Beyond Perturbation Theory in Inflation

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Outline: Beyond PT in Inflation

Motivations

- Inflation: $ds^2 = -dt^2 + a^2(t)d\boldsymbol{x}^2$ with $\ddot{a} > 0$

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R + X - V(\phi) + \dots\right]$$

$X = -(\partial\phi)^2/2$. The background eq. for $\phi_0(t)$ is

$$\ddot{\phi}_0 + 3H\dot{\phi}_0 + V'(\phi_0) + \dots = 0$$

- de-Sitter space: Flat slicing

$$ds^2 = \frac{1}{H^2\eta^2}(-d\eta^2 + d\boldsymbol{x}^2)$$

Motivations

- Quantum Fluctuation ζ, ζ-gauge

$$\delta\phi = 0\ , \quad h_{ij} = a^2(t)\left[e^{2\zeta}\delta_{ij} + \gamma_{ij}\right]$$

- Free action of ζ

$$S = \int d\eta d^3x \frac{1}{2\eta^2 P_\zeta}\left[\zeta'^2 - (\partial_i\zeta)^2\right]$$

where $P_\zeta \equiv H^2/(2\epsilon M_{\rm Pl}^2)$

- Quantization as usual: $\zeta_{\boldsymbol{k}}(\eta) \sim \zeta^{\rm cl}_{\boldsymbol{k}}(\eta)a^\dagger_{\boldsymbol{k}} + \zeta^{\rm cl}_{\boldsymbol{k}}(\eta)^* a_{-\boldsymbol{k}}$

$\Rightarrow$ Scale invariant power spectrum

$$\langle\zeta_{\boldsymbol{k}}\zeta_{\boldsymbol{k}'}\rangle' = \frac{P_\zeta}{k^3}\ , \qquad P_\zeta \ \sim 10^{-8}$$

Motivations

- Interacting Hamiltonian: In-In formalism (weakly coupling limit)

$$\langle Q(\eta)\rangle = \langle 0|\,\bar{T} e^{i\int_{-\infty(1-i\epsilon)}^{\eta} H^I_{int}(\eta')d\eta'}\, Q^I(\eta)\, T e^{-i\int_{-\infty(1+i\epsilon)}^{\eta} H^I_{int}(\eta'')d\eta''}\,|0\rangle$$

$i\epsilon$-prescription $\Rightarrow$ Bunch-Davies vacuum $|0\rangle$

Motivations

- Interacting Hamiltonian: In-In formalism (weakly coupling limit)

$$\langle Q(\eta)\rangle = \langle 0|\,\bar{T} e^{i\int_{-\infty(1-i\epsilon)}^{\eta} H_{int}^{I}(\eta')d\eta'}\, Q^{I}(\eta)\, T e^{-i\int_{-\infty(1+i\epsilon)}^{\eta} H_{int}^{I}(\eta'')d\eta''}\,|0\rangle$$

$i\epsilon$-prescription $\Rightarrow$ Bunch-Davies vacuum $|0\rangle$

- Example: $\mathcal{L} = \frac{1}{2\eta^2 P_\zeta}\left[\zeta'^2 - (\partial_i\zeta)^2\right] + \frac{\lambda}{4! P_\zeta^2}\zeta'^4,$

$$\frac{\langle\zeta\zeta\zeta\rangle}{P_\zeta^{3/2}} \sim f_{NL} P_\zeta^{1/2} \ll 1\,, \qquad \frac{\langle\zeta\zeta\zeta\zeta\rangle}{P_\zeta^{2}} \sim g_{NL} P_\zeta \sim \lambda \ll 1$$

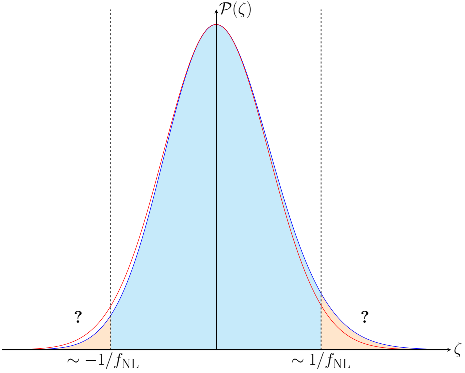
The expansion parameter is just λ

Motivations

- Tails of the distribution

$$\mathcal{P}_{\rm PT}[\zeta] \sim \exp\left\{ -\frac{\zeta^2}{2P_\zeta} - \frac{\langle\zeta\zeta\zeta\rangle}{2P_\zeta^3}\zeta^3 - \frac{\langle\zeta\zeta\zeta\zeta\rangle}{2P_\zeta^4}\zeta^4 \dots \right\}$$

