Integrability from four-dimensional Chern-Simons theory and the three-dimensional WZW model

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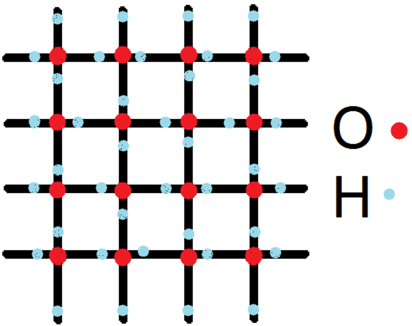
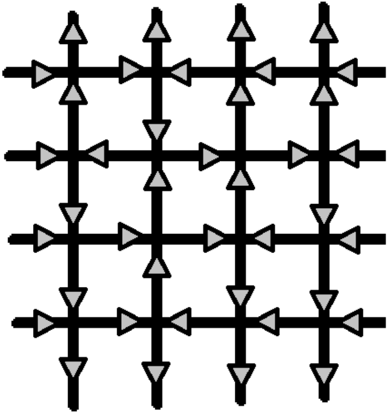
Kavli IPMU Postdoc Colloquium Series

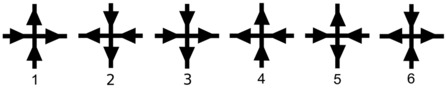
Introduction



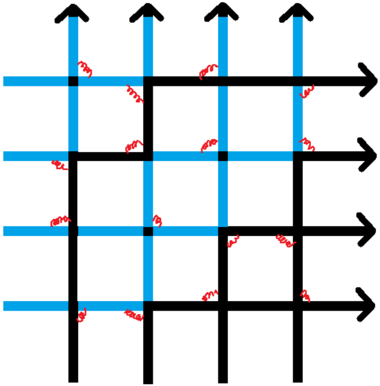
An ice crystal, where each bond contains one hydrogen atom



Such a crystal can be described by the six-vertex model



The six possible configurations at a vertex.



A configuration of Wilson lines in an SU(2) gauge theory, each in the fundamental representation.

Question: Can the statistical mechanics of ice be captured by a quantum gauge theory?

Answer: Yes! **Four-dimensional Chern-Simons theory** is able to do this.

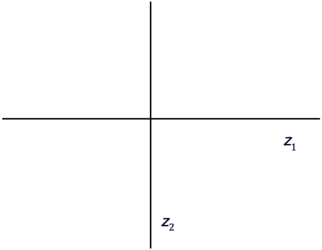
- 4d Chern-Simons theory has the action

$$S = \frac{1}{\hbar} \int_{\Sigma \times C} \omega \wedge \mathrm{Tr}\Big(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\Big), \qquad (1.1)$$

  where $\mathcal{A}$ is a complex-valued gauge field, Σ is a 2-manifold, and C is a Riemann surface endowed with a holomorphic one-form $\omega = \omega(z)dz$.

- Topological along Σ (modulo a framing anomaly), but has holomorphic dependence on C.
- It has a complex gauge group, denoted G.

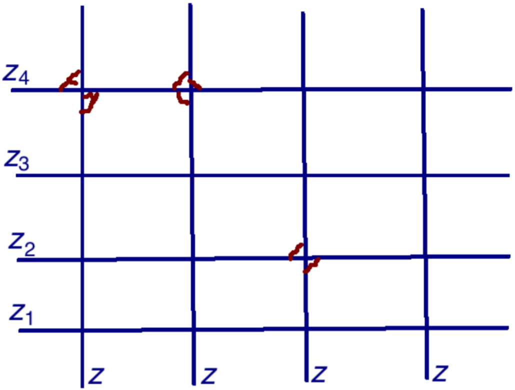
- Studied by Costello, Witten and Yamazaki,* who showed that **correlation functions of Wilson lines** realize solutions of the **Yang-Baxter equation with spectral parameters**.
- E.g., the rational R-matrix for Wilson lines on $\Sigma = \mathbb{R}^2$:



$$= I + \frac{\hbar\, c_{\rho,\rho'}}{z_1 - z_2} + \mathcal{O}(\hbar^2)$$

- Here the Wilson lines at z_1 and z_2 are respectively in representations ρ and ρ', with $c_{\rho,\rho'} = \sum_a T_{a,\rho} \otimes T_{a,\rho'}$.

