Making Everything Easier!™

Calabi-Yau Metrics

FOR

DUMMIES®

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Preface: Elliptic numerical relativity

- General relativity in more than 4 spacetime dimensions admits a striking variety of *static* and *stationary* solutions with no 4-dimensional counterparts:
  - Exotic black holes (uniform & non-uniform black branes, rings, Randall-Sundrum black holes, . . .)
  - Compactified spacetimes (e.g. on Calabi-Yau manifolds)
  - Black holes in compactified spaces

- Traditionally we rely on
  1. exact solutions (usually with lots of (super) symmetry)
  2. abstract existence theorems (Yau's theorem, . . .)

- Increasingly, we need answers beyond those provided by exact solutions and existence theorems, for
  - string phenomenology
  - applications of holography to nuclear, fluid, & condensed-matter systems
  - understanding black holes in compact spaces
  - study of mirror symmetry
  - . . .
- We therefore need numerical methods to solve the Einstein equation in its elliptic form. (Numerical relativity traditionally focuses on the hyperbolic Einstein equation.)
- The demand for numerical methods is highly elastic!

- In this talk, I will explain two new methods for this problem.
- As usual in geometry, the basic distinction is between
  - real (black holes, non-Kähler compactifications, . . .)
  - complex (CYs, del Pezzos, . . .)
- Let’s start with real case: 0905.1822 [gr-qc], with S. Kitchen & T. Wiseman

- How is the Einstein equation different from any other elliptic PDE?
- Diffeomorphism invariance
- Pure gauge modes lead to numerical instabilities.
- Gauge-fixing (choosing coordinates) a priori is often not optimal; e.g. can lead to coordinate singularities.
- We propose instead gauge-fixing a posteriori: imposing a differential gauge condition which we solve for simultaneously with Einstein equation.

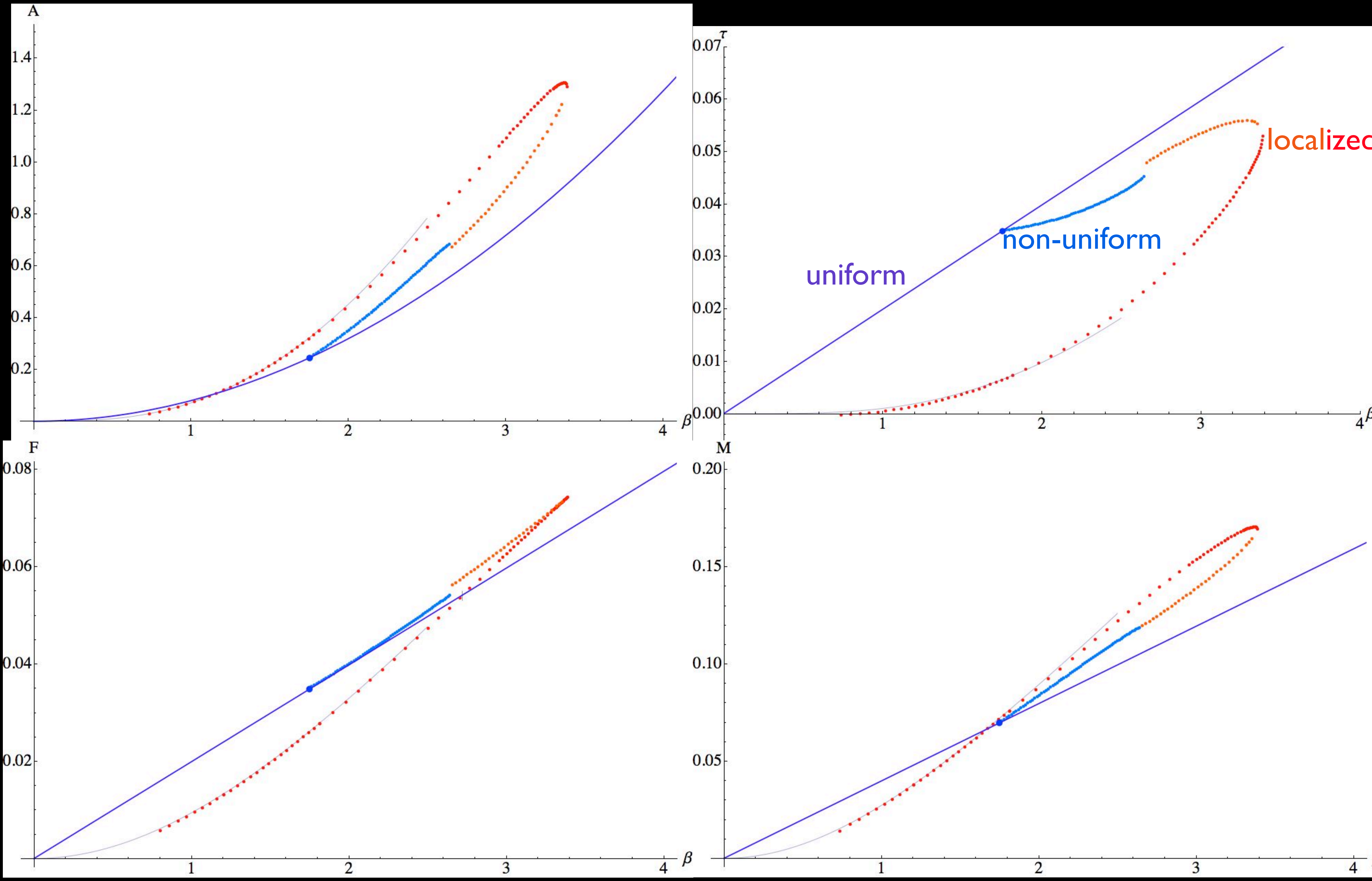
- Following DeTurck's ('83) method for "gauge-fixing" in Ricci flow, fix a background metric $\tilde{g}_{\mu\nu}$, define

$$\xi^\mu \equiv g^{\lambda\nu}\left(\Gamma^\mu_{\lambda\nu} - \tilde{\Gamma}^\mu_{\lambda\nu}\right)$$

