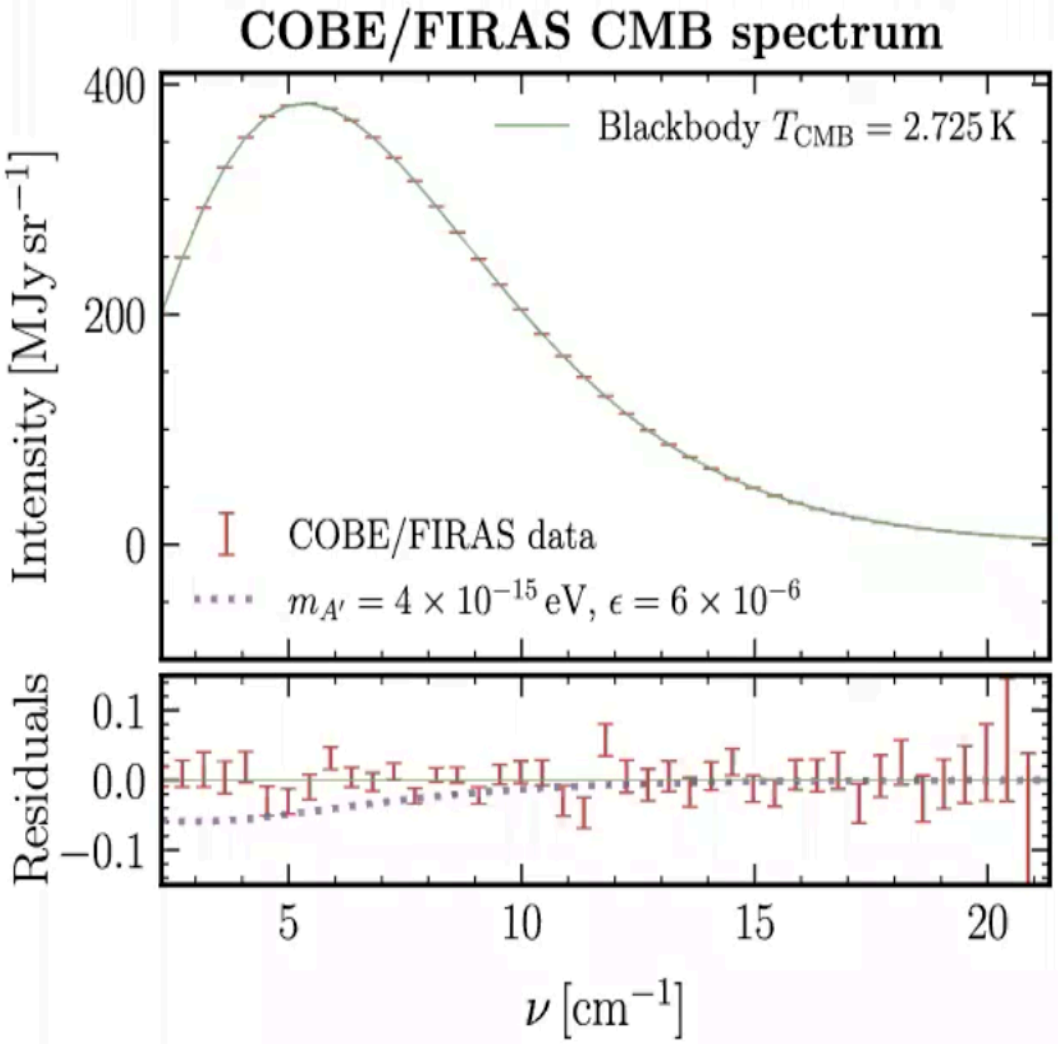
$$P_{\gamma\to A'} = \sum_i \frac{\pi\epsilon^2 m_{A'}^2}{\omega} \left| \frac{\mathrm{d}\ln m_\gamma^2}{\mathrm{d}t} \right|^{-1}_{t_i=t_{\mathrm{res}}}$$

1 crossing

3 crossings

0 crossings

CMB spectral distortions due to $\gamma \to A'$



$$I_\omega\left(m_{A'}, \epsilon; T_{\rm CMB}\right) = B_\omega\left(1 - P_{\gamma\to A'}\right)$$

Blackbody spectrum

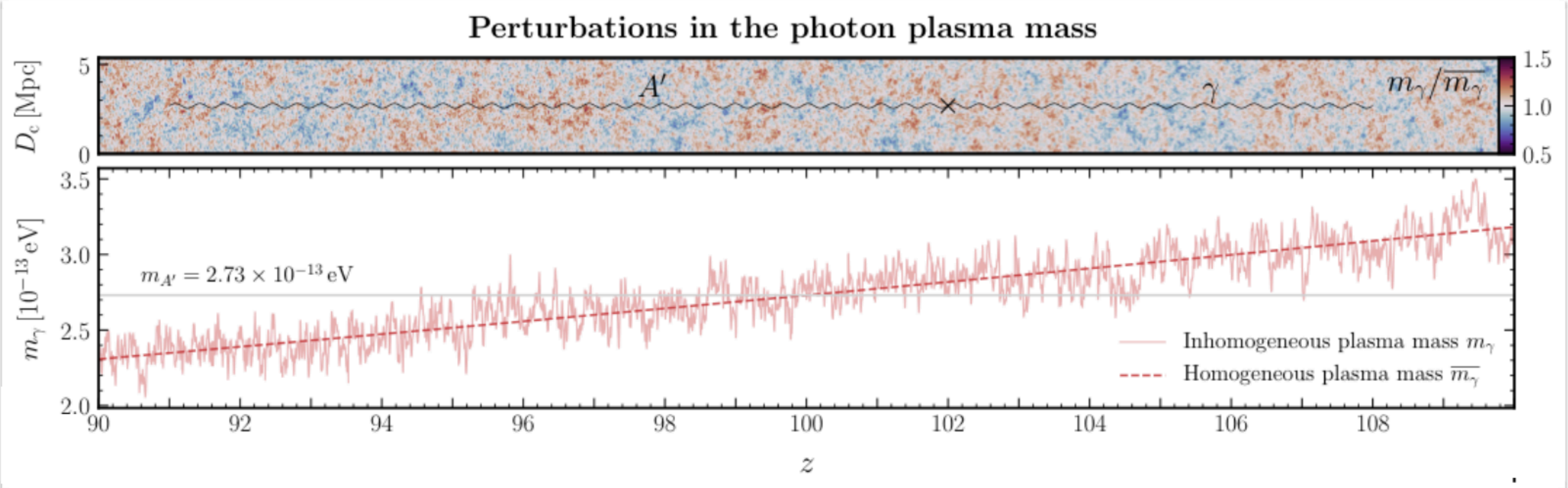
$$B_\omega = \frac{\omega^3}{2\pi^2}\left[\exp\left(\frac{\omega}{T_{\rm CMB}}\right) - 1\right]^{-1}$$

γ disappearance probability

$$P_{\gamma\to A'} \simeq \frac{\pi\epsilon^2 m_{A'}^2}{\omega\left(z_{\rm res}\right)}\left|\frac{{\rm d}\ln m_\gamma^2(t)}{{\rm d}t}\right|^{-1}_{z=z_{\rm res}}$$

Mirizzi, Redondo & Sigl 0901.0014

However there is an important piece missing. The Universe is not homogeneous!



A. Caputo, H. Liu, S. Mishra-Sharma & J. T. Ruderman 2002.05165 and 2004.06733

See also the related works: Bondarenko, Pradler & Sokolenko 2002.08942 A. A. Garcia+ 2003.10465 Witte+ 2003.13698

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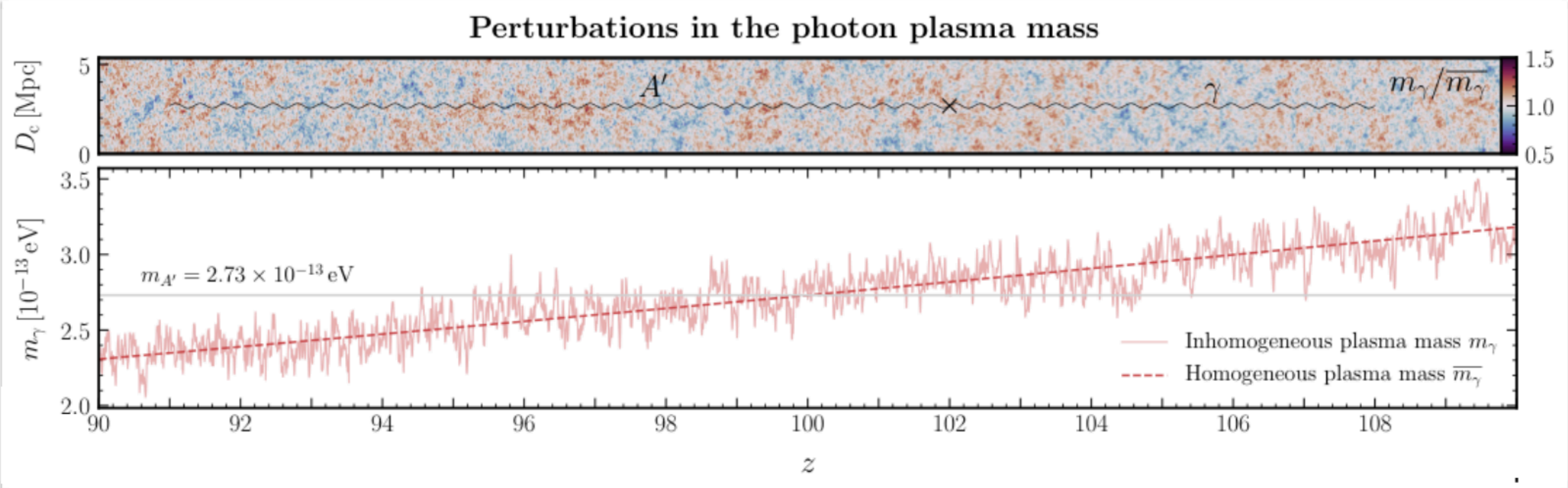
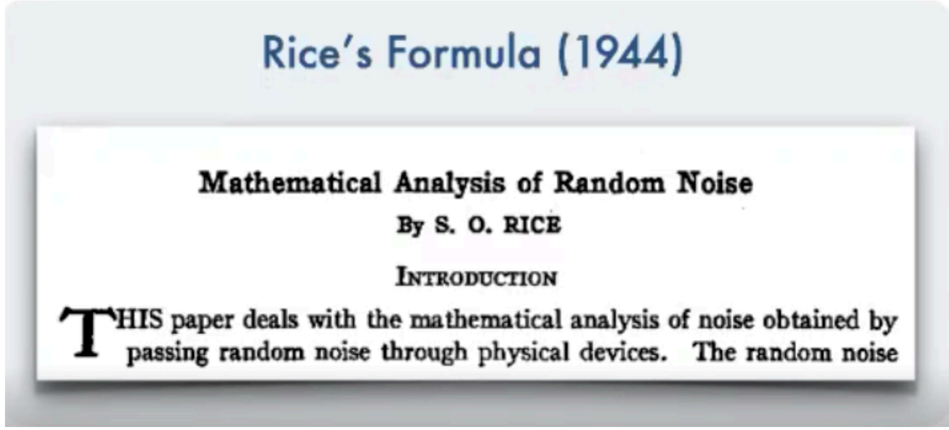


Fluctuations in electron density generate then fluctuations in the plasma mass of the photon. We have to make an average to get the real probability of conversion. We did this analytically!

A. Caputo, H. Liu, S. Mishra-Sharma & J. T. Ruderman 2002.05165 (PRL) and 2004.06733 (PRD)

The analytical formula


Rice's Formula (1944)

Mathematical Analysis of Random Noise

By S. O. RICE

INTRODUCTION

THIS paper deals with the mathematical analysis of noise obtained by passing random noise through physical devices. The random noise

$$P_{\gamma\to A'} = \int \mathrm{d}t\, \frac{\pi\epsilon^2 m_{A'}^2}{\omega(t)}\, \delta_{\mathrm{D}}(m_\gamma^2 - m_{A'}^2)\, m_\gamma^2$$

Average over distribution of m_γ^2

(time-dependent) probability density function

$$\langle P_{\gamma\to A'}\rangle = \int \mathrm{d}t \int \mathrm{d}m_\gamma^2\, f(m_\gamma^2; t)\, \frac{\pi\epsilon^2 m_{A'}^2}{\omega(t)}\, \delta_{\mathrm{D}}(m_\gamma^2 - m_{A'}^2)\, m_\gamma^2$$