*. K. Costello, E. Witten, M. Yamazaki, *Gauge Theory and Integrability, I, II*, arXiv:1709.09993, 1802.01579

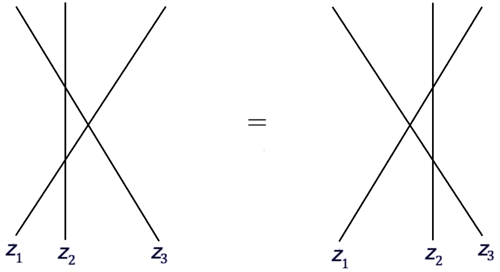


A correlation function of net of Wilson lines thus leads to the partition function of the lattice model.

- The YBE

$$R_{12}(z_1, z_2)R_{13}(z_1, z_3)R_{23}(z_2, z_3) = R_{23}(z_2, z_3)R_{13}(z_1, z_3)R_{12}(z_1, z_2)$$

is also realized due to the topological symmetry along Σ.

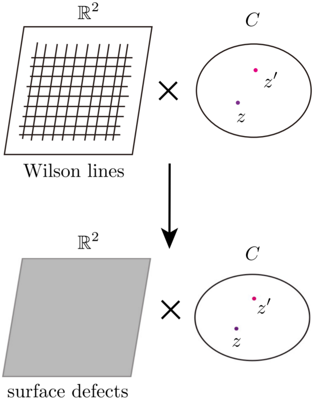


- No singular behaviour arises in moving a Wilson line, as long as z_1, z_2 and z_3 are distinct.

- The action involves only the ratio $\omega/\hbar$ - naively, a zero of ω corresponds to a point at which $\hbar \to \infty$.
- But the theory is only defined perturbatively, so ω cannot have zeros, though it may have poles (ω CAN have zeros if we consider surface operators).
- This restricts Σ to one of the following possibilities:

$$\begin{aligned} C &= \mathbb{C}\,, & \omega &= dz\,, \quad (\text{rational})\,, \\ C &= \mathbb{C}^\times = \mathbb{C}/\mathbb{Z}\,, & \omega &= \frac{dz}{z}\,, \quad (\text{trigonometric})\,, \\ C &= E = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})\,, & \omega &= dz\,, \quad (\text{elliptic})\,. \end{aligned} \tag{1.2}$$

- The three choices of C lead to rational, trigonometric and elliptic integrable lattice models.



Integrable field theories : Realizes the Gross-Nuveau, Thirring, principal chiral, Riemannian symmetric space sigma model (including the $AdS_5 \times S^5$ superstring), and affine Gaudin models.

3d "Chiral" WZW Model from 4d CS Theory

4d Chern-Simons theory defined on $D \times \mathbb{C}$, where D is a disk, is

$$S = \frac{1}{\hbar} \int_{D \times \mathbb{C}} dz \wedge \mathrm{Tr}\Big(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \Big), \qquad (2.1)$$

where $\mathcal{A}$ is the partial connection $\mathcal{A} = \mathcal{A}_r dr + \mathcal{A}_\varphi d\varphi + \mathcal{A}_{\bar{z}} d\bar{z}$. To have EOM free from boundary corrections, and to ensure gauge invariance, we impose $\mathcal{A}_{\bar{z}} = 0|_{\partial D}$.

Using $\mathcal{A}_{\bar{z}} = 0|_{\partial D}$, we find

$$S = \frac{1}{2\pi\hbar}\int dz \wedge dr \wedge d\varphi \wedge d\bar{z}\ \mathrm{Tr}\left(2\mathcal{A}_{\bar{z}}\mathcal{F}_{r\varphi} - \mathcal{A}_r\partial_{\bar{z}}\mathcal{A}_\varphi + \mathcal{A}_\varphi\partial_{\bar{z}}\mathcal{A}_r\right). \tag{2.4}$$

Varying $\mathcal{A}_{\bar{z}}$ gives $\mathcal{F}_{r\varphi} = 0$. Solved by

$$\mathcal{A}_r = -\partial_r g g^{-1}, \quad \mathcal{A}_\varphi = -\partial_\varphi g g^{-1}, \tag{2.5}$$

where $g : D \times \mathbb{C} \to G$.