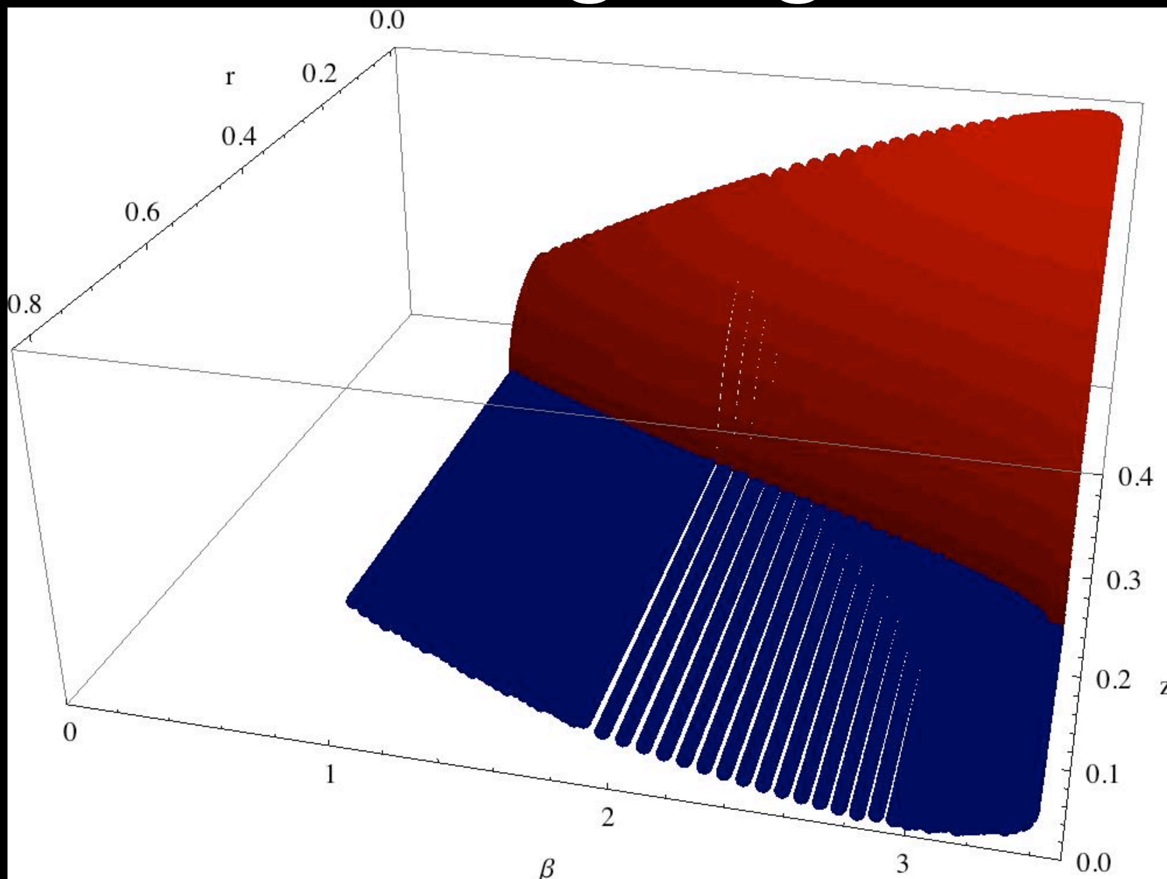
and solve $$R_{\mu\nu} - \nabla_{(\mu}\xi_{\nu)} = 0$$

  - implies $R_{\mu\nu} = 0$ (Perelman '02)
  - effectively imposes the gauge condition $\xi^\mu = 0$: classical equation of motion for sigma model from $(M, g_{\mu\nu})$ into $(M, \tilde{g}_{\mu\nu})$ (harmonic map); generalization of harmonic coordinates
  - is strictly elliptic; rather than removing pure gauge modes, we gave them a kinetic term
  - can be solved by standard methods (e.g. Newton's method)

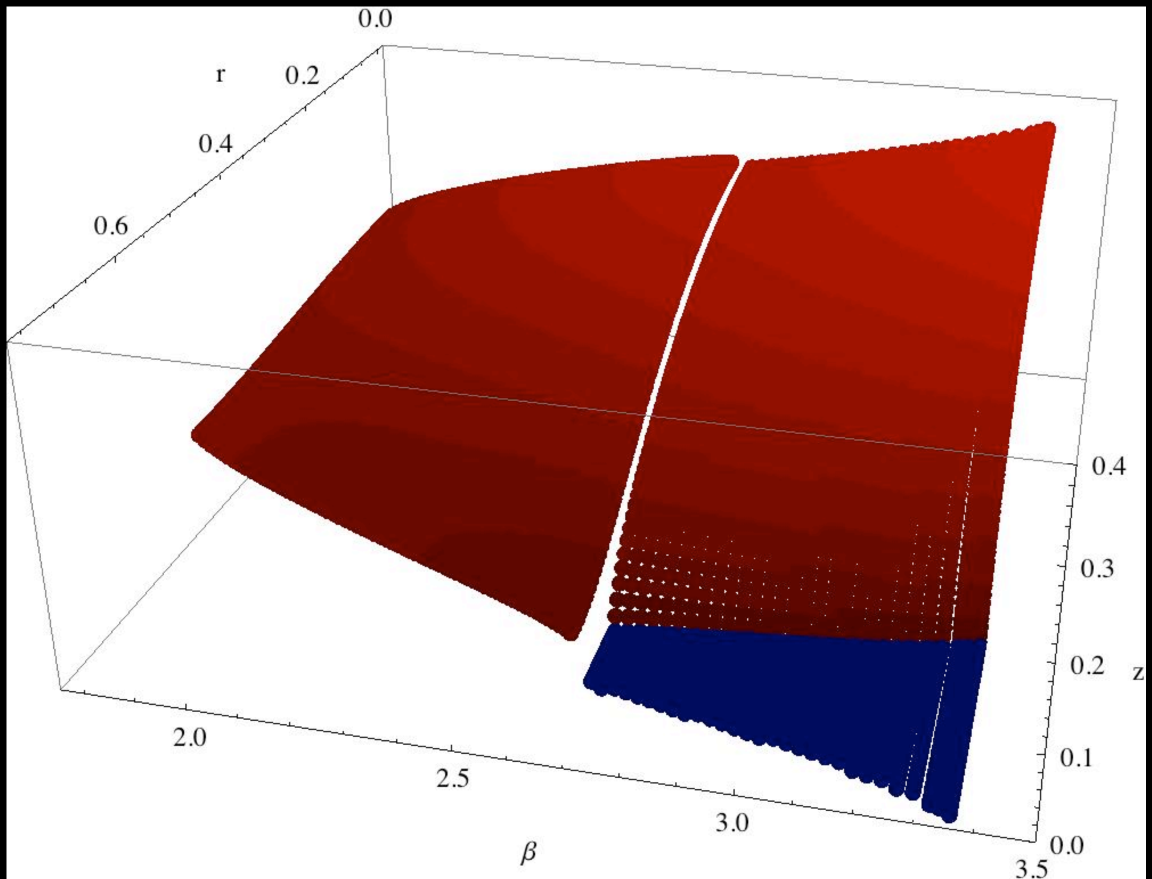
- Using this method, we studied black holes in 5-dimensional Kaluza-Klein theory