$$= \exp\left\{ -\frac{\zeta^2}{2P_\zeta}\left[1 + \frac{\langle\zeta\zeta\zeta\rangle}{P_\zeta^2}\zeta + \frac{\langle\zeta\zeta\zeta\zeta\rangle}{P_\zeta^3}\zeta^2 + \dots\right]\right\}$$



$$\frac{\langle\zeta\zeta\zeta\rangle}{P_\zeta^2}\zeta \sim f_{NL}\zeta,$$

$$\frac{\langle\zeta\zeta\zeta\zeta\rangle}{P_\zeta^3}\zeta^2 \sim g_{NL}\zeta^2$$

The expansion parameter now depends on size of ζ

Main Idea

Unlikely events at the tails

$$\|$$

Semi-classical limit ($\hbar \to 0$)

The wavefunction of the Universe (WFU) $\sim e^{iS/\hbar}$ will be computed semi-classically

Motivations

• Primordial black hole formation: occurs around horizon re-entry

The mass fraction of PBH is

$$\beta(M) = \int_{\zeta_c}^{\infty} \mathcal{P}[\hat{\zeta}] d\hat{\zeta}\,, \qquad \hat{\zeta}(\boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3} W(k)\zeta(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

$\Rightarrow$ The formation is sensitive to $\zeta \sim 1$

$\Rightarrow$ The pert. theory is still valid for $f_{NL}\zeta \sim f_{NL} \ll 1$

(Single field slow-roll: $f_{NL} \sim \mathcal{O}(\epsilon, \eta)$, K-Inflation: $f_{NL} \sim (1 - 1/c_s^2)$, $|f_{NL}^{equil}| < 80$)

$\Rightarrow$ To study PBH formation one needs to go beyond perturbation theory

Analogy in QM: Compute the wavefunction in the Semi-classical limit

Euclidean Path integral

- From Path integral to the wavefunction of the ground state

$$\langle x_f | e^{-H(\tau_f - \tau_i)/\hbar} | x_i \rangle = \sum_n e^{-E_n(\tau_f-\tau_i)/\hbar} \Psi_n(x_f)\Psi_n^*(x_i)$$

$$\Psi_0(x_f)\Psi_0^*(x_i) e^{-E_0 T/\hbar} = \lim_{T\to\infty} \int_{x(\tau_i)=x_i}^{x(\tau_f)=x_f} \mathcal{D}x(\tau) e^{-S_E[x(\tau)]/\hbar}$$

- Expand $x(\tau) = x_{cl}(\tau) + y(\tau)$

$$\Psi_0(x_f) = N\ e^{-S_E[x_{cl}]/\hbar} \int_{y(\tau_i)=0}^{y(\tau_f)=0} \mathcal{D}y(\tau) e^{-\frac{1}{\hbar}\left(\frac{1}{2}\frac{\delta^2 S}{\delta x^2}y^2 + \frac{1}{3!}\frac{\delta^3 S}{\delta x^3}y^3 + \dots\right)}$$

$$\Psi_0(x_f) \simeq\ \mathcal{I}[x_f]\ e^{-S_E[x_{cl}]/\hbar}$$

Anharmonic oscillator

- $V(x) = \hbar\omega\left[\frac{1}{2}(\frac{x}{d})^2 + \lambda(\frac{x}{d})^4\right], \quad d \equiv \sqrt{\hbar/m\omega}$

$\Rightarrow$ The PT breaks down when $\lambda x_f^2/d^2 \equiv \bar{x}^2/2 \sim 1$

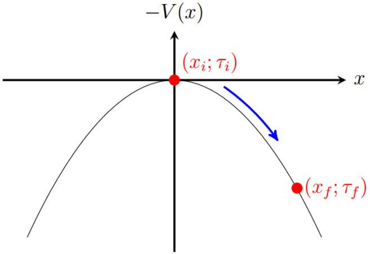
- In Euclidean space,

$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + V(x)$

- Real path connecting $x(\tau_i) = x_i$ and $x(\tau_f) = x_f$

- For $T = \tau_f - \tau_i \to \infty \Rightarrow E = 0$

$\Rightarrow$ The real path with infinite amount of times is the one with zero energy



$$x(\tau) = -\frac{d}{\sqrt{2\lambda}\sinh(\omega\tau)}$$

Anharmonic oscillator: Scaling argument

- Recall the wavefunction $\Psi_0(x_f) \simeq \mathcal{I}[x_f] e^{-S_E[x_{\rm cl}]/\hbar}$
- The Euclidean action is

$$S_E[x(\tau)] = \int_{\tau_{\rm i}}^{\tau_{\rm f}} d\tau \left\{ \frac{1}{2} m\dot{x}^2 + \hbar\omega \left[\frac{1}{2}\left(\frac{x}{d}\right)^2 + \lambda \left(\frac{x}{d}\right)^4 \right] \right\}$$