Then, substituting into S, we obtain the **3d “chiral" WZW model**

$$S(g) = \frac{1}{2\pi\hbar}\int_{S^1\times\mathbb{C}} d\varphi \wedge dz \wedge d\bar{z}\ \mathrm{Tr}(\partial_\varphi g g^{-1}\partial_{\bar{z}} g g^{-1}) + \frac{1}{6\pi\hbar}\int_{D\times\mathbb{C}} dz \wedge \mathrm{Tr}\,(dgg^{-1}\wedge dgg^{-1}\wedge dgg^{-1}). \tag{2.6}$$

This model has a $G \times G$ symmetry under

$$g(\varphi, z, \bar{z}) \to \tilde{\Omega}(\varphi, z) g \Omega(z, \bar{z}). \tag{2.7}$$

$\tilde{\Omega}$ and Ω correspond, respectively, to the conserved currents $J_\varphi = -\frac{1}{\pi\hbar}\partial_\varphi g g^{-1}$ and $J_{\bar{z}} = -\frac{1}{\pi\hbar} g^{-1}\partial_{\bar{z}} g$, that obey $\partial_\varphi J_{\bar{z}} = 0$ and $\partial_{\bar{z}} J_\varphi = 0$.

We can use J_φ to derive a current algebra.

R-matrix from Local Boundary Operators

Consider Wilson lines along D ending on ∂D. These can be expressed in terms of **local boundary operators** since $\mathcal{A}|_D$ is pure gauge.

E.g., for a Wilson line in representation R,

$$\mathcal{P}e^{\int_{t_i}^{t_f}\mathcal{A}} = g_R^{-1}(t_f)\mathcal{P}e^{\int_{t_i}^{t_f}\mathcal{A}'}g_R(t_i) \tag{3.1}$$

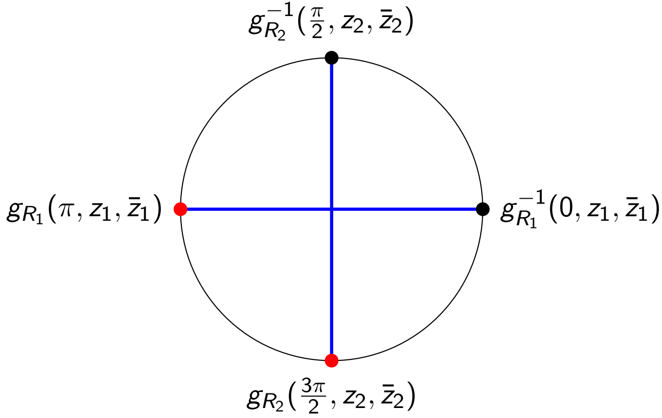
where $\mathcal{A} = g\mathcal{A}'g^{-1} - dgg^{-1}$. Setting $\mathcal{A}' = 0$, we find that

$$\mathcal{P}e^{\int_{t_i}^{t_f}(-dgg^{-1})} = g_R^{-1}(t_f)g_R(t_i). \tag{3.2}$$

We can thus compute correlation functions of Wilson lines via correlators of such boundary operators.

Let us try to retrieve the R-matrix, using

$$
\begin{aligned}
&\langle \mathcal{P} e^{\int_{\pi, z_1, \bar{z}_1}^{0, z_1, \bar{z}_1} \mathcal{A}_{R_1}} \otimes \mathcal{P} e^{\int_{3\pi/2, z_2, \bar{z}_2}^{\pi/2, z_2, \bar{z}_2} \mathcal{A}_{R_2}} \rangle \\
=&\langle g_{R_1}^{-1}(0, z_1, \bar{z}_1) g_{R_1}(\pi, z_1, \bar{z}_1) \otimes g_{R_2}^{-1}(\pi/2, z_2, \bar{z}_2) g_{R_2}(3\pi/2, z_2, \bar{z}_2) \rangle .
\end{aligned}
$$



Perpendicular Wilson lines on D.

Bulk R-matrix computation (to order $\hbar$) used perturbation theory around $\mathcal{A} = 0$ and free field propagators.

So we consider perturbation theory around $g = \mathbb{1}$:

$$g = e^{\phi_a T^a} = \mathbb{1} + \phi_a T^a + \ldots$$

whereby the 3d WZW kinetic term is

$$\begin{aligned} &\frac{1}{2\pi\hbar}\int_{S^1\times\Sigma} d\varphi \wedge dz \wedge d\bar{z}\,\mathrm{Tr}(\partial_\varphi g g^{-1}\partial_{\bar{z}} g g^{-1}) \\ = &-\frac{1}{2\pi\hbar}\int_{S^1\times\Sigma} d\varphi \wedge dz \wedge d\bar{z} \quad \phi^a \partial_\varphi \partial_{\bar{z}} \phi_a + \ldots . \end{aligned} \tag{3.3}$$