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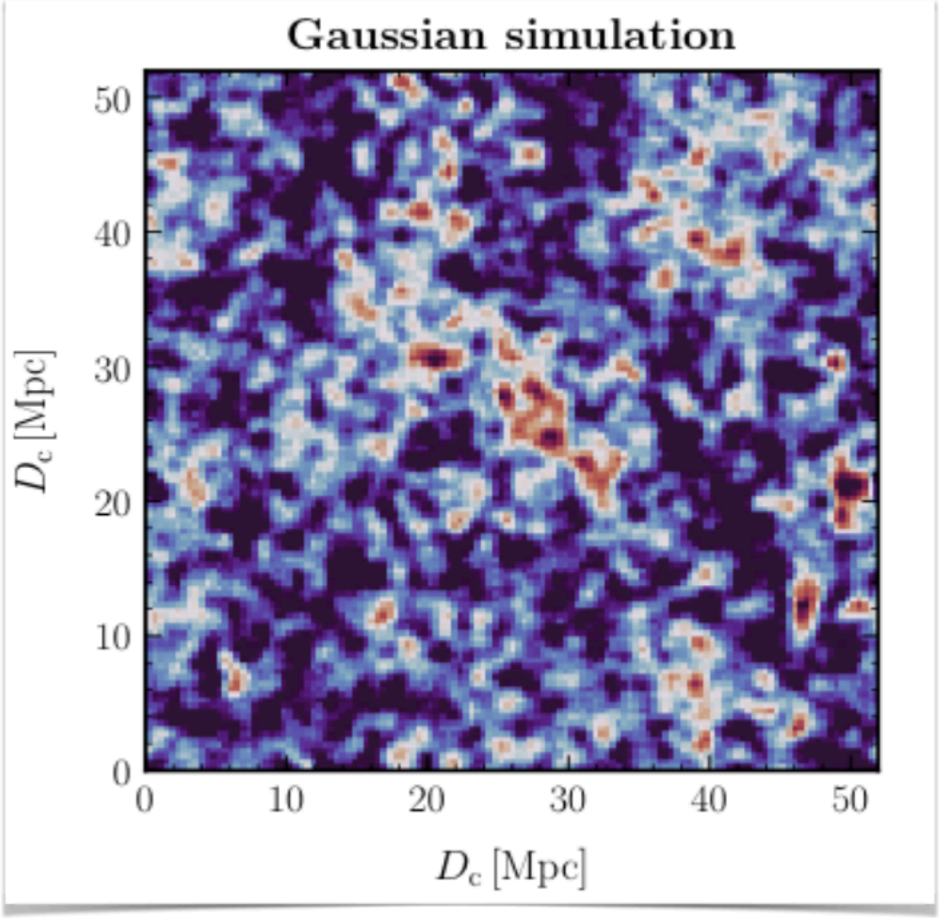
Then integrating over the mass using the delta function

$$\langle P_{\gamma\to A'}\rangle = \int \mathrm{d}t\, f(m_\gamma^2 = m_{A'}^2; t)\, \frac{\pi\epsilon^2 m_{A'}^4}{\omega(t)}$$

The average conversion probability is related the PDF of the plasma mass squared

One point PDF



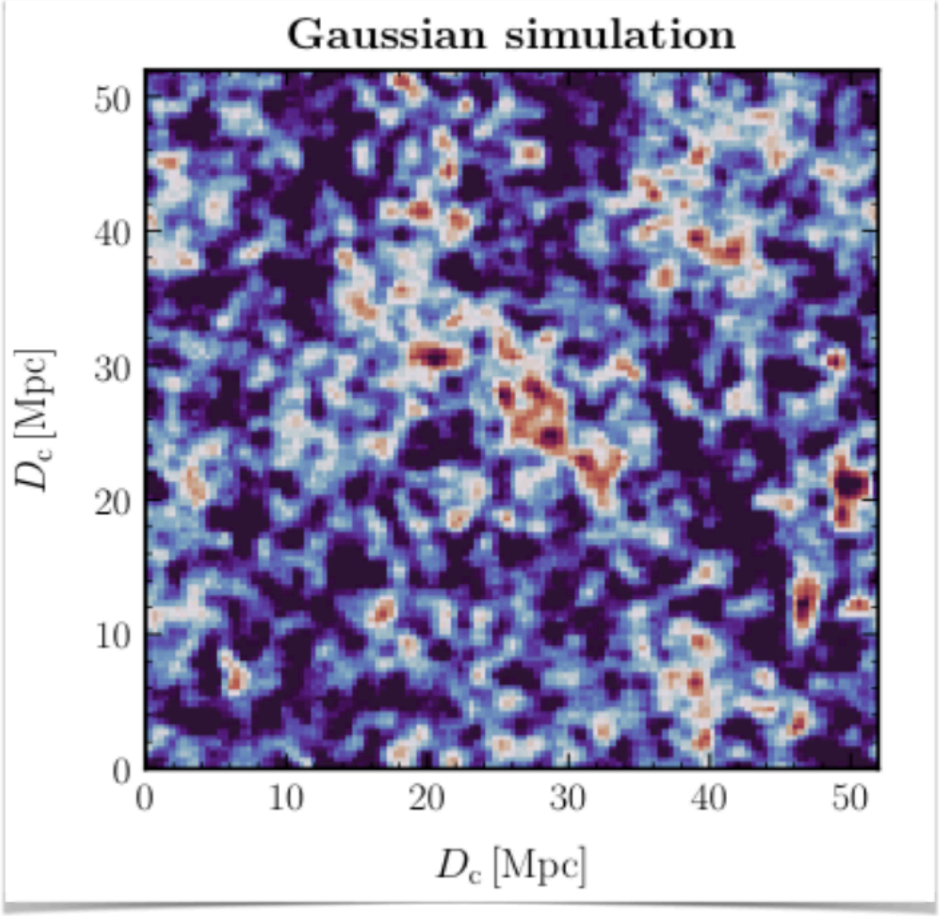
$$m_\gamma^2 \propto n_e \implies f(m_\gamma^2; t) \propto \mathscr{P}(\delta_b; t)$$

one-point PDF of baryon fluctuations

$$\delta_b \equiv \frac{\rho_b - \overline{\rho_b}}{\overline{\rho_b}}$$

m_γ^2 fluctuations directly related to **baryon density** fluctuations, a well-defined **cosmological parameter**.

One point PDF



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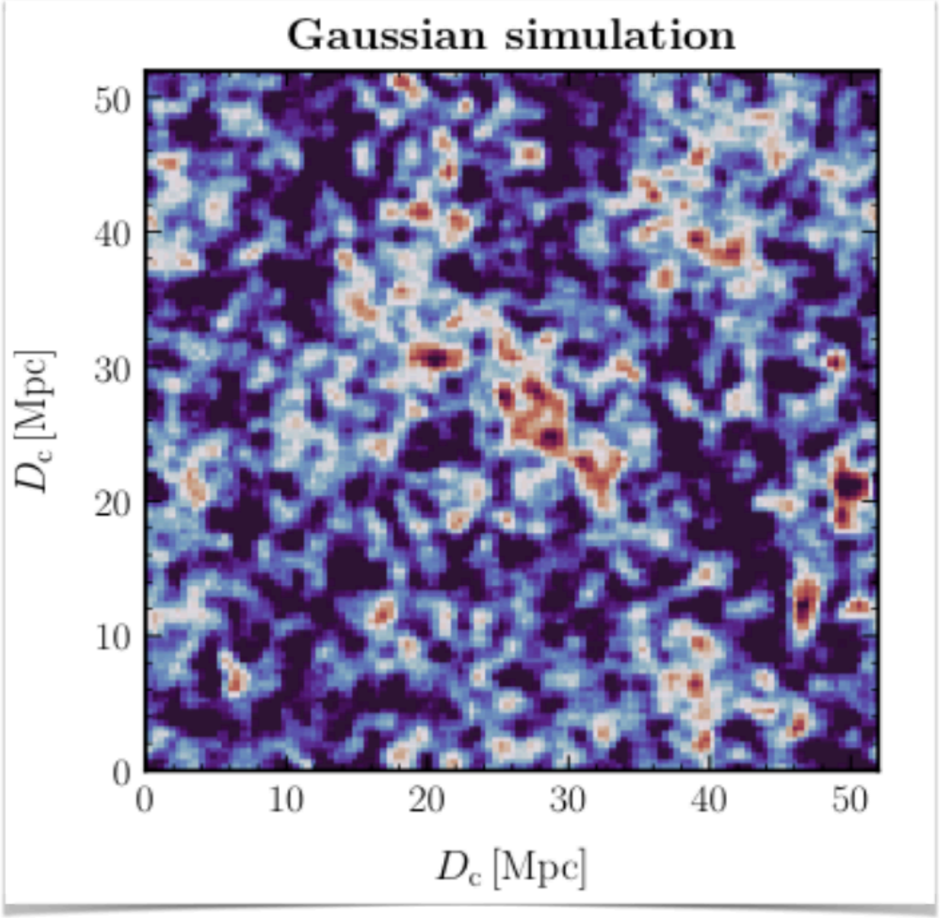
Electron and baryon fluctuations

$$\bar{n}_e(1+\delta_e) = \bar{x}_e(1+\delta_{x_e})\,\bar{n}_H(1+\delta_b)$$

$$\implies \delta_e = \delta_b + \delta_{x_e} + \delta_{x_e}\delta_b$$

If $\delta_{x_e} \ll \delta_b \implies \delta_e \approx \delta_b$

One point PDF



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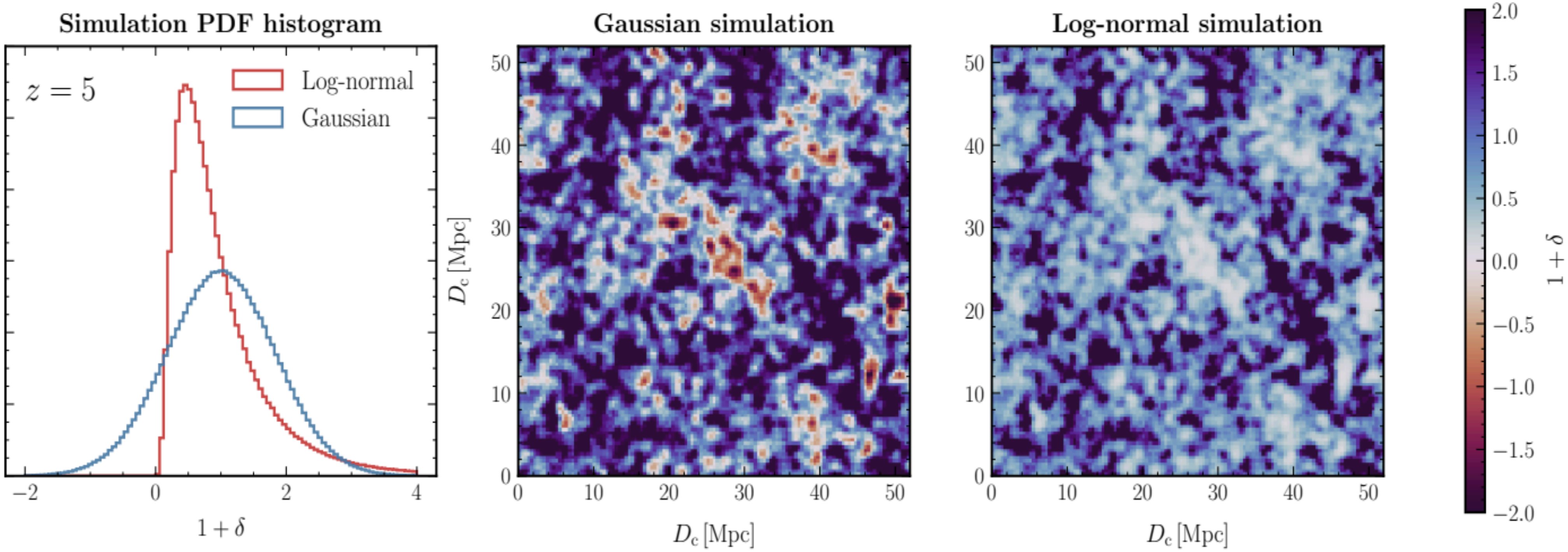
For example in the Gaussian case (z > 20) we would have

$$\mathscr{P}(\delta_b; z) = \frac{1}{\sqrt{2\pi\sigma_b^2(z)}} \exp\left(-\frac{\delta_b^2}{2\sigma_b^2(z)}\right)$$

Log-normal PDF

$$\mathscr{P}_{\mathrm{LN}}\left(\delta_{\mathrm{b}}; z\right) = \frac{\left(1+\delta_{\mathrm{b}}\right)^{-1}}{\sqrt{2\pi\,\Sigma^2(z)}} \exp\left(-\frac{\left[\ln\left(1+\delta_{\mathrm{b}}\right)+\Sigma^2(z)/2\right]^2}{2\Sigma^2(z)}\right)$$



Alternative PDF prescriptions

Log-normal PDF

Log-normal PDF with nonlinear baryon power spectrum

$$\mathscr{P}_{\rm LN}\left(\delta_{\rm b};z\right)=\frac{\left(1+\delta_{\rm b}\right)^{-1}}{\sqrt{2\pi\,\Sigma^2(z)}}\exp\left(-\frac{\left[\ln\left(1+\delta_{\rm b}\right)+\Sigma^2(z)/2\right]^2}{2\Sigma^2(z)}\right)$$

"Analytic" PDF

Non-linear spherical collapse of linear matter field

Ivanov, Kaurov, Sibiryakov [1811.07913]

$$\mathscr{P}_{\rm an}\left(\delta_{\rm b};z\right)=\frac{\hat{C}\left(\delta_{\rm b}\right)}{\sqrt{2\pi\sigma^2_{R_{\rm J}}(z)}}\exp\left[-\frac{F^2\left(\delta_{\rm b}\right)}{2\sigma^2_{R_{\rm J}}(z)}\right]$$