Rescaling $x \to (\sqrt{\hbar/\lambda})x$, then

$$\frac{S_E[x_{\rm cl}(\tau)]}{\hbar} \sim \frac{1}{\lambda} F(\lambda x_f^2/d^2)$$

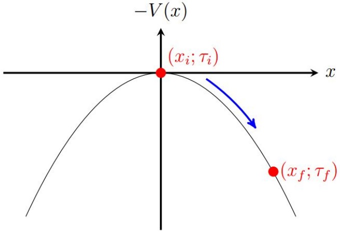
- The prefactor of $\mathcal{I}[x_f]$ goes as $\lambda^0 G(\lambda x_f^2/d^2) \Leftrightarrow$ 1-loop diagrams
- Neglect the higher-order in $\lambda \Leftrightarrow$ higher-loop diagrams

Anharmonic oscillator: Ground-state wavefunction

- The on-shell action with zero energy

$$\frac{S_E[x_{cl}(\tau)]}{\hbar} = \frac{1}{\hbar}\int_{\tau_i}^{\tau_f} d\tau \; m\dot{x}^2$$
$$= \frac{1}{6\lambda}\left[(1+\bar{x}^2)^{3/2} - 1\right]$$

$$\mathcal{I}(x_f) = \mathcal{N}\sqrt{\frac{m}{2\pi i\hbar v_i v_f \int_0^{x_f} \frac{\mathrm{d}x'}{v^3(x')}}}, \qquad \bar{x}^2 \equiv 2\lambda x_f^2/d^2$$



$$x(\tau) = -\frac{d}{\sqrt{2\lambda}\,\sinh(\omega\tau)}$$

$$\Psi_0(x_f) = \mathcal{N}\frac{\exp\left\{-\frac{1}{6\lambda}\left[\left(1+\bar{x}^2\right)^{3/2} - 1\right]\right\}}{(1+\bar{x}^2)^{1/4}(1+\sqrt{1+\bar{x}^2})^{1/2}}\Big(1+\mathcal{O}(\lambda)f(\bar{x})\Big)$$

This is valid for arbitrary $\bar{x}$.

Free theory

- The wavefunction of the Universe $\Psi[\zeta_0(\boldsymbol{x})] = \int_{BD}^{\zeta_0(\boldsymbol{x})} \mathcal{D}\zeta \; e^{iS[\zeta]/\hbar}$
- The free saddle point is (Maldacena 02)

$$\zeta_{\boldsymbol{k}}^{cl}(\eta) = \zeta_{\boldsymbol{k}}^{0} \frac{(1 - ik\eta)e^{ik\eta}}{(1 - ik\eta_f)e^{ik\eta_f}}$$

$\Rightarrow$ $i\epsilon$-prescrip. selects the correct BC at early times

$$iS[\zeta_{cl}] = i \int_{\boldsymbol{k}} \frac{1}{2P_\zeta \eta_f^2} \zeta_{-\boldsymbol{k}}^{cl} \partial_\eta \zeta_{\boldsymbol{k}}^{cl} \Big|_{\eta=\eta_f} = \int_{\boldsymbol{k}} \frac{1}{2P_\zeta} \left(\frac{ik^2}{\eta_f} - k^3 + \dots \right) \zeta_{-\boldsymbol{k}}^{0} \zeta_{\boldsymbol{k}}^{0}$$

$\int_{\boldsymbol{k}} = \int d^3k/(2\pi)^3$

$\Rightarrow$ The WFU is a Gaussian distribution

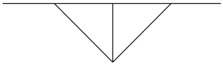
Interacting theory

- In perturbation theory, the WFU can be expanded as

$$\Psi = \exp\left[\frac{1}{2}\int d^3x d^3y \langle \mathcal{O}(\boldsymbol{x})\mathcal{O}(\boldsymbol{y})\rangle \zeta(\boldsymbol{x})\zeta(\boldsymbol{y}) + \frac{1}{6}\int d^3x d^3y d^3z \langle \mathcal{O}(\boldsymbol{x})\mathcal{O}(\boldsymbol{y})\mathcal{O}(\boldsymbol{z})\rangle \zeta(\boldsymbol{x})\zeta(\boldsymbol{y})\zeta(\boldsymbol{z}) + \dots\right]$$