The propagator which obeys $\partial_\varphi \partial_{\bar{z}} \Delta^{ab}(x) = \delta^{ab}\delta(x)$ is given explicitly by

$$\Delta^{ab}(x) = \delta^{ab} \frac{1}{2\pi i} \frac{1}{z} \widetilde{\Delta}_\varphi. \tag{3.6}$$

where,

$$\widetilde{\Delta}_\varphi = \frac{1}{2\pi} \left(\sum_{k=1}^{\infty} \frac{e^{ik\varphi}}{ik} + \varphi + \sum_{k=-\infty}^{-1} \frac{e^{ik\varphi}}{ik} \right), \tag{3.7}$$

defined with a branch cut. The two point function for ϕ is

$$\langle \phi^a(x) \phi^b(y) \rangle = -\pi i \hbar \Delta^{ab}(x-y). \tag{3.8}$$

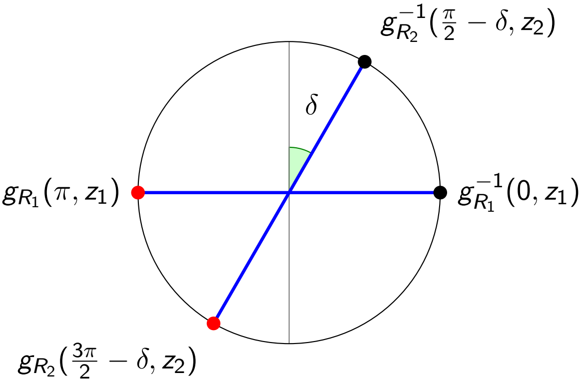
Using the 2 pt. function for ϕ we have

$$\begin{aligned}&\langle g_{R_1}^{-1}(0,z_1,\bar{z}_1)g_{R_1}(\pi,z_1,\bar{z}_1)\otimes g_{R_2}^{-1}(\pi/2,z_2,\bar{z}_2)g_{R_2}(3\pi/2,z_2,\bar{z}_2)\rangle\\ =&\mathbb{1}+\frac{\hbar}{z_1-z_2}(\widetilde{\Delta}_{\frac{\pi}{2}}-\widetilde{\Delta}_{-\frac{\pi}{2}})T^a_{R_1}\otimes T_{aR_2}+\mathcal{O}(\hbar^2)\\ =&\mathbb{1}+\frac{\hbar}{z_1-z_2}T^a_{R_1}\otimes T_{aR_2}+\mathcal{O}(\hbar^2),\end{aligned}$$

via

$$\widetilde{\Delta}_{\frac{\pi}{2}}=\frac{1}{2\pi}\frac{\pi}{2}+\frac{1}{\pi}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\dots\right)=\frac{1}{2}, \qquad (3.10)$$

and $\widetilde{\Delta}_{\frac{\pi}{2}}=-\frac{1}{2}$. We find **precise agreement** with the computation of Costello, Witten and Yamazaki.



Non-perpendicular Wilson lines on D.

Here, the four-point function is

$$\begin{aligned}&\langle g_{R_1}^{-1}(0,z_1)g_{R_1}(\pi,z_1)\otimes g_{R_2}^{-1}(\pi/2-\delta,z_2)g_{R_2}(3\pi/2-\delta,z_2)\rangle\\ =&\mathbb{1}+\frac{\hbar}{z_1-z_2}(\widetilde{\Delta}_{\frac{\pi}{2}+\delta}-\widetilde{\Delta}_{-\frac{\pi}{2}+\delta})T^a_{R_1}\otimes T_{R_2 a}+O(\hbar^2),\end{aligned}$$

where

$$\widetilde{\Delta}_{\frac{\pi}{2}+\delta}=\frac{\frac{\pi}{2}+\delta}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin(k\frac{\pi}{2})\cos(k\delta)+\cos(k\frac{\pi}{2})\sin(k\delta)}{k},\quad (3.11)$$

and

$$\widetilde{\Delta}_{-\frac{\pi}{2}+\delta}=\frac{-\frac{\pi}{2}+\delta}{2\pi}-\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin(k\frac{\pi}{2})\cos(k\delta)-\cos(k\frac{\pi}{2})\sin(k\delta)}{k}.\quad (3.12)$$