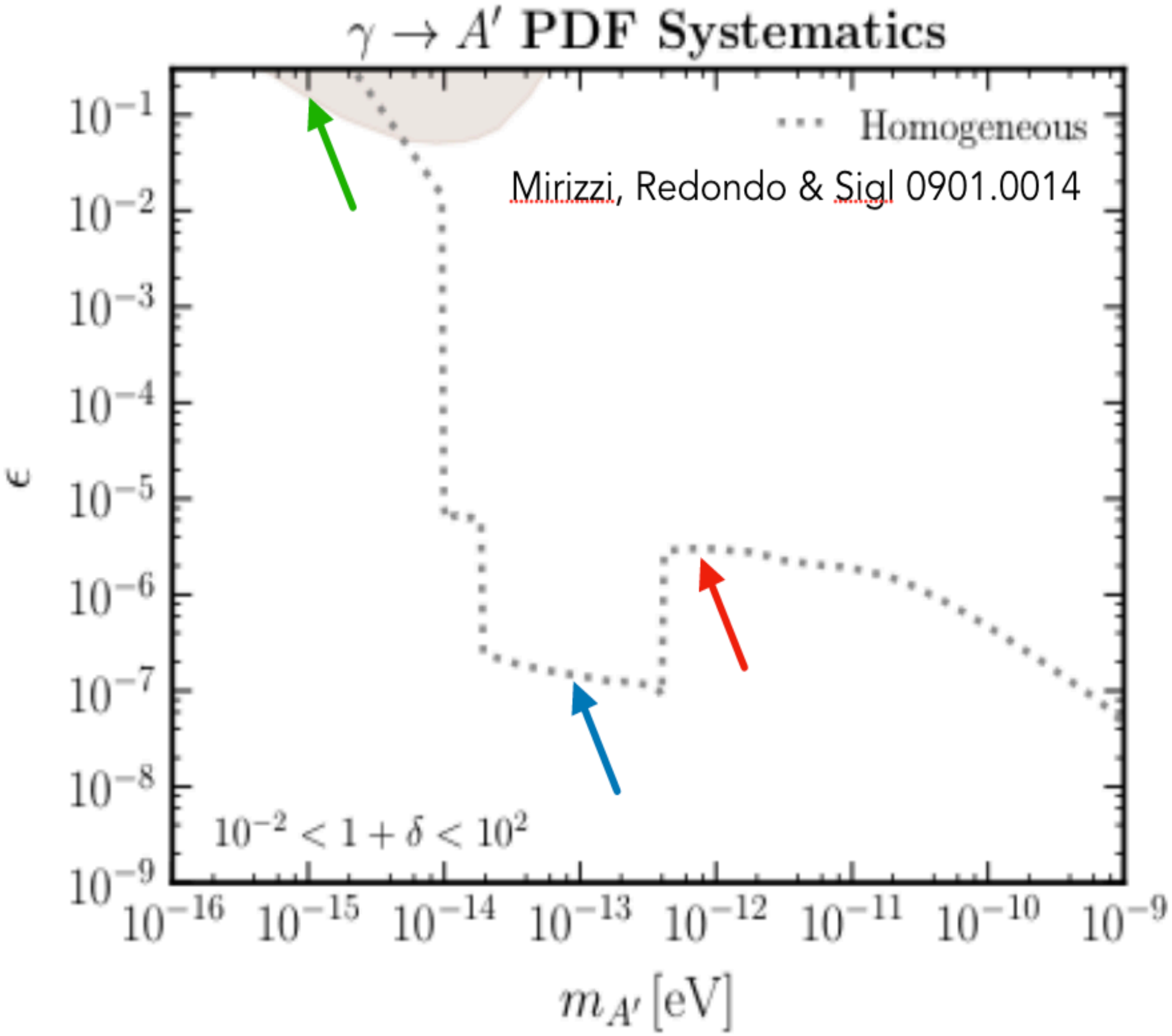
Cosmic voids PDF

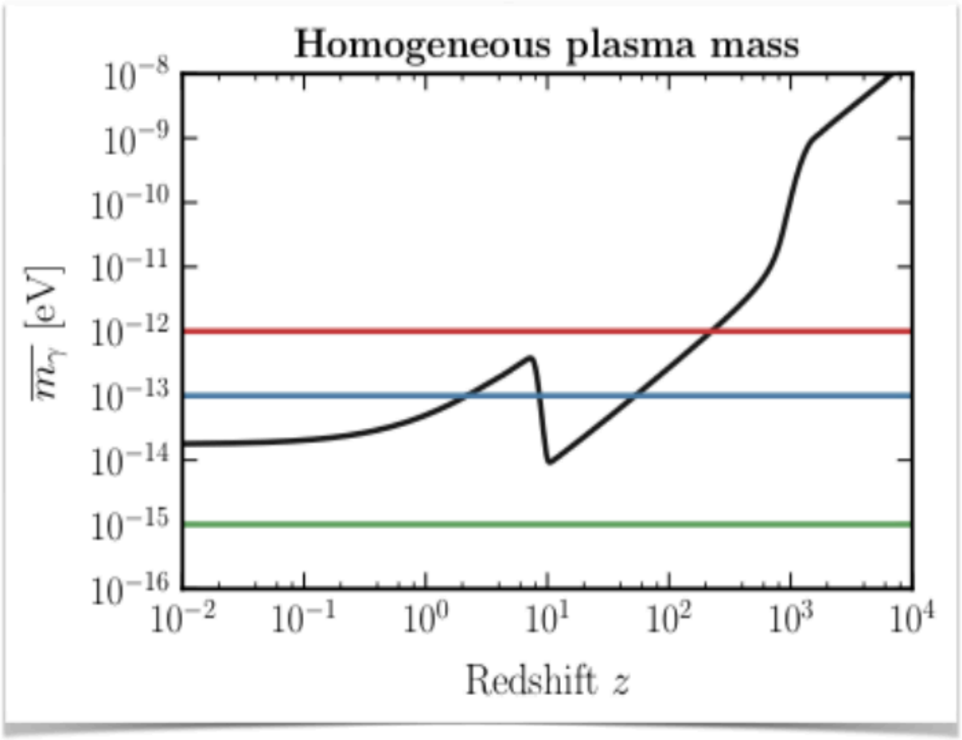
PDF of matter underdensities

Adermann et al [1703.04885, 1807.02938]

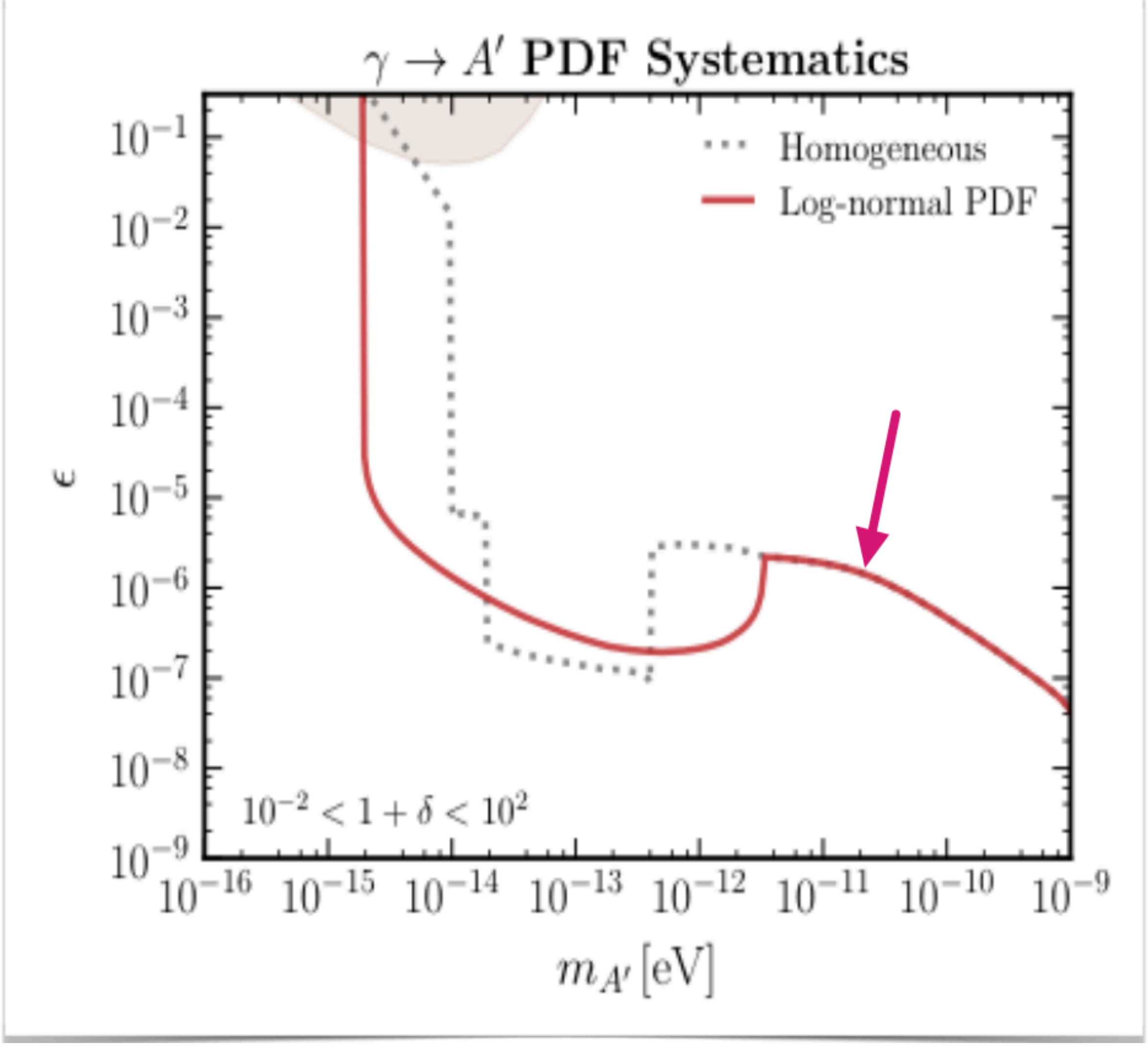
$$\mathscr{P}_{\rm voids}\left(\delta_{\rm b};z\right)\sim\text{from simulations}$$