- Embedding diagrams for horizon

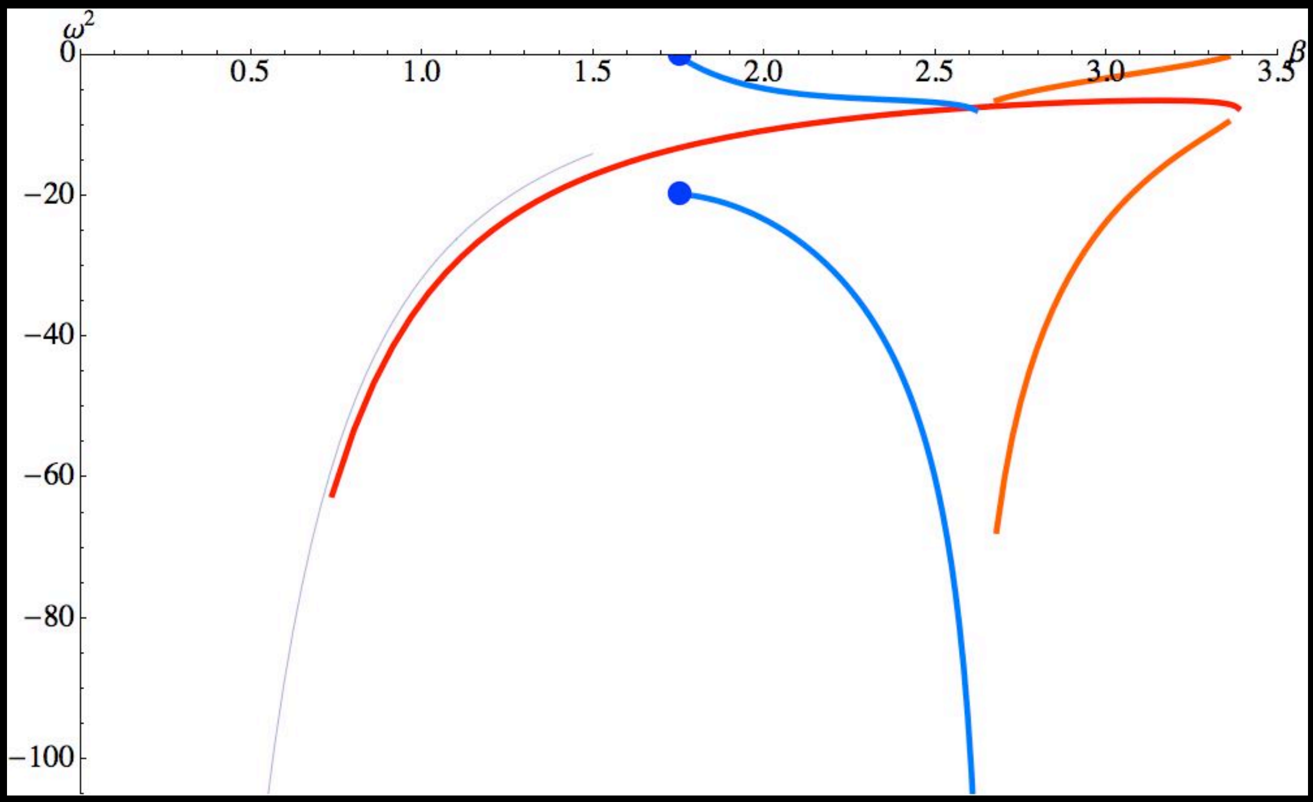


small localized black holes



non-uniform strings & large localized black holes

- Negative Lichnerowicz eigenvalues



- Calabi-Yau manifolds and their Ricci-flat metrics play an important role in complex differential geometry and in string theory.
- We know these metrics exist by Yau's theorem.
- Yau's theorem is not constructive; no (no-flat) RFMs are known explicitly.
- In string theory compactified on a CY, some quantities can be computed (using supersymmetry/algebraic geometry) without knowing RFM explicitly, others cannot.
- If the 4-dimensional theory has $\mathcal{N} = 1$ SUSY, roughly speaking the superpotential can be computed without knowing RFM, but the Kähler potential cannot.

- If we cannot find exact solutions, can we find useful approximations numerically?
- This question has been addressed by a few groups in the last several years:
  - MH & Wiseman ’05
  - Donaldson ’05
  - Douglas, Karp, Lukic, Reinbacher ’06, ‘06
  - Braun, Brelidze, Douglas, Ovrut ’07, ’08
  - Lukic & Keller ’09
  - . . .
- Kähler structure offers many advantages compared to real geometry, especially:

1. Natural gauge fixing (complex coordinates)
2. Simplification of the Einstein equation

- 0908.2635 [hep-th], with A. Nassar: new method that
  - is fast and relatively easy to implement
  - gives significantly better results than some previous methods
  - involves some interesting math.

Let us recall the setup:

- Let X be our CY n-fold, with coordinates $x^i,\ \bar{x}^{\bar{\imath}}$
- The metric is given locally in terms of the Kähler potential:

$$g_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} K\,, \qquad J = i\partial\bar{\partial} K$$

- The associated volume form is $\mu_J = J^n/n!$ (which we will assume is normalized: $\int_X \mu_J = 1$).
- The holomorphic $(n,0)$ form gives rise to a natural volume form $\mu_\Omega = \Omega \wedge \bar{\Omega}$ (which we also take to be normalized: $\int_X \mu_\Omega = 1$).
- Define

$$\eta = \frac{\mu_J}{\mu_\Omega}$$

- Ricci tensor is

$$R_{i\bar{\jmath}} = -\partial_i \partial_{\bar{\jmath}} \ln \eta \,,$$

which vanishes if and only if

$$\eta = 1$$

everywhere on X.

- This is an elliptic PDE of Monge-Ampère type for K.
- Two basic issues in solving this numerically:

1. how to represent K (challenges: X is 4- or 6-dimensional ⇒ storage problem?; complicated topology)
2. how to solve the MA equation (challenge: nonlinear)

- The first numerical method to solve this (MH & Wiseman '05) was based on a lattice representation of K with finite differencing, and a Gauss-Seidel relaxation method to solve the MA equation.
- Straightforward; a bit messy with coordinate patches; limited in accuracy by storage problem.
- Subsequently, Donaldson ('05) introduced a new method based on algebraic metrics.
- The method works for X embedded as an algebraic variety in CP^N, with Kähler class induced from CP^N.
- This is a limitation, but includes many metrics of interest.