Single-valuedness of the propagators requires that $-\frac{\pi}{2} < \delta < \frac{\pi}{2}$, implying

$$\sum_{k=1}^{\infty} \frac{\sin(\frac{k\pi}{2})\cos(k\delta)}{k} = \frac{\pi}{4}, \tag{3.13}$$

and

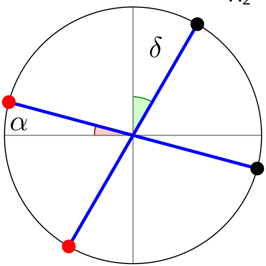
$$\sum_{k=1}^{\infty} \frac{\cos(\frac{k\pi}{2})\sin(k\delta)}{k} = -\frac{\delta}{2}, \tag{3.14}$$

whereby

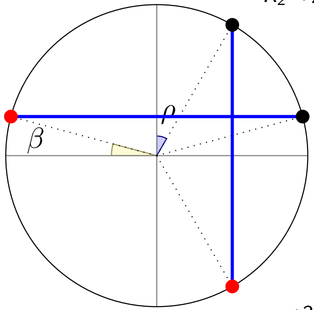
$$\begin{aligned} \widetilde{\Delta}_{\frac{\pi}{2}+\delta} &= \frac{1}{2} \\ \widetilde{\Delta}_{-\frac{\pi}{2}+\delta} &= -\frac{1}{2}. \end{aligned} \tag{3.15}$$

Once again, we have precise agreement with CWY.

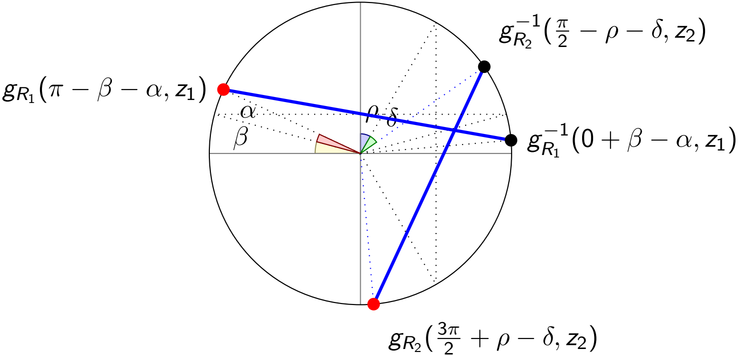
This can be generalized further:



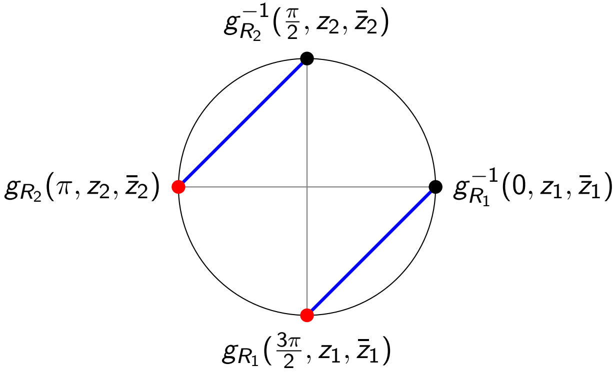
$$\begin{aligned} &\mathbb{1} + \frac{\hbar}{z_1 - z_2}(\widetilde{\Delta}_{\frac{\pi}{2}-\alpha+\delta} - \widetilde{\Delta}_{-\frac{\pi}{2}-\alpha+\delta}) T^a_{R_1} \otimes T_{R_2 a} + O(\hbar^2) \\ =&\mathbb{1} + \frac{\hbar}{z_1 - z_2} T^a_{R_1} \otimes T_{R_2 a} + O(\hbar^2), \end{aligned} \tag{3.16}$$



$$
\begin{aligned}
&\mathbb{1} + \frac{\hbar}{z_1 - z_2} \frac{1}{2} (\widetilde{\Delta}_{\frac{\pi}{2} + \beta - \rho} - \widetilde{\Delta}_{-\frac{\pi}{2} + \beta + \rho} + \widetilde{\Delta}_{\frac{\pi}{2} - \beta + \rho} - \widetilde{\Delta}_{-\frac{\pi}{2} - \beta - \rho}) T^a_{R_1} \otimes T_{R_2 a} \\
&+ O(\hbar^2) \\
=&\mathbb{1} + \frac{\hbar}{z_1 - z_2} T^a_{R_1} \otimes T_{R_2 a} + O(\hbar^2).
\end{aligned}
$$



$$
\begin{aligned}
&\mathbb{1}+\frac{\hbar}{z_1-z_2}\frac{1}{2}(\widetilde{\Delta}_{\frac{\pi}{2}+\beta-\rho-\alpha+\delta}-\widetilde{\Delta}_{-\frac{\pi}{2}+\beta+\rho-\alpha+\delta} \\
&+\widetilde{\Delta}_{\frac{\pi}{2}-\beta+\rho-\alpha+\delta}-\widetilde{\Delta}_{-\frac{\pi}{2}-\beta-\rho-\alpha+\delta})T^a_{R_1}\otimes T_{R_2 a}+O(\hbar^2) \qquad (3.18) \\
=&\mathbb{1}+\frac{\hbar}{z_1-z_2}T^a_{R_1}\otimes T_{R_2 a}+O(\hbar^2).
\end{aligned}
$$