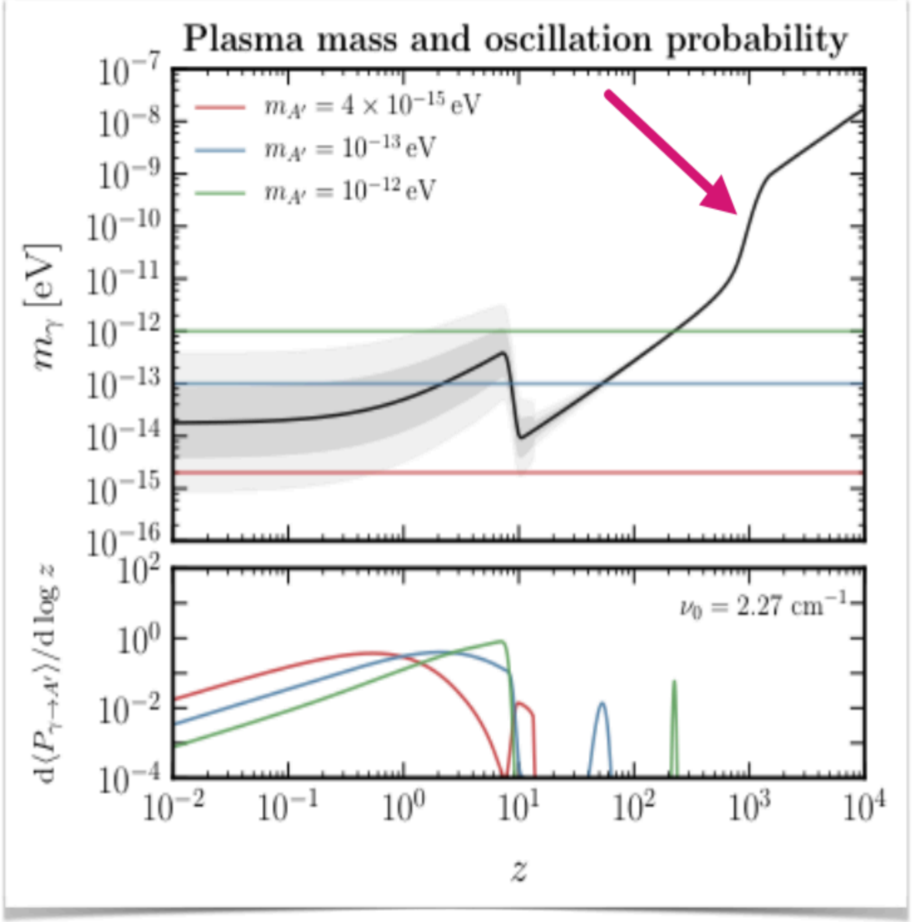
CMB Constraints





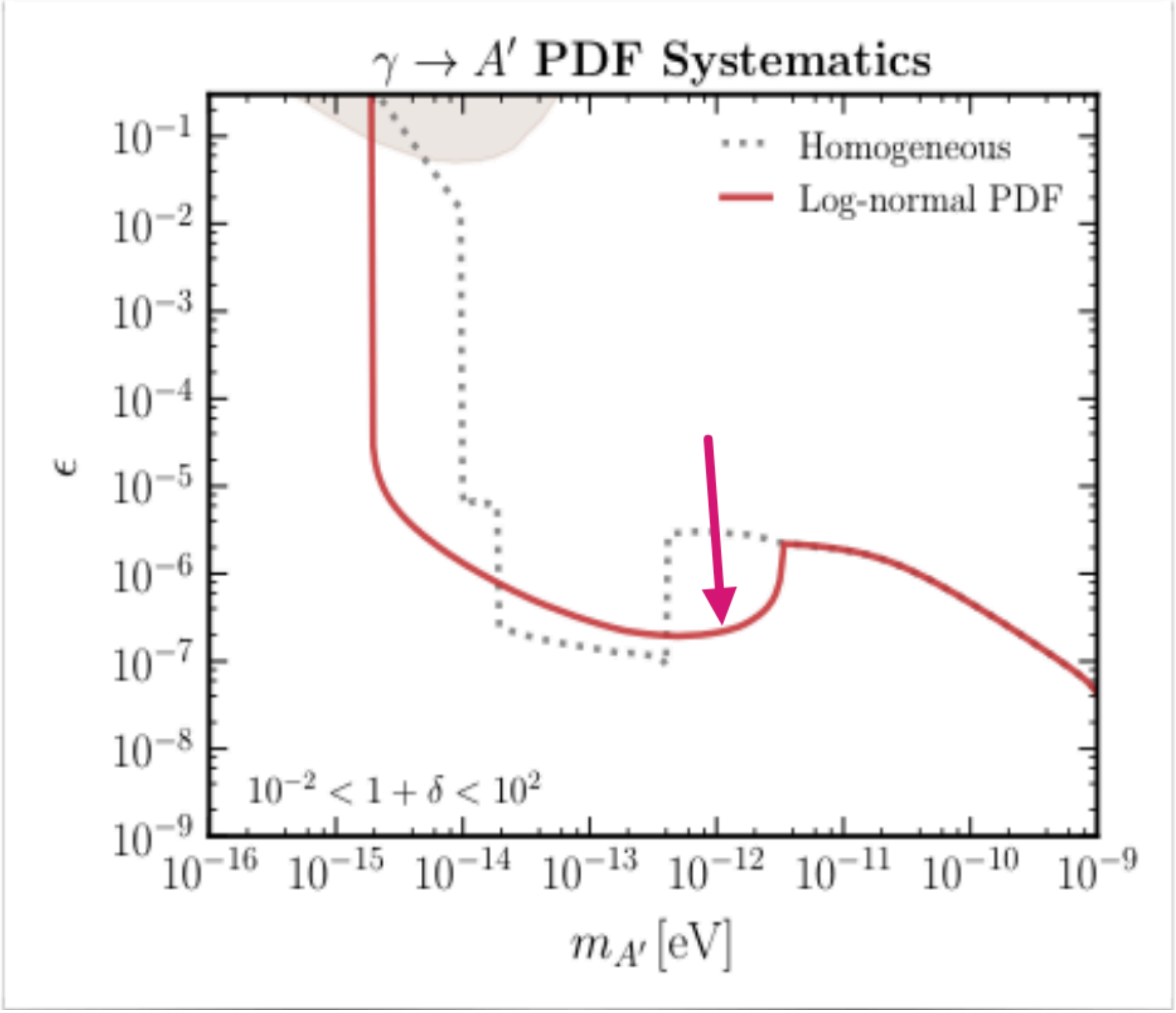
CMB Constraints

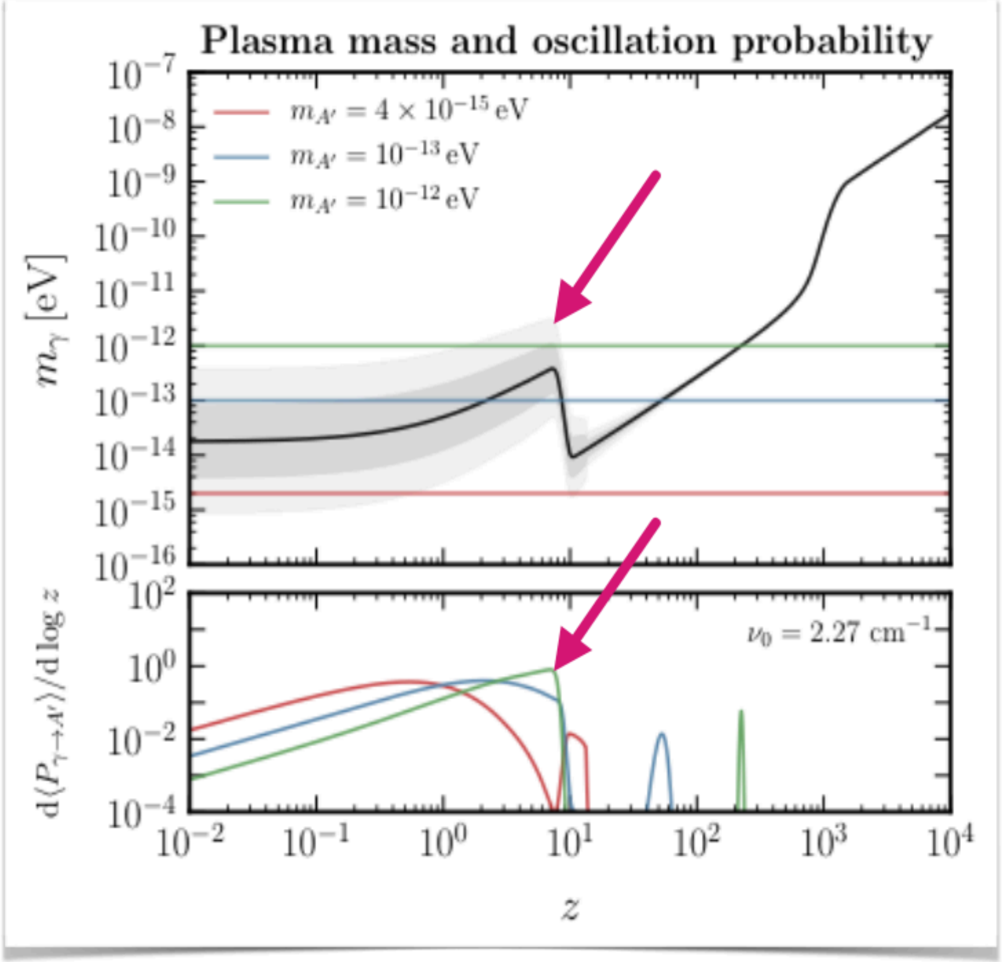




Small fluctuations at high redshift: similar to the homogeneous case

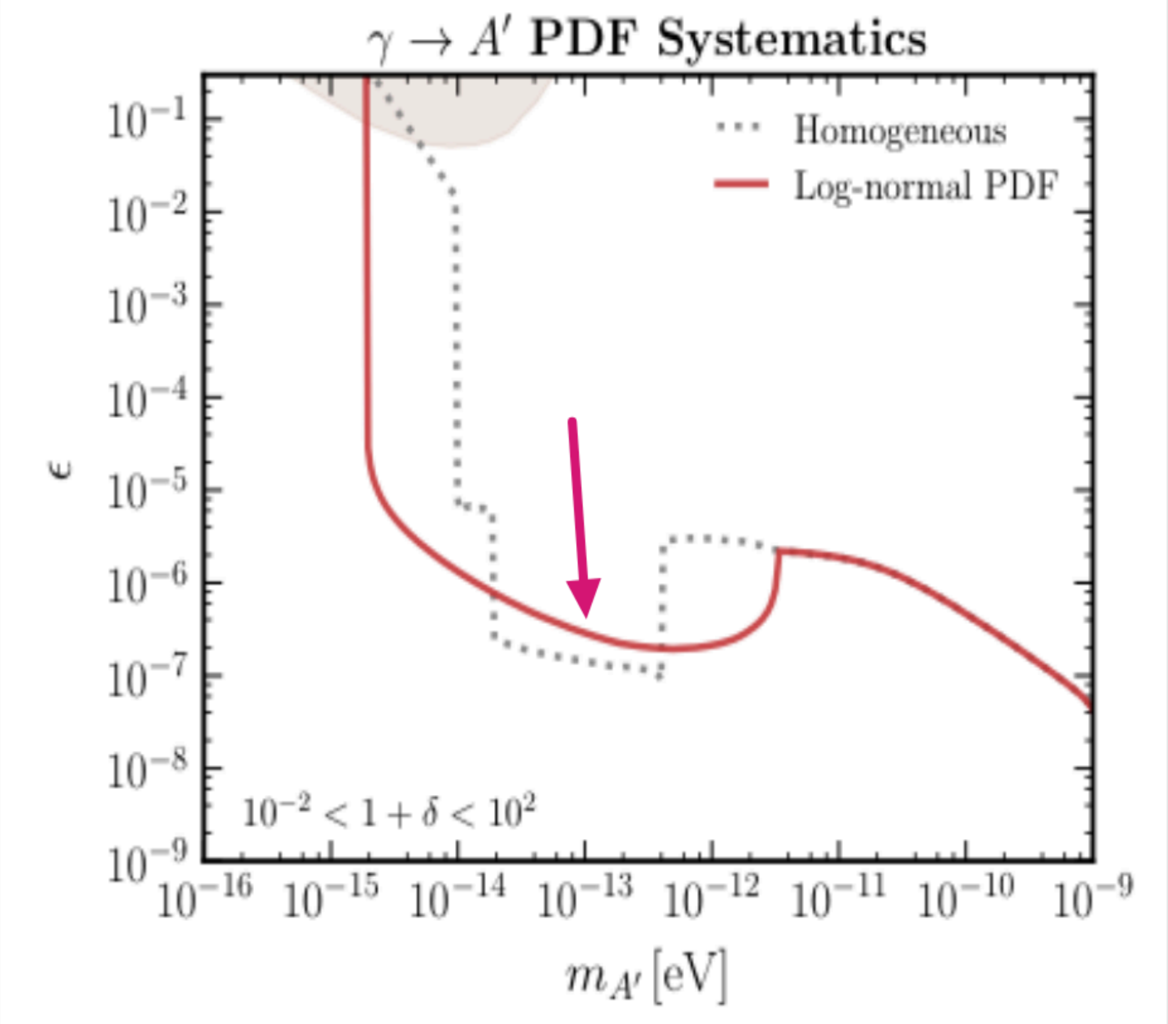
CMB Constraints

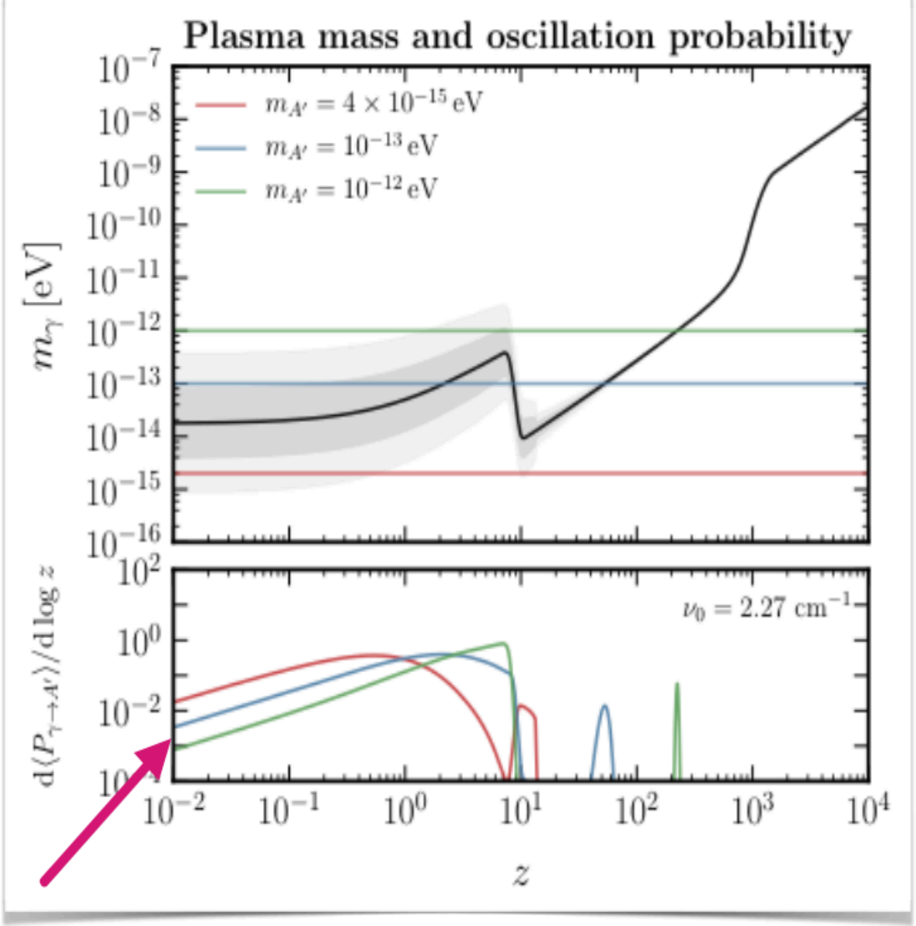




Later conversions available

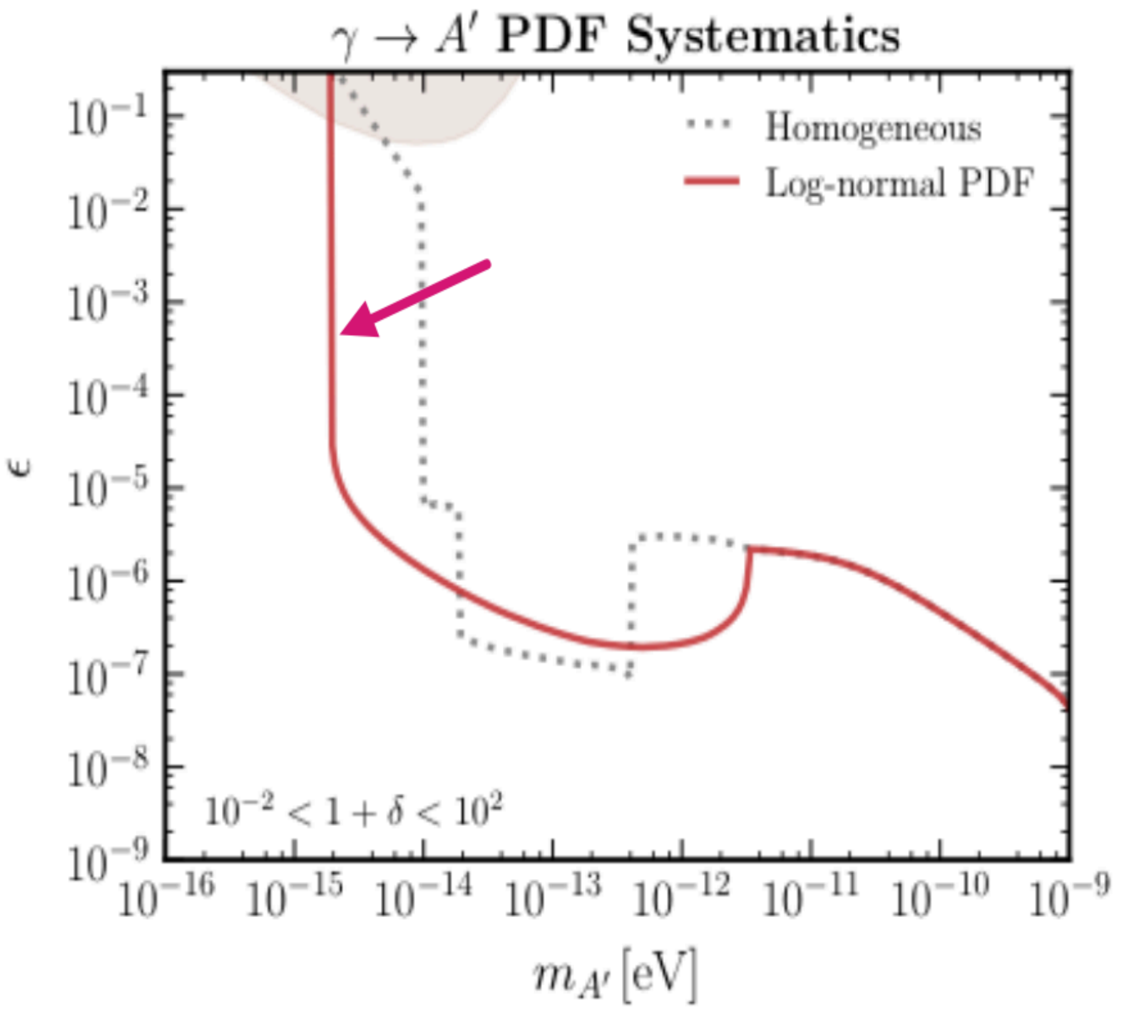
CMB Constraints

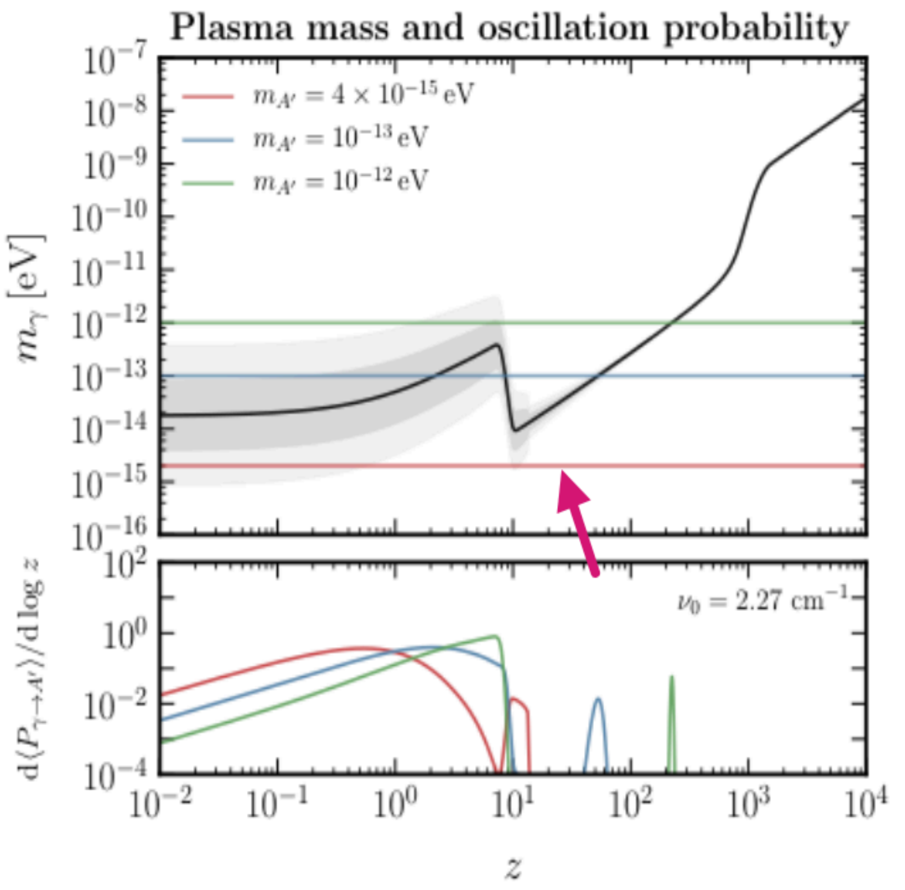




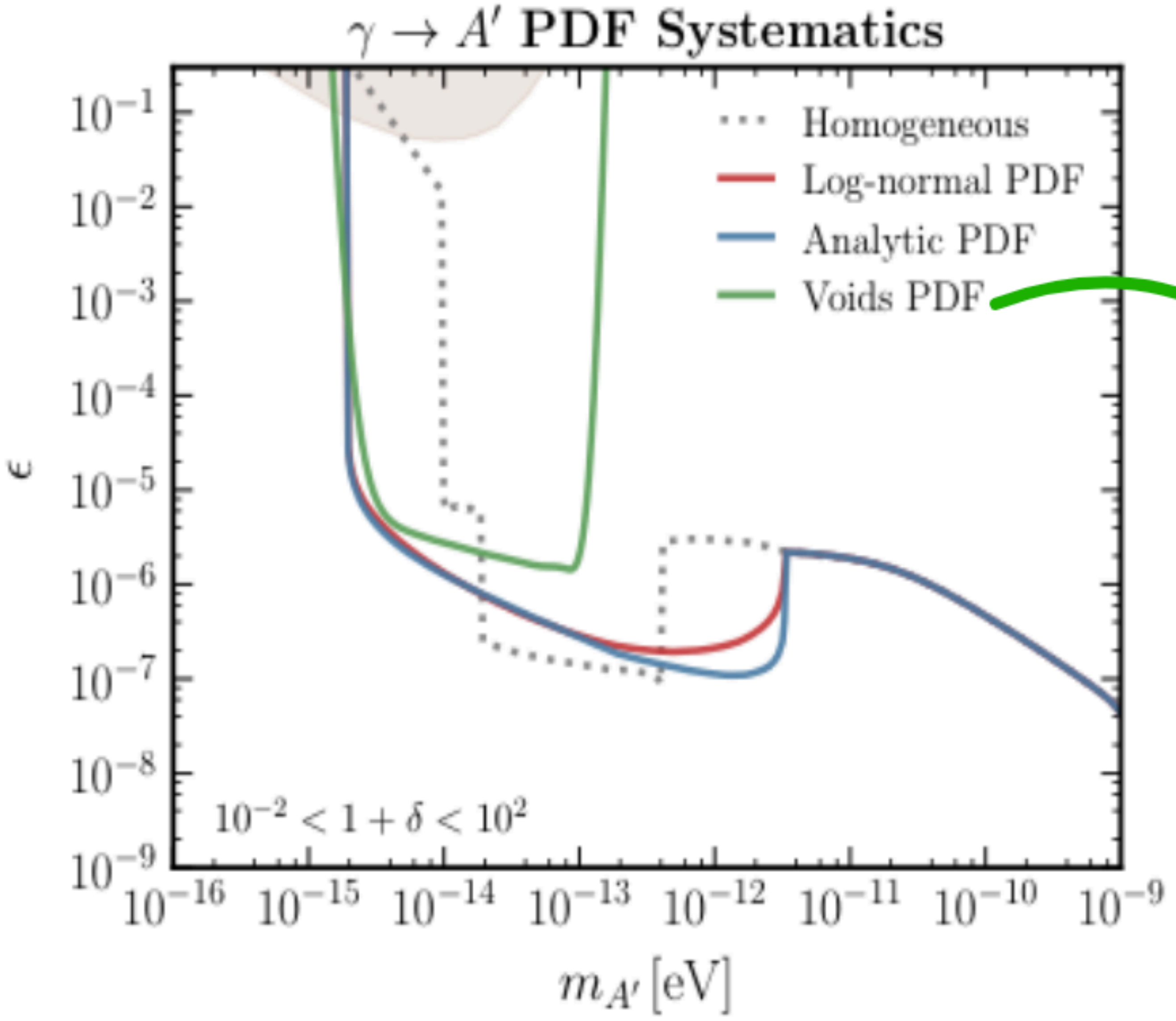
Constraints weakened here, as some conversion probability is in the future

CMB Constraints





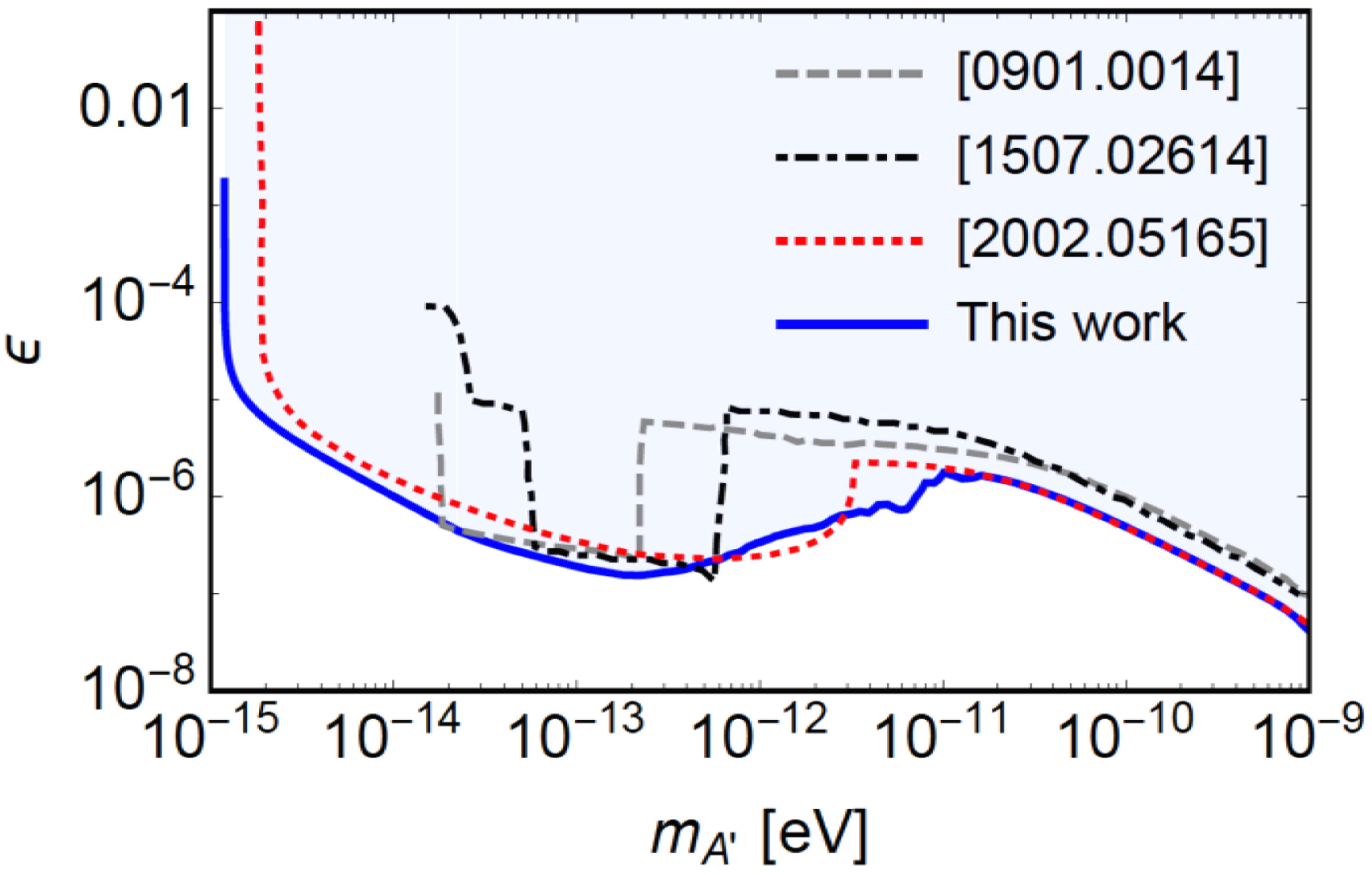
Limits extend to lower masses because of under-fluctuations



check for underdensities

E. Adermann et al, (2018)

Comparison with numerical approach



Bondarenko, Pradler, Sokolenko [2002.08942]
Garcia et al [2003.10465]

An extra ingredient: dark matter axion

Now what we want to do is instead to put this kind of coupling but with the dark photon

$$\mathcal{L} = \frac{1}{2}(\partial_\mu a)^2 - \frac{m_a^2}{2}a^2 + \frac{a}{4f_a}F'_{\mu\nu}\tilde{F}'^{\mu\nu} + \mathcal{L}_{AA'}$$

Kinetic mixing

Axion-dark photon coupling

Pospelov et al, 2018

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Kinetic mixing

Axion-dark photon coupling

Essentially now the axion will decay into two dark photons

$$\Gamma_a = \frac{m_a^3}{64\pi f_a^2} = \frac{3\times 10^{-4}}{\tau_{\rm U}}\left(\frac{m_a}{10^{-4}\,{\rm eV}}\right)^3\left(\frac{100\,{\rm GeV}}{f_a}\right)^2.$$