- What is an algebraic metric?
- Let z^a be homogeneous coordinates for CP^N.
- Recall that the Fubini-Study metric has Kähler potential in patch a given by

$$K^{\mathrm{FS}}_{(a)} = \ln \frac{\sum_b |z^b|^2}{|z^a|^2}$$

- We want to generalize this by replacing $\sum_b |z^b|^2$ by an arbitrary degree (k,k) homogeneous polynomial p:

$$K^{p}_{(a)} = \ln \frac{p(z,\bar{z})^{1/k}}{|z^a|^2}$$

- The larger k, the more coefficients in p, so the more freedom in choosing the metric.

- This is a *spectral representation*: the functions
$$e^{k(K^p - K^{\rm FS})} = \frac{p(z,\bar z)}{(\sum_b |z^b|^2)^k}$$
are spanned by the spherical harmonics on CP^N up to $l = k$.

- They give a complete basis for functions on CP^N as $k \to \infty$.

- Pulled back to X, they are overcomplete, but it's easy to remove the superfluous ones: those where p is proportional to the defining polynomial of X.

- As with any Fourier expansion, any smooth metric (in the correct class) can be approximated exponentially well in k by algebraic metrics.

- Advantages:
  - elegant
  - no patches
  - based on polynomials, so easy to work with both theoretically and computationally
  - can potentially solve the storage problem: due to exponential convergence, can represent the RFM very accurately with a relatively small number of coefficients (particularly for very smooth CYs, i.e. far from singular points in moduli space).
- Problem: how to solve the MA equation? Given k, how to find the algebraic metric closest to the RFM?
- Nonlinear PDEs become complicated written in momentum space.

- Donaldson proposed approximating the RFM by the *balanced metric*, an algebraic metric that satisfies a certain integral equation.
- This method has been tested and generalized by Douglas et al.
- Unfortunately, as Donaldson pointed out, while the balanced metric goes to the RFM as $k \to \infty$. it does so only like $1/k$, not exponentially.

- A common strategy for numerically solving elliptic PDEs is to recast them as optimization problem:
  1. find an energy functional that is bounded below and minimized (only) on the solution
  2. minimize it over a finite-dimensional function space, using a standard numerical minimizer.
- Example: Rayleigh-Ritz variational method.
- Such functionals do not exist for all PDEs.
- For example, there is no such functional for the Riemannian Einstein equation.
- However, thanks to the magic of Kähler geometry there are many well-behaved energy functionals in the CY case.

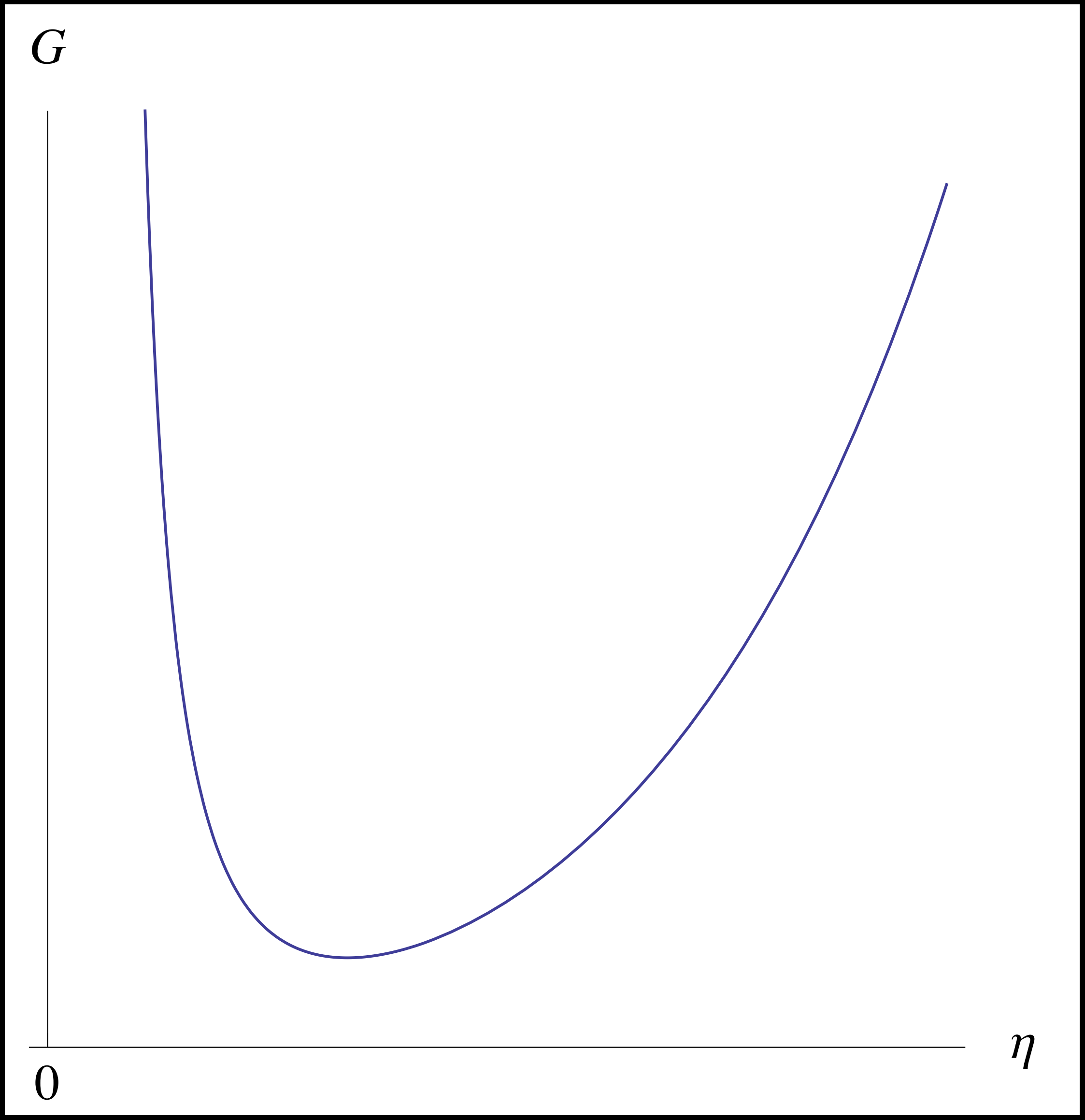
- Two examples:

$$H_1[K] \equiv \int_X \mu_\Omega (\eta - 1)^2$$

$$H_2[K] \equiv \int_X \mu_\Omega \, |\partial \ln \eta|^2 = - \int_X \mu_\Omega \, R$$

- Both are non-negative, zero only on RFM.
- By minimizing H_1 or H_2, we obtain for each k an "optimal" algebraic metric (slightly different for H_1 and H_2). Expect minimum value to decrease exponentially with k.
- Can also be employed to obtain approximate solutions within any convenient family of metrics.