The OPEs of parallel Wilson lines in 4d CS do not have the same singular behaviour. This is reflected in the boundary dual:

$$\langle g_{R_1}^{-1}(0, z_1, \bar{z}_1) g_{R_1}(3\pi/2, z_1, \bar{z}_1) \otimes g_{R_2}^{-1}(\pi/2, z_2, \bar{z}_2) g_{R_2}(\pi, z_2, \bar{z}_2)\rangle = \mathbb{1} + O(\hbar^2).$$

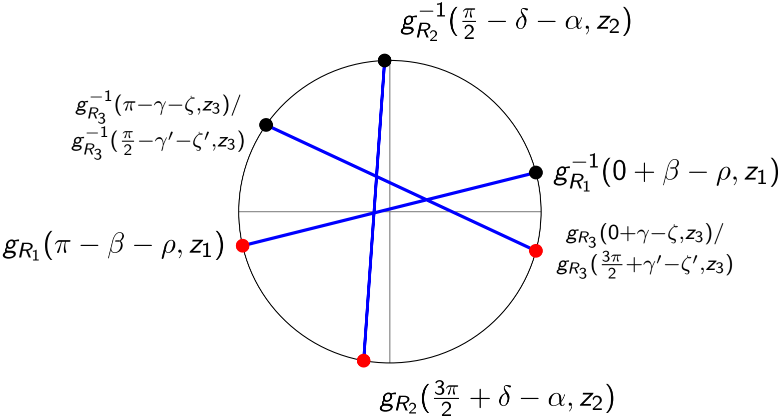
The computation for crossed Wilson lines can be extended to higher order in $\hbar$:

$$\begin{aligned}&\langle g_{R_1}^{-1}(0,z_1)g_{R_1}(\pi,z_1)\otimes g_{R_2}^{-1}(\pi/2,z_2)g_{R_2}(3\pi/2,z_2)\rangle\\ =&\mathbb{1}+\frac{\hbar}{z_1-z_2}T^a_{R_1}\otimes T_{R_2a}\\ &+\frac{\hbar^2}{4(z_1-z_2)^2}\left(T^a_{R_1}T^b_{R_1}\otimes T_{R_2a}T_{R_2b}+T^a_{R_1}T^b_{R_1}\otimes T_{R_2b}T_{R_2a}\right)+\mathcal{O}(\hbar^3)\end{aligned} \quad (3.20)$$

This holds for the boundary duals of arbitarily crossed Wilson lines, but only modulo the framing anomaly.

Finally we consider three crossed Wilson lines, which corresponds to the boundary correlator

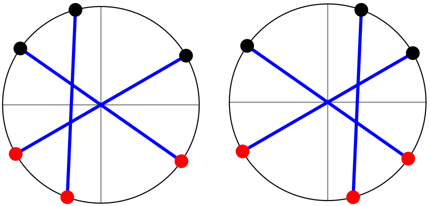
$$\langle g_{R_1}^{-1}(0+\beta-\rho)g_{R_1}(\pi-\beta-\rho)\otimes g_{R_2}^{-1}(\frac{\pi}{2}-\delta-\alpha)g_{R_2}(\frac{3\pi}{2}+\delta-\alpha) \\ \otimes g_{R_3}^{-1}(\pi-\gamma-\zeta)g_{R_3}(0+\gamma-\zeta)\rangle. \tag{3.21}$$


$g_{R_2}^{-1}(\frac{\pi}{2} - \delta - \alpha, z_2)$
$g_{R_3}^{-1}(\pi - \gamma - \zeta, z_3)/$
$g_{R_3}^{-1}(\frac{\pi}{2} - \gamma' - \zeta', z_3)$
$g_{R_1}^{-1}(0 + \beta - \rho, z_1)$
$g_{R_1}(\pi - \beta - \rho, z_1)$
$g_{R_3}(0 + \gamma - \zeta, z_3)/$
$g_{R_3}(\frac{3\pi}{2} + \gamma' - \zeta', z_3)$
$g_{R_2}(\frac{3\pi}{2} + \delta - \alpha, z_2)$