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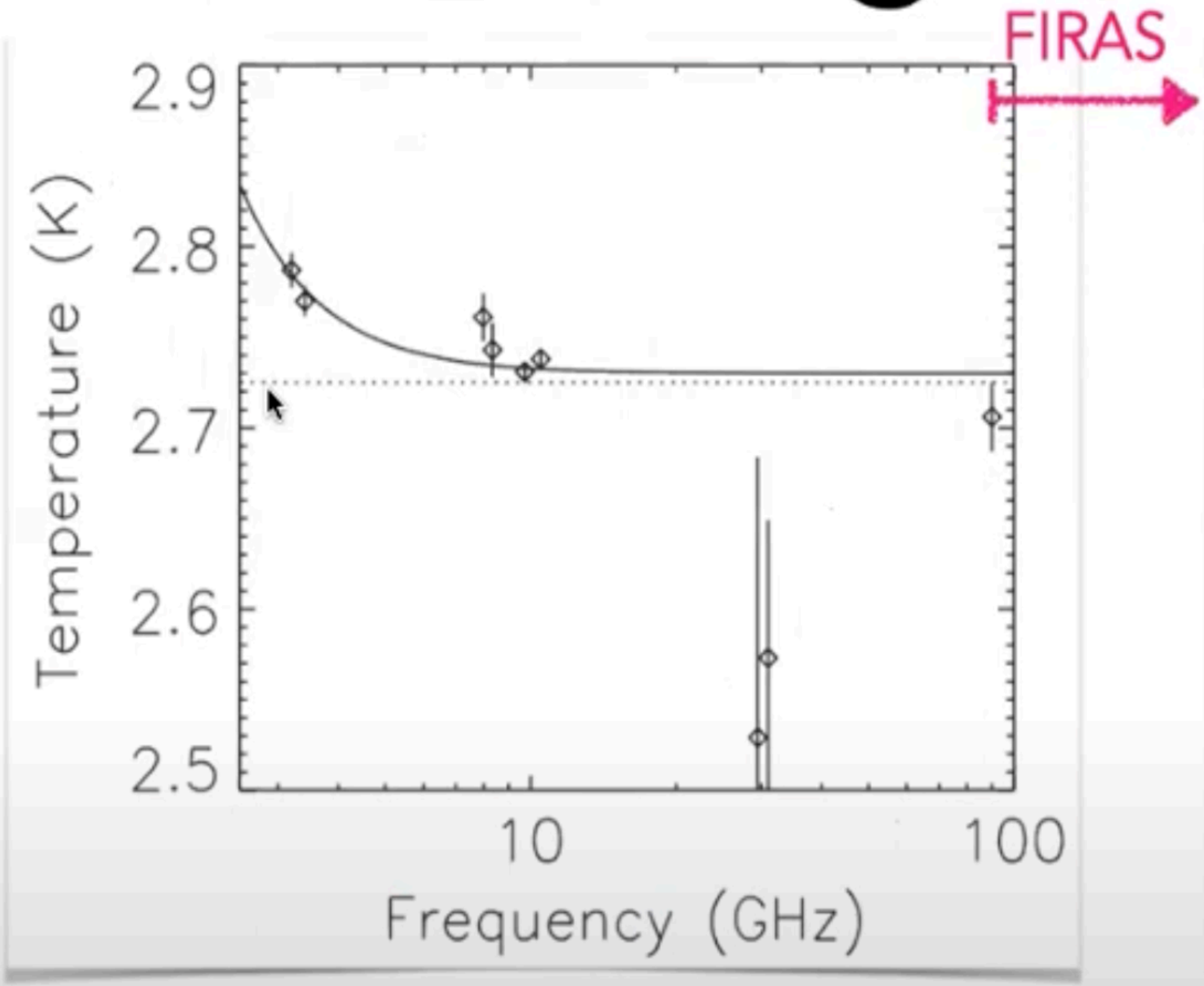
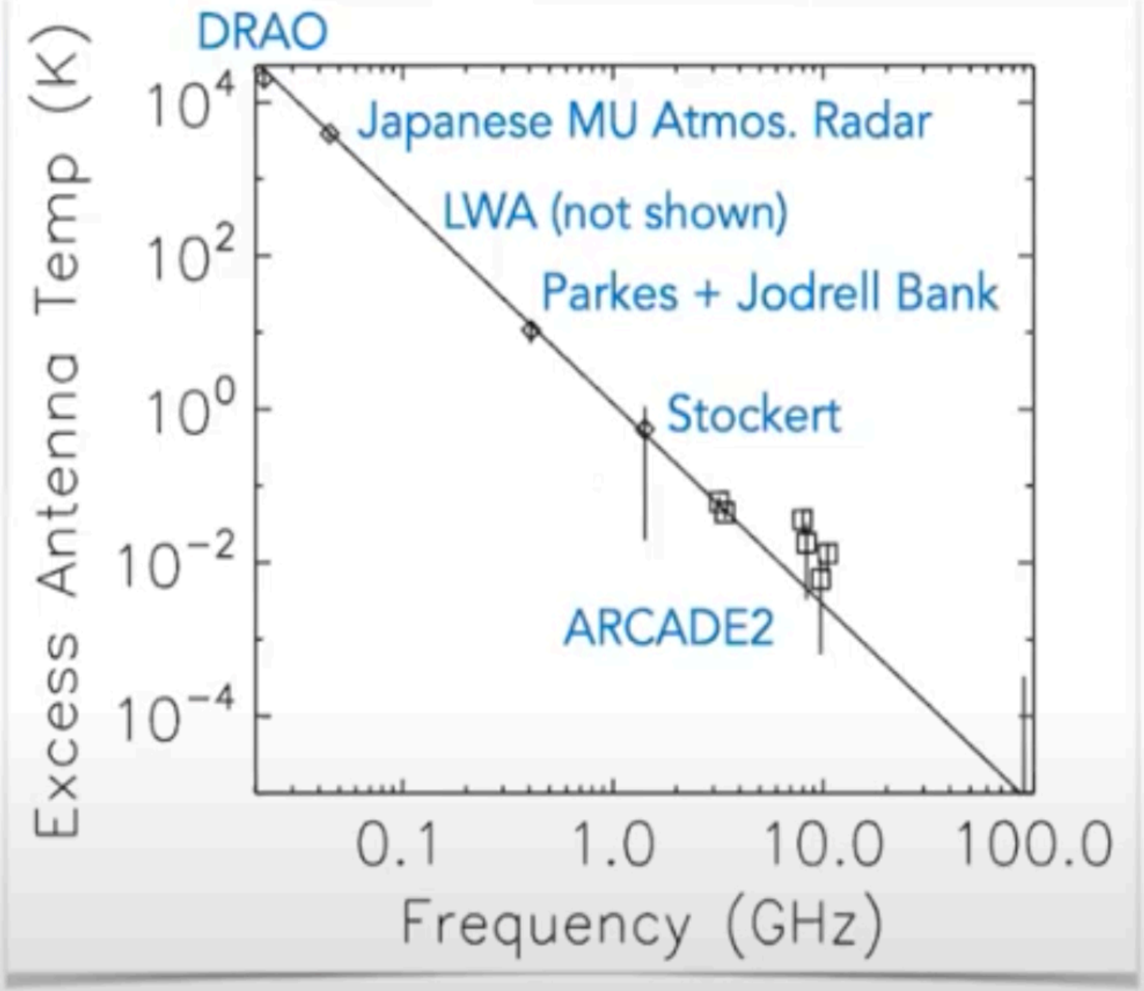
Then the dark photon will possibly oscillate to ordinary photon, essentially modifying the apparent number count of CMB radiation

$$\frac{dn_A}{d\omega} \to \frac{dn_A}{d\omega}\times P_{A\to A} + \frac{dn_{A'}}{d\omega}\times P_{A'\to A}$$

Radio Background

Fixsen+ 0901.0555





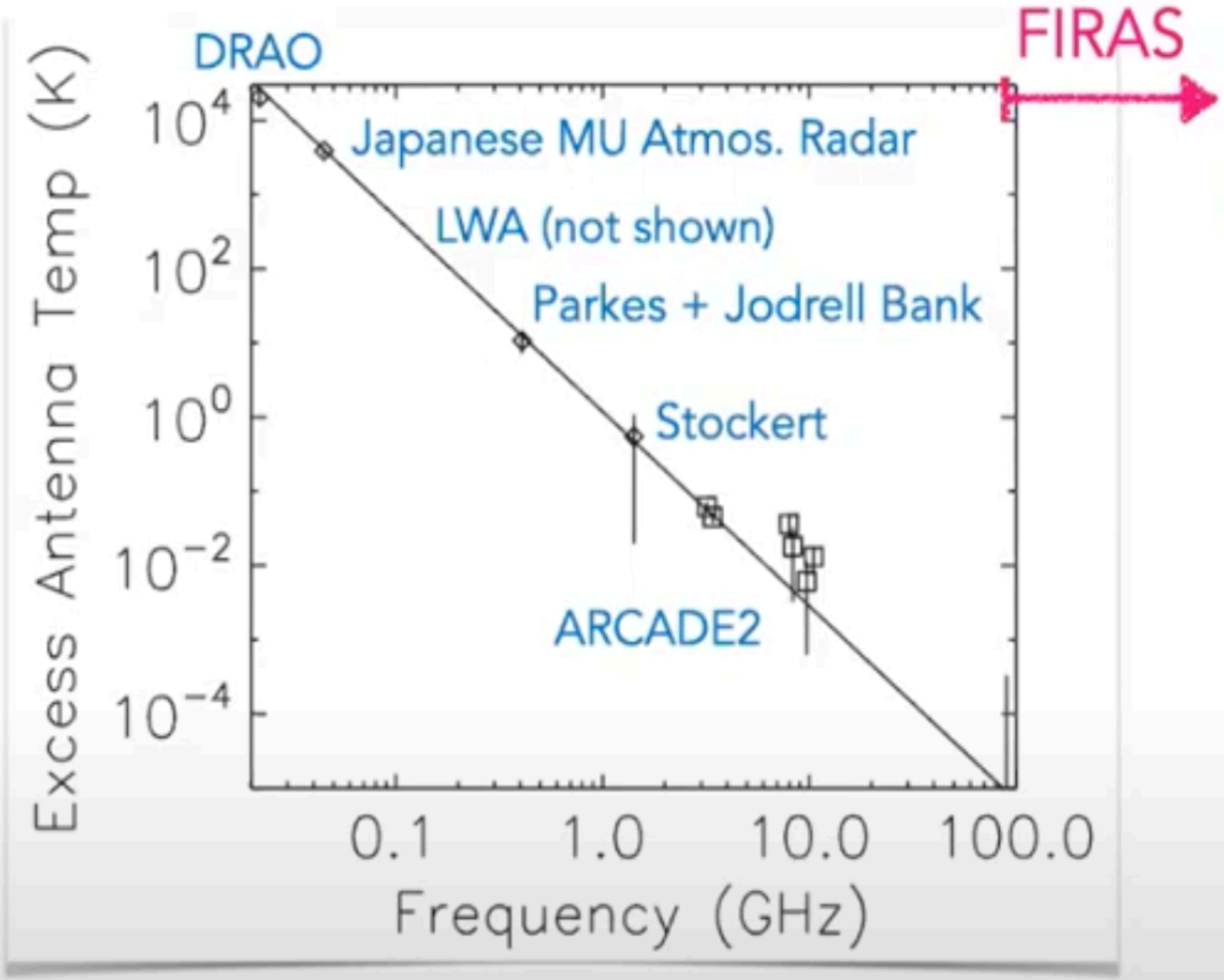
Broad series of **radio experiments** see a **power law excess** in the sky temperature (after subtracting the modelled galactic contribution).

ARCADE Excess

Fixsen+ 0901.0555

$$T = 2.729\,\mathrm{K} + 1.19\,\mathrm{K}\left(\frac{\nu}{1\,\mathrm{GHz}}\right)^{-2.62}$$

$$T = \frac{\pi^2}{\omega}\frac{dn_\gamma}{d\omega} \implies \frac{dn_\gamma}{d\omega} \propto \omega^{-1.62}$$



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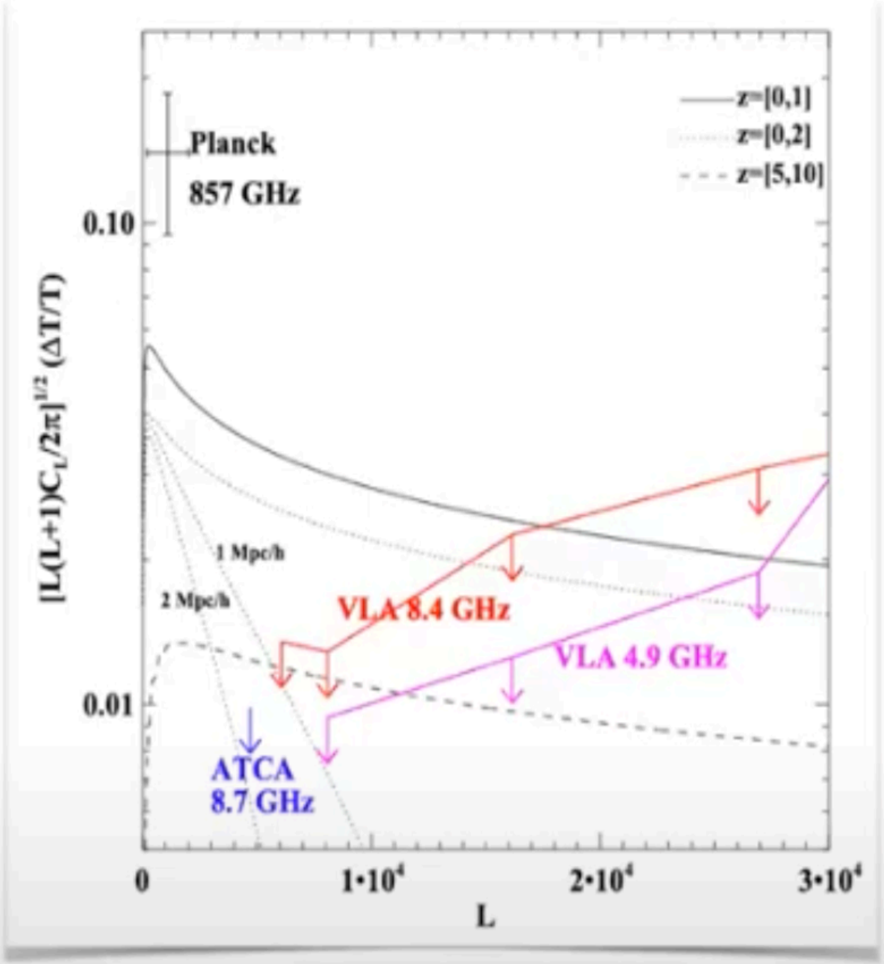
Smooth Emission

Holder 1207.0856



The emission is extremely **smooth**: assuming perfect correlation with structure, it **cannot be emitted when** $z \lesssim 5$, or **dominated by large structures** ($\gtrsim$ few Mpc).