- Aside: We can alter H_1 to obtain a heuristic proof of Yau's theorem.
- Define
$$H_G[K] \equiv \int_X \mu_\Omega \, G(\eta)$$
where $G(\eta)$ is convex and goes to ∞ as $\eta \to 0, \infty$, fast enough so that H_G goes to ∞ (generically) on the boundary of the space of metrics in each Kähler class, i.e. whenever the metric degenerates.



- Since H_G is bounded below, it must attain a minimum somewhere, which must be a RFM.

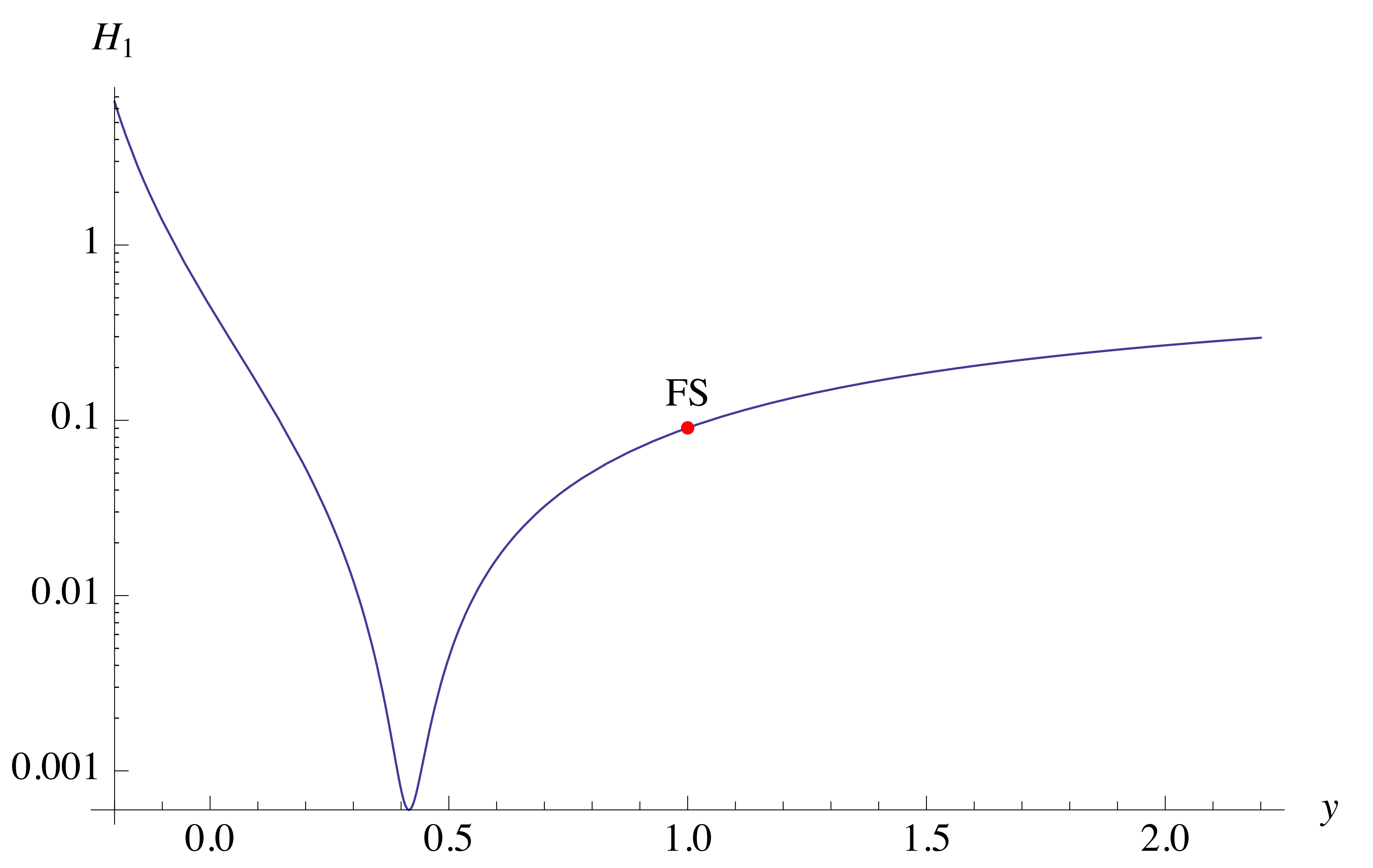
- Example: Let X be the Fermat quartic in CP^3:

$$\sum_{a=1}^{4} (z^a)^4 = 0$$

- The symmetry group of X restricts the polynomials we should consider.
- With $k = 1$, FS is the only symmetrical algebraic metric.
- With $k = 2$, there is a 1-parameter family:

$$p = y \left(\sum_a |z^a|^2 \right)^2 + (1 - y) \sum_a |z^a|^4$$

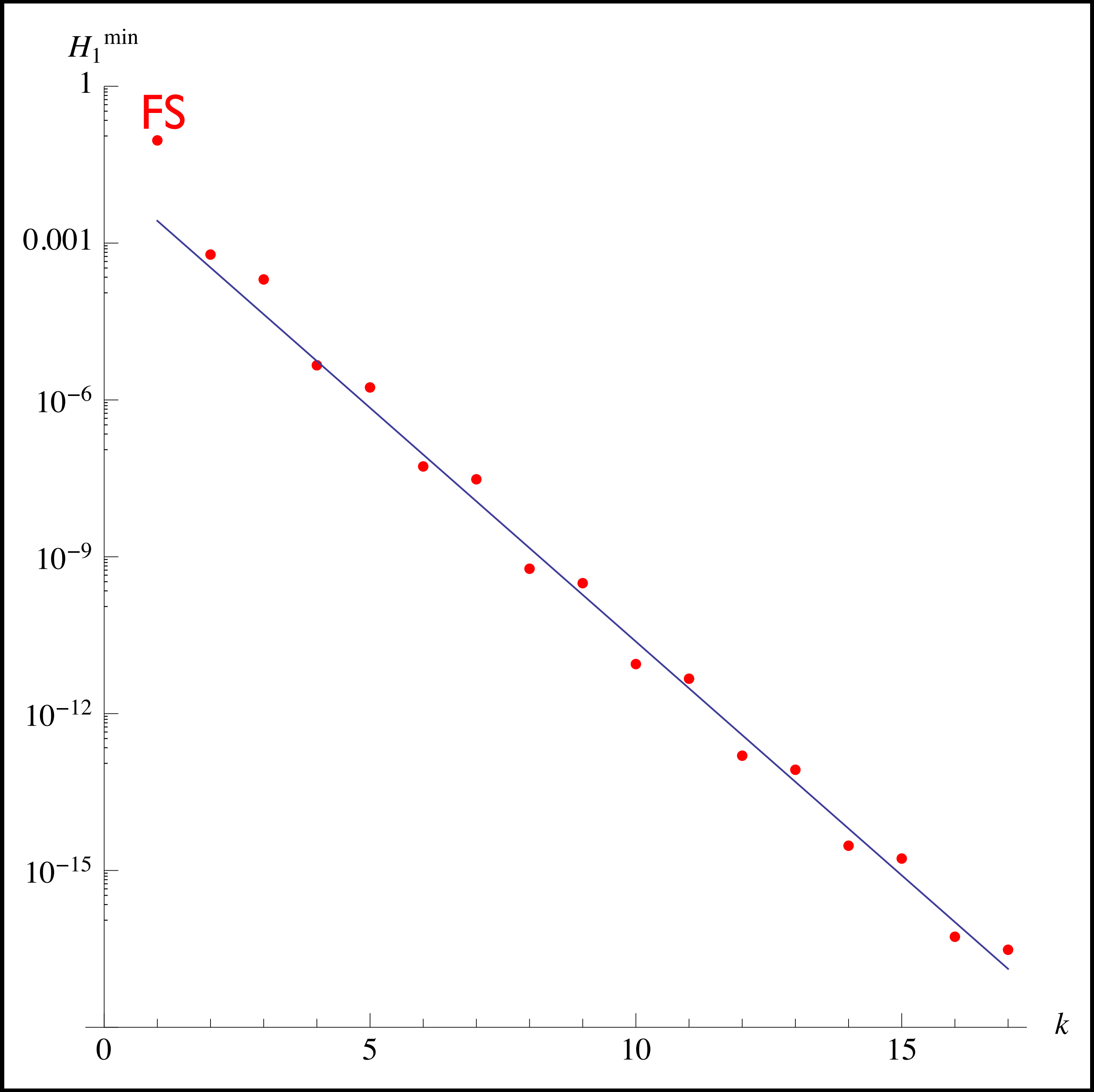
($y = 1$ returns to FS).



- H_1 decreases by a factor of ~ 100 between FS and minimum.

- We did all calculations in *Mathematica* (a few pages of code).
- H_1 is straightforward to calculate (main difficulty is integral over X, done by Monte Carlo).
- Minimization of H_1 done by built-in function *FindMinimum;* takes less than a second for low k, to a few hours for $k \sim 15$.

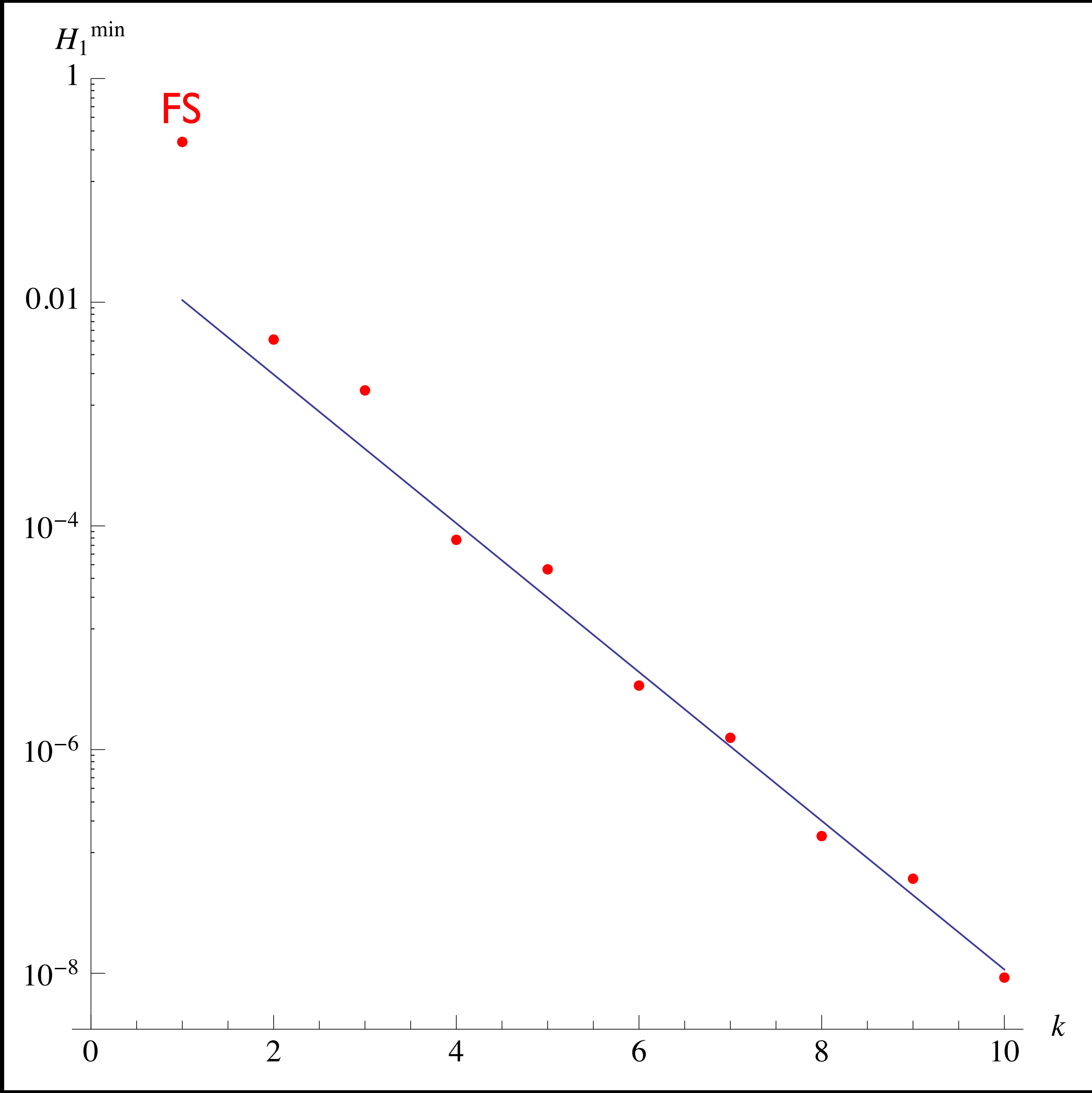
- Minimum value of H_1 vs k (Fermat quartic):



- Beautiful exponential decrease, starting around $k = 4$.
- Fit is $H_1^{\min} \approx 0.02 \times 8^{-k}$ $(k \gtrsim 4)$.