$\Rightarrow$ The on-shell action amounts to computing tree-level Witten diagrams

ψ_3



ψ_4

Cosmological correlators:

$$\langle \zeta_{\boldsymbol{k}}\zeta_{-\boldsymbol{k}}\rangle' = \frac{-1}{2Re\langle \mathcal{O}_{\boldsymbol{k}}\mathcal{O}_{-\boldsymbol{k}}\rangle'}$$

$$\langle \zeta_{\boldsymbol{k}_1}\zeta_{\boldsymbol{k}_2}\zeta_{\boldsymbol{k}_3}\rangle' = \frac{2Re\langle \mathcal{O}_{\boldsymbol{k}_1}\mathcal{O}_{\boldsymbol{k}_2}\mathcal{O}_{\boldsymbol{k}_3}\rangle'}{\Pi_i(-2Re\langle \mathcal{O}_{\boldsymbol{k}_i}\mathcal{O}_{-\boldsymbol{k}_i}\rangle')}$$

Non-linear WFU

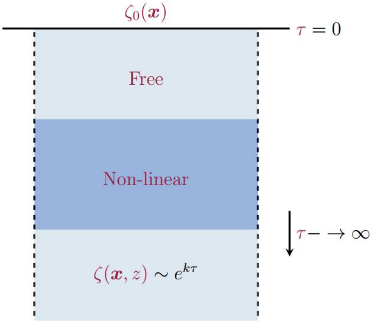
- Boundary conditions: BD at early times and ζ_0 at late times
- Find the non-linear classical solution to the EoM
- Compute the WFU in the semi-classical limit

$$\Psi[\zeta_0(\boldsymbol{x})] \sim e^{iS[\zeta_{cl}]/\hbar}$$

- The derivative coupling e.g. ζ'^4

$\Rightarrow$ Deep inside horizon: free theory

$\Rightarrow$ Outside horizon: non-linear term is switched off



EFT of Inflation

- The single coupling ζ'^4 can be justified in EFT of Inflation for large quartic operator (Senatore & Zaldarriaga 11)

$$\mathcal{L}_{EFT} = -M_{\rm Pl}^2 \dot{H} \left(\dot{\pi}^2 - \frac{(\partial_i \pi)^2}{a^2} \right) + M_4^4 (16 \dot{\pi}^4 - 32 \dot{\pi}^3 (\partial_\mu \pi)^2 + \ldots)$$

The coeff. of cubic operators can be set to zero: $M_2^4 (\delta g^{00})^2$, $M_3^4 (\delta g^{00})^3$

- $\pi \to \pi_c$, $\mathcal{O}(\pi_c^{N>4})$ are suppressed by $g_{\rm NL} \sim M_4^4/(|\dot{H}| M_{\rm Pl}^2) \lesssim 10^6$

$$iS = i \int d^3x d\eta \left\{ \frac{1}{2\eta^2 P_\zeta} \left[\zeta'^2 - (\partial_i \zeta)^2 \right] + \frac{\lambda}{4! P_\zeta^2} \zeta'^4 \right\}, \quad \zeta = -H \pi_c / \dot{\phi}_0$$

The Euclidean EoM reads

$$-\zeta'' + \frac{2}{\tau} \zeta' - \partial_i^2 \zeta - \frac{\lambda}{2 P_\zeta} \tau^2 \zeta'^2 \zeta'' = 0$$

Scaling argument of $S[\zeta_{\rm cl}]$

- Recall the WFU: $\Psi[\zeta_0(\boldsymbol{x})] \sim e^{-S_E[\zeta_{\rm cl}]}$
- The Euclidean action ($\eta = -i\tau$)

$$S_E \equiv -\int d^3x d\tau \left\{ \frac{1}{2\tau^2 P_\zeta} \left[\zeta'^2 + (\partial_i \zeta)^2 \right] + \frac{\lambda}{4! P_\zeta^2} \zeta'^4 \right\}$$

- Rescaling $\zeta \to \zeta/\sqrt{\lambda}$, then

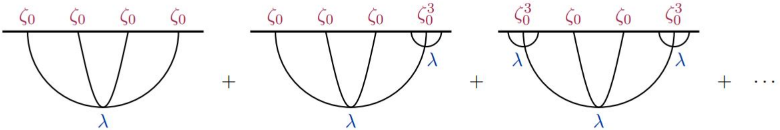
$$S_E[\zeta_{\rm cl}] = \frac{1}{\lambda} F\left(\lambda \zeta_0^2 / P_\zeta\right)$$