Astrophysical explanations do not seem to work

- Galactic origin? No, because it would:

1. overproduce the observed X-ray background through inverse Compton emission;
2. make our Galaxy anomalous among nearby similar spiral galaxies;
3. Imply overproduction of the observed level of emission from CII.

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1. overproduce the observed X-ray background through inverse Compton emission;
2. make our Galaxy anomalous among nearby similar spiral galaxies;
3. Imply overproduction of the observed level of emission from CII.

- Extragalactic origin? Seems to be the case, but still difficult with ordinary astrophysics explanations:

1. would require an incredibly numerous new population of radio sources far below the flux densities currently probed;
2. far-infrared background would be overproduced

Dark matter explanation?

Synchrotron contribution from DM annihilation (or decay) products, see for example Fornengo et al. (2011), Hooper et al. (2012), Fang, Linden (2014).

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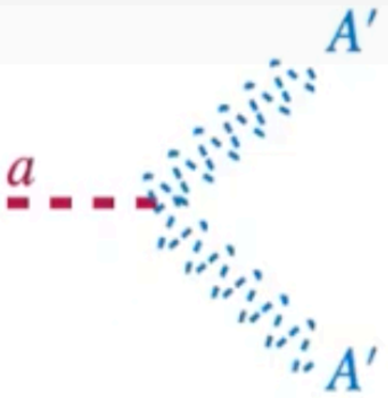
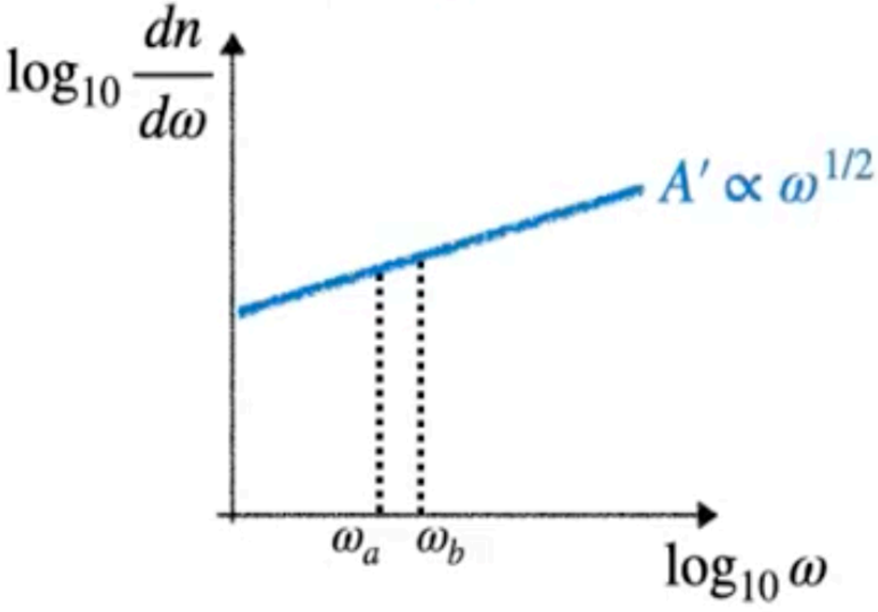
Typically requires too large magnetic fields and/or exotic magnetic field configurations. Also, the best fit spectrum is typically too soft (e.g in Fang, Linden (2014)).

Dark Photon Spectrum



Goal: $dn/d\omega \propto \omega^{-3/2}$



$$\omega \frac{dn_{A'}}{d\omega} = (1+z_\star)\frac{dn_{A'}}{d(1+z)}(z_\star)$$

$$= (1+z_\star) \times \frac{dn_{A'}}{dt}(z_\star) \times \frac{dt}{dz}(z_\star)$$

$$= (1+z_\star) \times \frac{2\Gamma\rho_a}{m_a} \times \frac{1}{H(z_\star)(1+z_\star)}$$

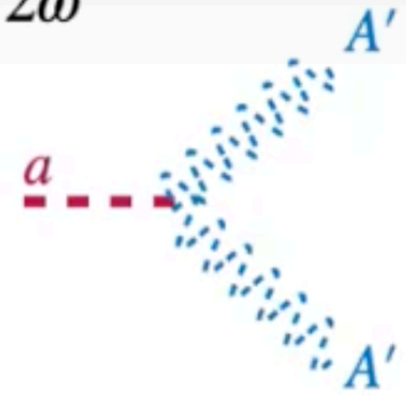
$$\frac{dn_{A'}}{d\omega} \propto \omega^{1/2}$$

A' with energy ω was produced at $(1+z_\star) = m_a/2\omega$.

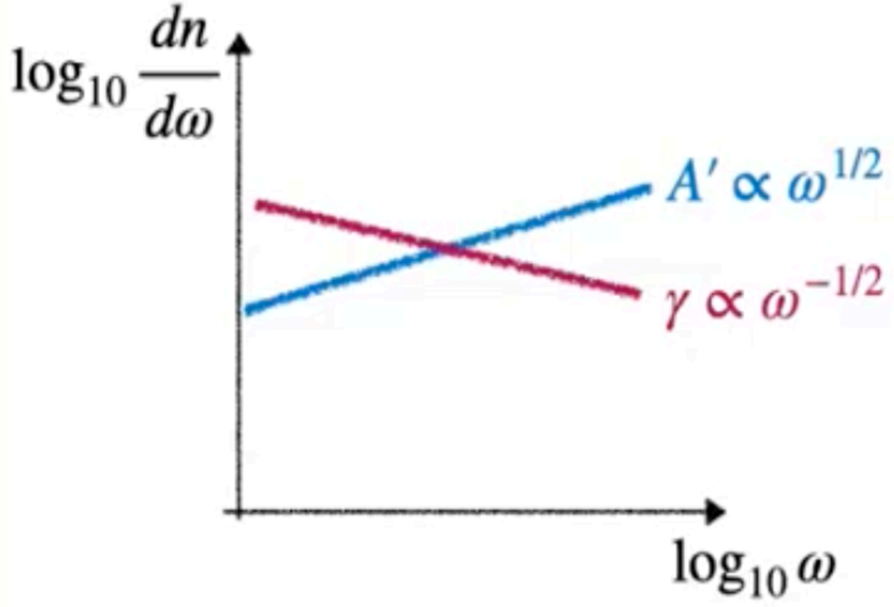
$$\frac{\omega_a}{\omega_b} = \frac{1+z_b}{1+z_a} \implies d\log\omega = -\,d\log z$$

Photon Spectrum

$$1 + z_\star = m_a/2\omega$$



Goal: $dn/d\omega \propto \omega^{-3/2}$



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$$= (1 + z_\star) \times \frac{dn_{A'}}{dt}(z_\star) \times \frac{dt}{dz}(z_\star)$$

$$= (1 + z_\star) \times \frac{2\Gamma\rho_a}{m_a} \times \frac{1}{H(z_\star)(1 + z_\star)}$$

$$\frac{dn_{A'}}{d\omega} \propto \omega^{1/2}$$

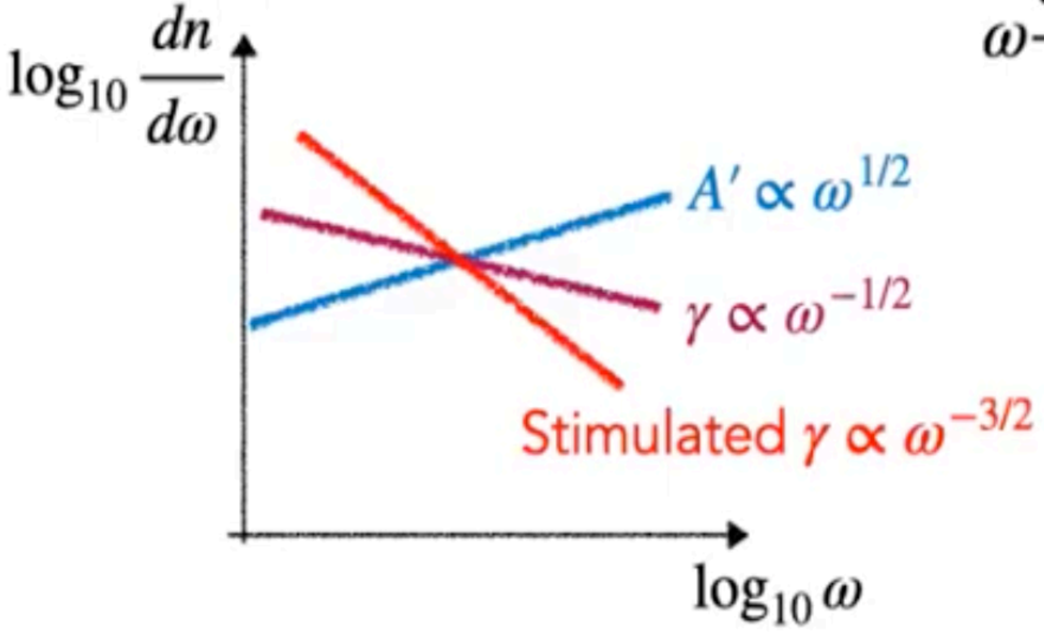
Probability of $\gamma \to A'$ conversion (homogeneous):

$$P_{\gamma \to A'} = \sum_i \frac{\pi \epsilon^2 m_{A'}^2}{\omega} \left| \frac{\mathrm{d} \ln m_\gamma^2}{\mathrm{d}t} \right|^{-1}_{t_i = t_{\mathrm{res}}}$$

Stimulated Decay

$$1 + z_\star = m_a/2\omega$$

Goal: $dn/d\omega \propto \omega^{-3/2}$

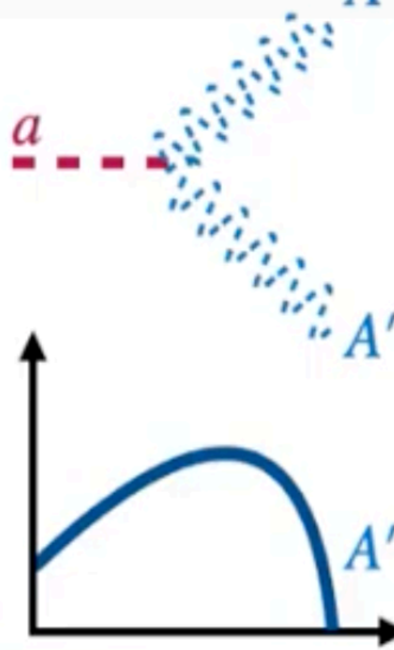


$$\omega \frac{dn_{A'}}{d\omega} = (1+z_\star) \frac{dn_{A'}}{d(1+z)}(z_\star)$$

$$= (1+z_\star) \times \frac{dn_{A'}}{dt}(z_\star) \times \frac{dt}{dz}(z_\star)$$

$$= (1+z_\star) \times \frac{2\Gamma \rho_a}{m_a} \times \frac{2T_0'}{\omega} \times \frac{1}{H(z_\star)(1+z_\star)}$$