We find

$$
\begin{aligned}
&\mathbb{1} + \frac{\hbar}{z_1 - z_2} T^a_{R_1} \otimes T_{R_2 a} \otimes \mathbb{1} + \frac{\hbar}{z_1 - z_3} T^a_{R_1} \otimes \mathbb{1} \otimes T_{R_3 a} + \frac{\hbar}{z_2 - z_3} \mathbb{1} \otimes T^a_{R_2} \otimes T_{R_3 a} \\
&+ \frac{\hbar^2}{4(z_1 - z_2)^2} \left(T^a_{R_1} T^b_{R_1} \otimes T_{R_2 a} T_{R_2 b} \otimes \mathbb{1} + T^a_{R_1} T^b_{R_1} \otimes T_{R_2 b} T_{R_2 a} \otimes \mathbb{1} \right) \\
&+ \frac{\hbar^2}{4(z_1 - z_3)^2} \left(T^a_{R_1} T^b_{R_1} \otimes \mathbb{1} \otimes T_{R_3 a} T_{R_3 b} + T^a_{R_1} T^b_{R_1} \otimes \mathbb{1} \otimes T_{R_3 b} T_{R_3 a} \right) \\
&+ \frac{\hbar^2}{4(z_2 - z_3)^2} \left(\mathbb{1} \otimes T^a_{R_2} T^b_{R_2} \otimes T_{R_3 a} T_{R_3 b} + \mathbb{1} \otimes T^a_{R_1} T^b_{R_1} \otimes T_{R_2 b} T_{R_2 a} \right) \\
&+ \frac{\hbar^2}{2(z_1 - z_2)(z_1 - z_3)} \left(T^a_{R_1} T^b_{R_1} \otimes T_{R_2 a} \otimes T_{R_3 b} + T^a_{R_1} T^b_{R_1} \otimes T_{R_2 b} \otimes T_{R_3 a} \right) \\
&+ \frac{\hbar^2}{2(z_1 - z_2)(z_2 - z_3)} \left(T^a_{R_1} \otimes T_{R_2 a} T_{R_2 b} \otimes T^b_{R_3} + T^a_{R_1} \otimes T^b_{R_2} T_{R_2 a} \otimes T_{R_3 b} \right) \\
&+ \frac{\hbar^2}{2(z_1 - z_3)(z_2 - z_3)} \left(T^a_{R_1} \otimes T^b_{R_2} \otimes T_{R_3 a} T_{R_3 b} + T^a_{R_1} \otimes T^b_{R_2} \otimes T_{R_3 b} T_{R_3 a} \right) \\
&+ O(\hbar^3).
\end{aligned}
$$

In fact we get the same answer for both of the following configurations



We find agreement with

$$\widetilde{R}_{12}\widetilde{R}_{13}\widetilde{R}_{23} = \widetilde{R}_{23}\widetilde{R}_{13}\widetilde{R}_{12}, \tag{3.23}$$

where

$$\begin{aligned}\widetilde{R}_{ij} =& \mathbb{1} + \frac{\hbar}{z_i - z_j} T^a_{R_i} \otimes T_{R_j a} \otimes \mathbb{1} \\ &+ \frac{\hbar^2}{4(z_i - z_j)^2}\left(T^a_{R_i} T^b_{R_i} \otimes T_{R_j a} T_{R_j b} \otimes \mathbb{1} + T^a_{R_i} T^b_{R_i} \otimes T_{R_j b} T_{R_j a} \otimes \mathbb{1}\right) + \mathcal{O}(\hbar^3),\end{aligned} \tag{3.24}$$

upon using the identity

$$\frac{[T^a_{R_1}, T^b_{R_1}] \otimes T_{R_2 a} \otimes T_{R_3 b}}{(z_1 - z_2)(z_1 - z_3)} + \frac{T^a_{R_1} \otimes [T_{R_2 a}, T^b_{R_2}] \otimes T_{R_3 b}}{(z_1 - z_2)(z_2 - z_3)} + \frac{T^a_{R_1} \otimes T^b_{R_2} \otimes [T_{R_3 a}, T_{R_3 b}]}{(z_1 - z_3)(z_2 - z_3)} = 0. \tag{3.25}$$

Thus, the 6 pt. function is **in agreement with the bulk correlation function of three Wilson lines** up to order $\hbar^2$.