- Quintic: $$\sum_{a=1}^{5}(z^a)^5 - \frac{1}{2}\prod_{a=1}^{5} z^a = 0$$

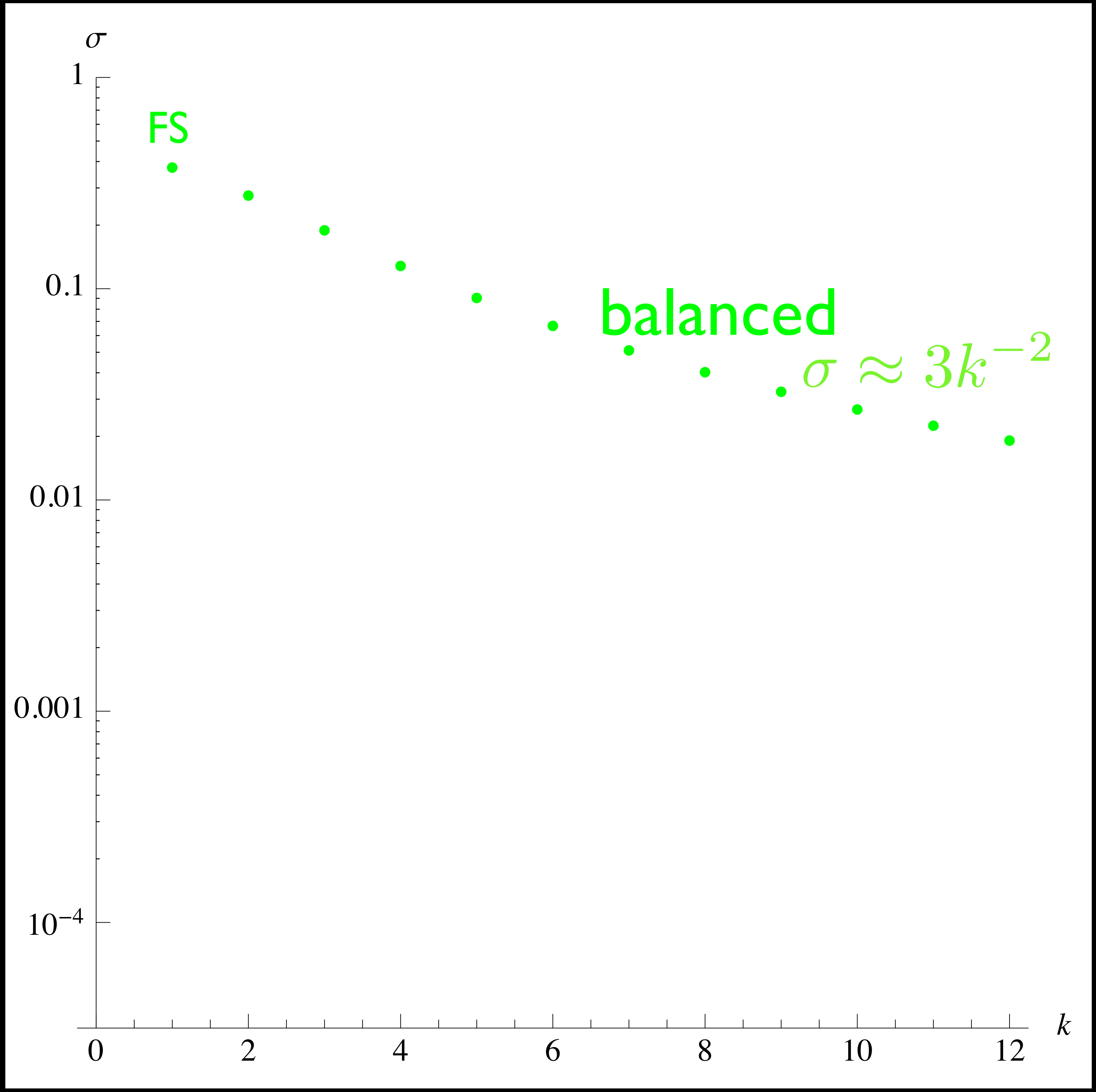
(less symmetric than Fermat)



$$H_1^{\min} \approx 0.05 \times 5^{-k}$$
$$(k \gtrsim 4)$$

- We would like to compare to the balanced metric.
- Douglas, Karp, Lukic, Reinbacher computed the balanced metrics on the quintic for $k = 3$ to 12.
- They estimated the quality of the approximation to the RFM by computing