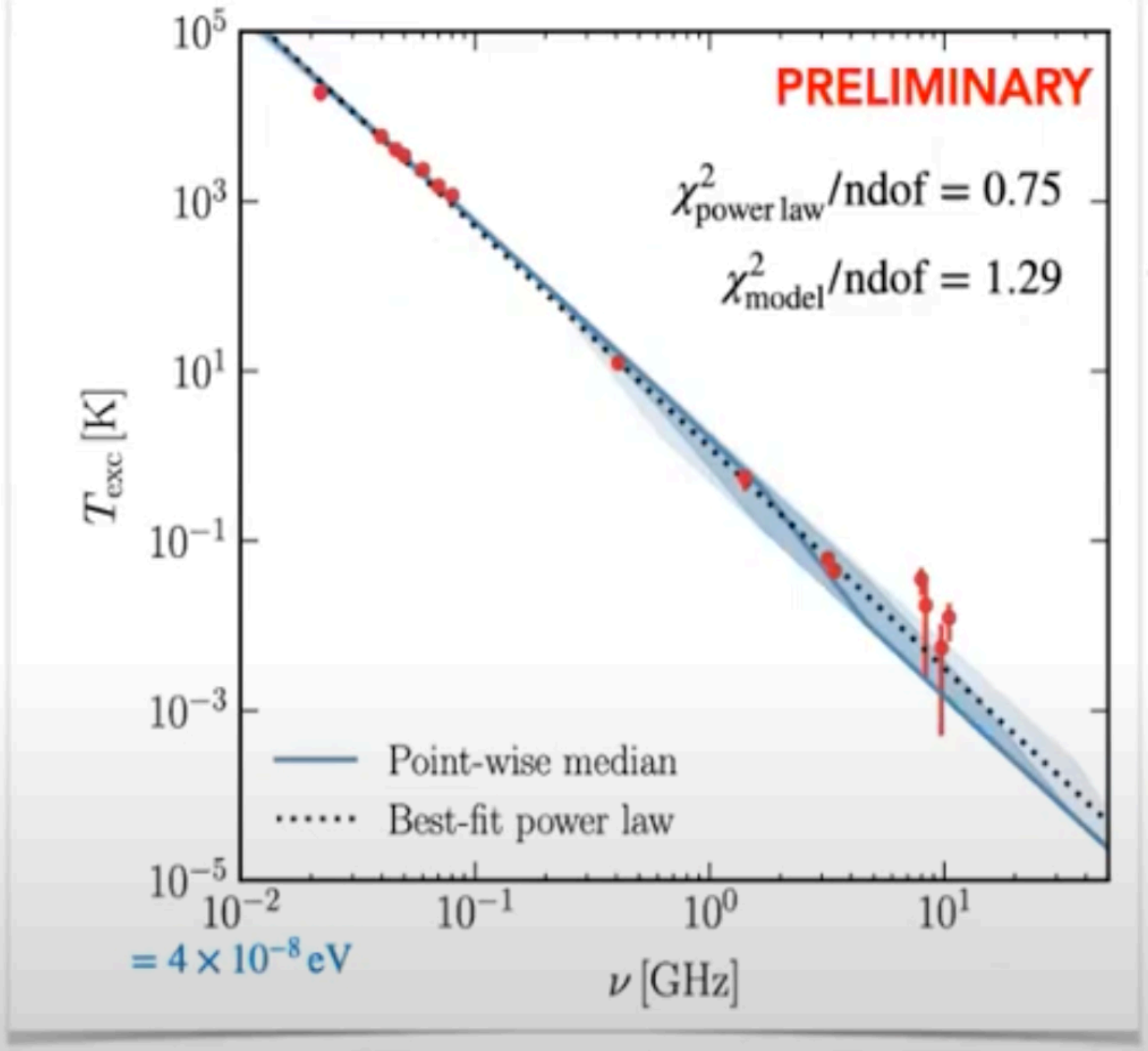
$$\propto \omega^{-1/2}$$



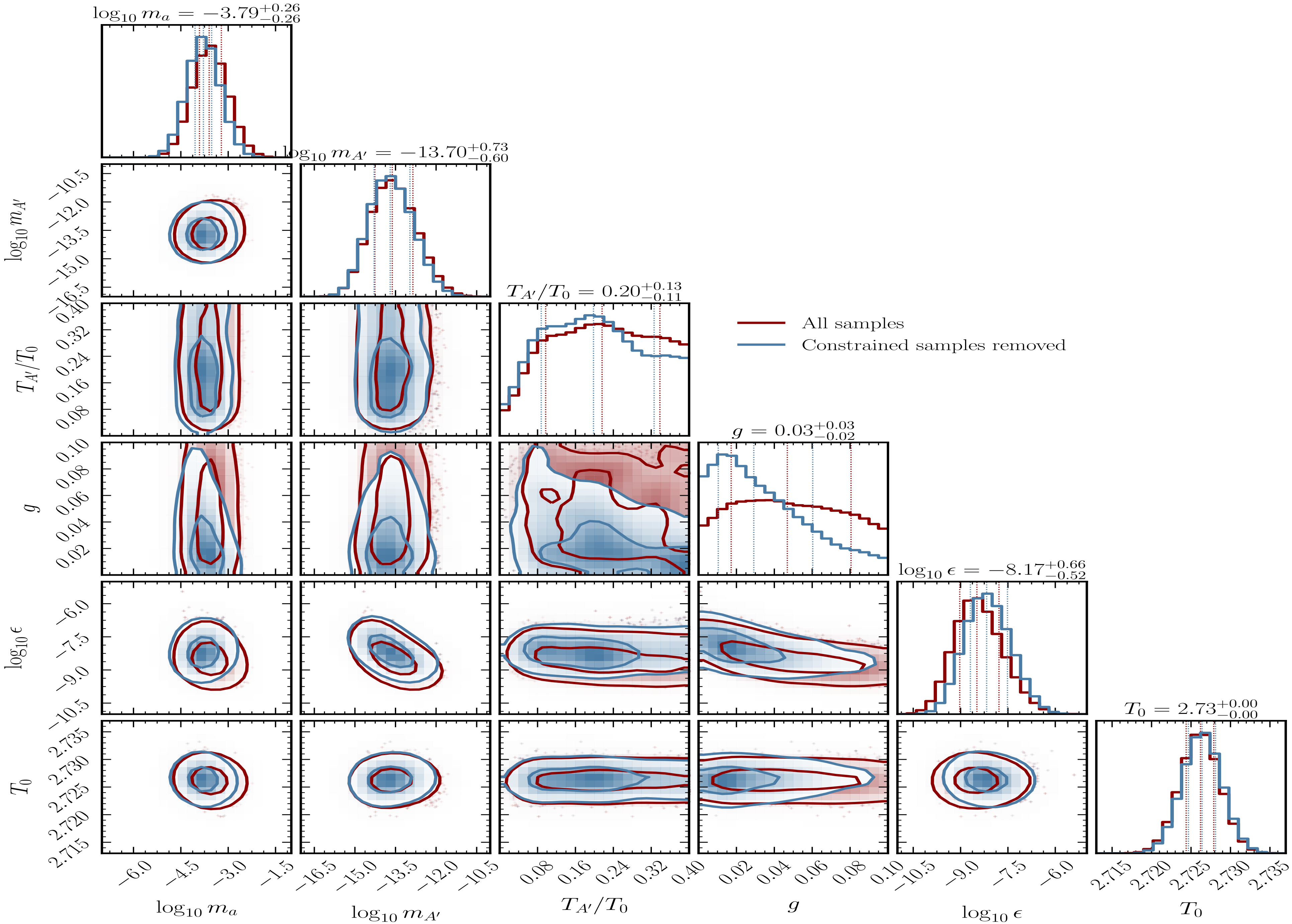
$$\frac{dn_\gamma}{d\omega} \sim \frac{dn_{A'}}{d\omega} P_{\gamma \to A'} = \propto \omega^{-3/2}$$

Spectral Fit

$$m_a = 4.9 \times 10^{-5}\,\mathrm{eV}$$
$$m_{A'} = 3.3 \times 10^{-14}\,\mathrm{eV}$$
$$g = 0.03\,\mathrm{GeV}^{-1}$$
$$\epsilon = 4.6 \times 10^{-9}$$
$$T_{A'} = 0.22 T_{\mathrm{CMB}}$$



Includes **inhomogeneities**, thermal $A' \to \gamma$ oscillations, and 20% extragalactic background.



$\log_{10} m_a = -3.79^{+0.26}_{-0.26}$
$\log_{10} m_{A'} = -13.70^{+0.73}_{-0.60}$
$T_{A'}/T_0 = 0.20^{+0.13}_{-0.11}$
$g = 0.03^{+0.03}_{-0.02}$
$\log_{10} \epsilon = -8.17^{+0.66}_{-0.52}$
$T_0 = 2.73^{+0.00}_{-0.00}$
All samples
Constrained samples removed
$\log_{10} m_a$
$\log_{10} m_{A'}$
$T_{A'}/T_0$
g
$\log_{10} \epsilon$
T_0

What about the isotropy of the signal?

$$\frac{\langle TT\rangle(\theta)}{\langle T\rangle^2} = \frac{\left\langle \int_z^{z_*} dz' \frac{dP_{\gamma\to A'}}{dz'} \int_z^{z_*} dz'' \frac{dP_{\gamma\to A'}}{dz''} \right\rangle_\theta}{\left\langle \int_z^{z_*} dz' \frac{dP_{\gamma\to A'}}{dz'} \right\rangle^2}$$

We want to compute this quantity

$$f_G(\vec{\delta}) = \frac{D(z')}{D(z)} \frac{1}{2\pi\sqrt{|\Sigma|}} \exp\left[-\frac{1}{2}\vec{\delta}^{\mathsf{T}}\Sigma^{-1}\vec{\delta}\right]$$

We need the joined probability distribution for conversion in two different point in space

$$\Sigma = \begin{pmatrix} \sigma^2(z') & \xi(z', |\vec{r} - \vec{r}'|) \\ \xi(z', |\vec{r} - \vec{r}'|) & \sigma^2(z') \end{pmatrix}$$

→ Covariance matrix

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→ Covariance matrix

We have anisotropies safely smaller than Holder's data

Conclusions

- The interplay between particle physics, astrophysics and cosmology is crucial;
- CMB for example can put strong constraints on dark photon models; for these is of particular importance to treat universe inhomogeneities;
- We propose a simple model to explain longstanding radio excess such as ARCADE-2, and it seems to work!

Thanks for the attention

Backup slides

Statistical Analysis

We construct a Gaussian log-likelihood as

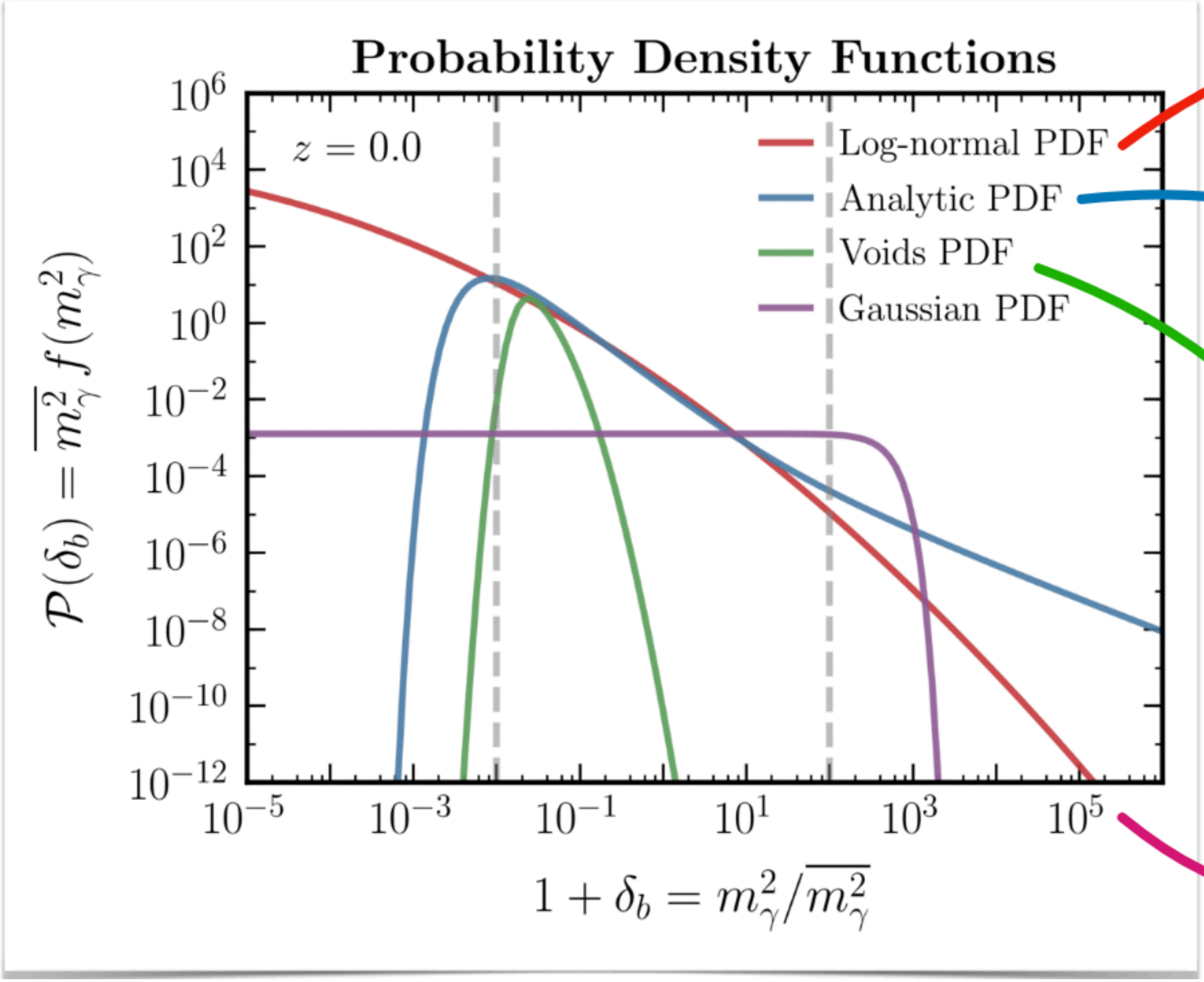
$$\ln \mathcal{L}(d|m_{A'}, \epsilon) = \max_{T_{\rm CMB}} \left[-\frac{1}{2} \Delta\vec{I}^T \, \mathsf{C}_{I_d}^{-1} \, \Delta\vec{I} \right], \qquad \text{(A2)}$$

where $\Delta\vec{I} = \left(\vec{I}\,(m_{A'}, \epsilon; T_{\rm CMB}) - \vec{I}_d\right)$ is the residual between the distorted CMB spectrum $\vec{I}\,(m_{A'}, \epsilon; T_{\rm CMB}) = \{I_{\omega_1}, I_{\omega_2}, \ldots\}$ and the FIRAS data vector $\vec{I}_d$, and C_{I_d} is the data covariance matrix. We treat the CMB temperature as a nuisance parameter and profile over it by maximizing the log-likelihood for $T_{\rm CMB}$ at each $\{m_{A'}, \epsilon\}$ point. We define our test-statistic as

$$\text{TS}(m_{A'}, \epsilon) = 2 \left[\ln \mathcal{L}(d|m_{A'}, \epsilon) - \ln \mathcal{L}(d|m_{A'}, \hat{\epsilon}) \right], \qquad \text{(A3)}$$

where $\hat{\epsilon}$ is the value of ϵ that maximizes the log-likelihood for a given $m_{A'}$, and obtain our limit by finding the value of ϵ at which TS $= -2.71$ corresponding to 95% containment for the one-sided χ^2 distribution.

PDF Functional Form



phenomenological

theoretically motivated PDF from first principles

Ivanov, Kaurov & Sibiryakov 1811.07913

fiducial

from simulations of voids: useful for underdensities

good agreement between fiducial for $10^{-2} \leq 1 + \delta_b \leq 10^2$.