Current Algebra

To compute Poisson brackets of $J_\varphi = -\frac{1}{\pi\hbar}\partial_\varphi g g^{-1}$, we shall first take $\bar{z}$ to be the time direction.

We compute the Poisson brackets $[X, Y]_{PB}$, and canonically quantize by making the replacement

$$[X, Y]_{PB} \to -i\tilde{\hbar}[X, Y] + \mathcal{O}(\tilde{\hbar}^2)$$

.

In this manner, we arrive at the **current algebra** (setting $\tilde{\hbar}=1$)

$$\begin{aligned}\left[\mathrm{Tr} AJ_{\varphi}(\varphi, z), \mathrm{Tr} BJ_{\varphi}(\varphi', z')\right] =& i\delta(\varphi-\varphi')\delta(z-z')\mathrm{Tr}[A,B]J_{\varphi}(\varphi,z) \\ & - i\frac{1}{\pi\hbar}\delta'(\varphi-\varphi')\delta(z-z')\mathrm{Tr}AB \\ & + q.c.,\end{aligned}$$

where $A, B \in \mathfrak{g}$.

Now let $z = \epsilon t + i\theta$, and compactify the θ direction to be valued in $[0, 2\pi]$, and take $\epsilon \to 0$. Expanding currents in Fourier modes along $S^1 = \partial D$ and the θ direction we find

$$\begin{aligned}\left[\mathrm{Tr} A J_\varphi^{n,\tilde{n}}, \mathrm{Tr} B J_\varphi^{m,\tilde{m}}\right] =& i \mathrm{Tr}[A, B] J_\varphi^{n+m,\tilde{n}+\tilde{m}} \\ &+ \frac{4\pi}{\hbar} n \delta_{m+n,0} \delta_{\tilde{m}+\tilde{n},0} \mathrm{Tr} AB \\ &+ q.c.\end{aligned} \tag{4.2}$$

This is a two-toroidal Lie algebra. Hence the current algebra of the 3d "chiral" WZW model is an **"analytically-continued" toroidal Lie algebra**.

Conclusion and Future Directions

- We have shown that a 3d WZW model dual to 4d CS theory exists, that admits a novel toroidal Lie algebra.
- 3d WZW model can also be obtained via methods of Costello and Yamazaki,†

†. K. Costello, M. Yamazaki, *Gauge Theory and Integrability, III*, arXiv:1908.02289

- $$\begin{aligned} S =& \frac{i}{12\pi}\int_{\Sigma\times\mathbb{C}P^1} \omega\wedge\langle\widehat{g}^{-1}dg, \widehat{g}^{-1}d\widehat{g}\wedge\widehat{g}^{-1}d\widehat{g}\rangle \\ &+ \frac{i}{4\pi}\int_{\Sigma\times\mathbb{C}P^1} d\omega\wedge\langle\widehat{g}^{-1}d\widehat{g}, \mathcal{L}\rangle - \frac{i}{4\pi}\int_{\partial\Sigma\times\mathbb{C}P^1} \omega\wedge\langle\widehat{g}^{-1}d\widehat{g}, \mathcal{L}\rangle \end{aligned}$$

  can be obtained from 4d CS via $A = -d\widehat{g}\widehat{g}^{-1} + \widehat{g}\mathcal{L}\widehat{g}^{-1}$, where $\mathcal{L}$ is interpreted as a Lax connection.
- To obtain the 3d WZW model, set $\omega = dz$, and $\mathcal{L} = -\partial_\varphi\widetilde{g}\widetilde{g}^{-1}d\varphi$ for a map $\widetilde{g}\to\partial\Sigma\times\mathbb{C}P^1$, where $\mathcal{L}_\varphi$ obeys $\partial_r\mathcal{L}_\varphi = 0$ and $\partial_{\bar{z}}\mathcal{L}_\varphi = 0$ on-shell.

- Therefore the 3d WZW model can easily be generalized via more general choices of ω, and **ought to be related to 2d integrable sigma models with boundary actions.**
- Moreover, the 3d WZW model can be obtained as an integrable field theory via reduction from 5d Chern-Simons theory.[‡]
- With Masahito Yamazaki, I am trying to understand the discretization of 2d and 3d integrable field theories, in the spirit of the quantum inverse scattering method, from the perspective of their parent Chern-Simons theories
- In particular we have found evidence that the Yangian emerges as a symmetry if one discretizes the circle direction of the 3d WZW model.

‡. R. Bittleston, D. Skinner, *Twistors, the ASD Yang-Mills equations, and 4d Chern-Simons theory*