$$\sigma[K] \equiv \int_X \mu_\Omega |\eta - 1|$$



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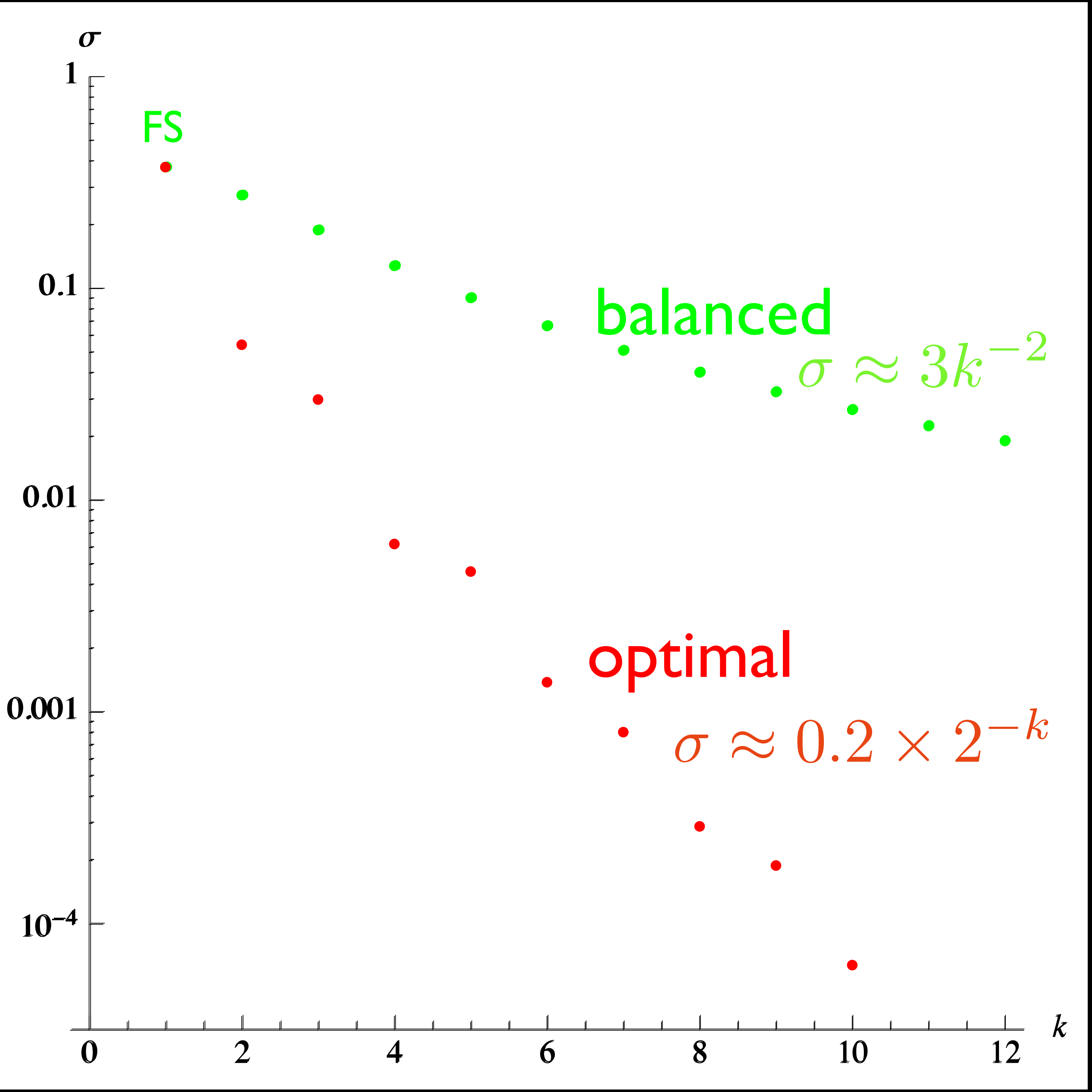
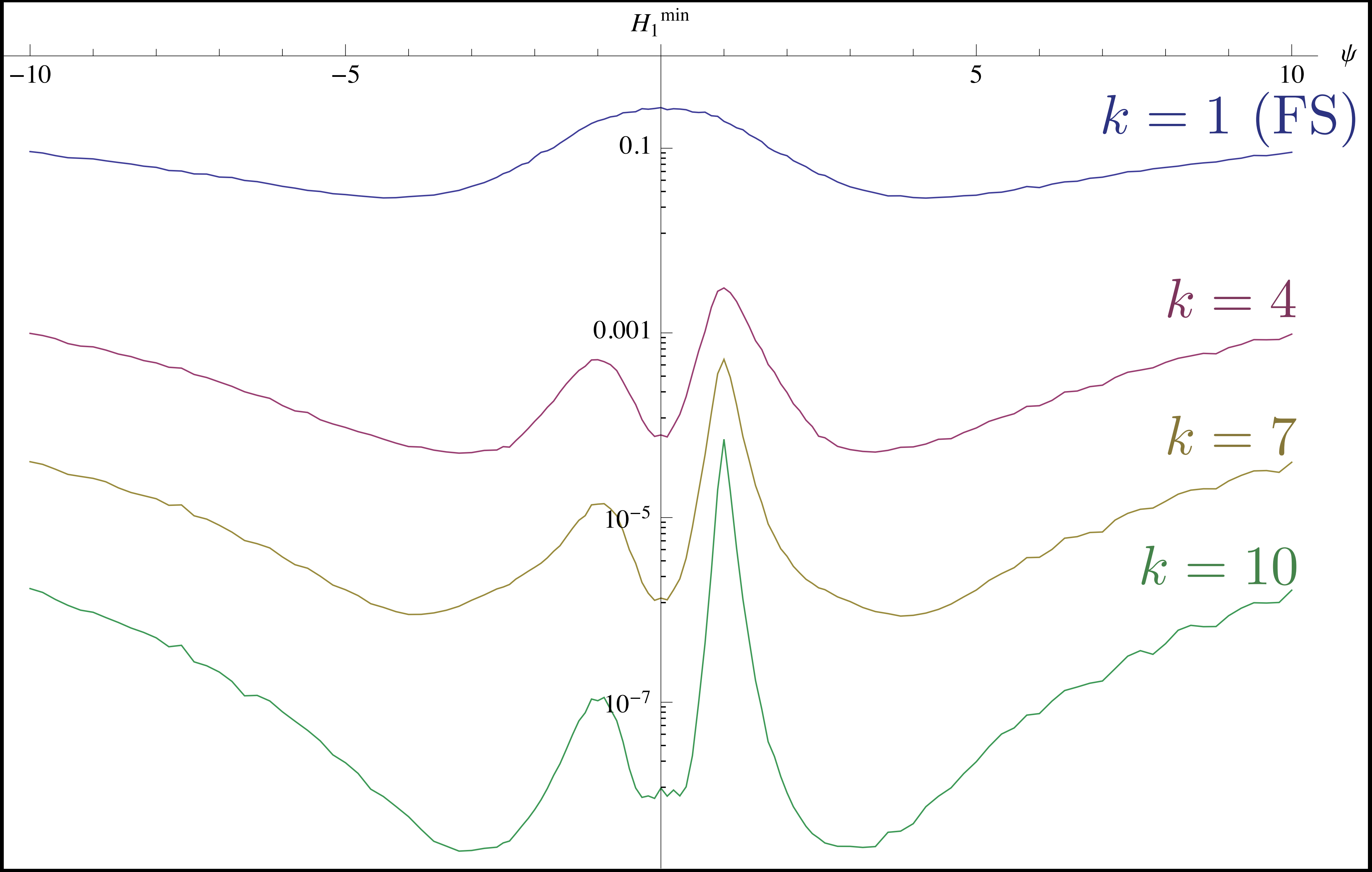


- More generally, $\sum_{a=1}^{5} (z^a)^5 - 5\psi \prod_{a=1}^{5} z^a = 0$



- Future directions:
  - Computing spectra, etc
  - Finite-element method, using simplicial decompositions of the CY
  - α' corrections
  - Matter (branes, fluxes, generalized geometries, $\ldots$)
  - Scanning moduli space
  - $\ldots$