- The relevant expansion parameter is $\lambda \zeta_0^2 / P_\zeta$
- Neglect the prefactor, $\lambda^0 G(\lambda \zeta_0^2 / P_\zeta)$, and the higher orders in λ

Witten diagrams

- Tree-level graphs, captured by semiclassical method:

$$\frac{F(\lambda\zeta_0^2)}{\lambda} \sim \zeta_0^2 + \lambda\zeta_0^4 + \lambda^2\zeta_0^6 + \lambda^3\zeta_0^8 + \dots$$



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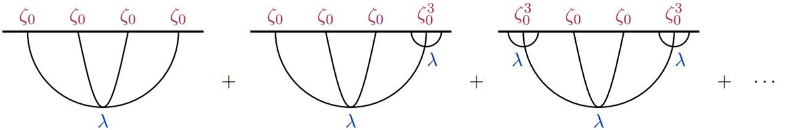
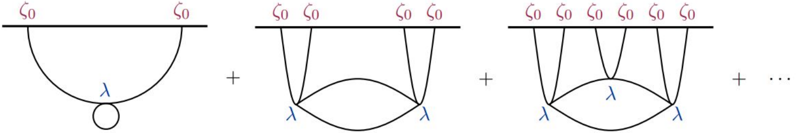


- 1-loop graphs, would be captured by the prefactor:

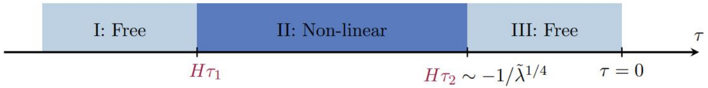
$$\lambda^0 G(\lambda\zeta_0^2) \sim \lambda\zeta_0^2 + \lambda^2\zeta_0^4 + \lambda^3\zeta_0^6 + \ldots$$



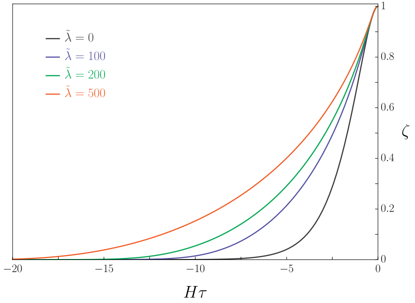
Approximation using ODE

- The derivative coupling only affects the modes of similar wavelength

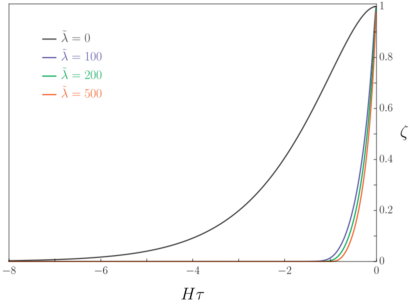
$$-\zeta'' + \frac{2}{\eta}\zeta' + H^2\zeta - \frac{\lambda}{2P_\zeta}\eta^2\zeta'^2\zeta'' = 0\,, \quad \tilde{\lambda} = \lambda\zeta_0^2/P_\zeta$$



$\zeta \to \zeta_0\zeta$



$\tau \to \tilde{\lambda}^{1/2}\tau$



Approximation using ODE

- The rescaling in τ gives the behaviour for large λ

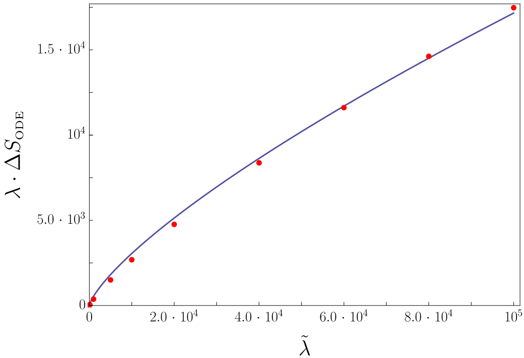
$$\Delta S_{ODE} = -\frac{\zeta_0^2}{P_\zeta}\int_{\tau_\mathrm{i}}^{\tau_\mathrm{f}} d\tau \left\{ \frac{1}{2\tau^2}\left[\zeta'^2 + H^2(\zeta^2 - 1)\right] + \frac{\tilde{\lambda}}{4!}\zeta'^4 \right\} = \frac{1}{\lambda}F(\tilde{\lambda})$$

$\tilde{\lambda} = \lambda\zeta_0^2/P_\zeta$

$$\Delta S_{ODE} \sim \frac{1}{\lambda}\tilde{\lambda}^{3/4}$$

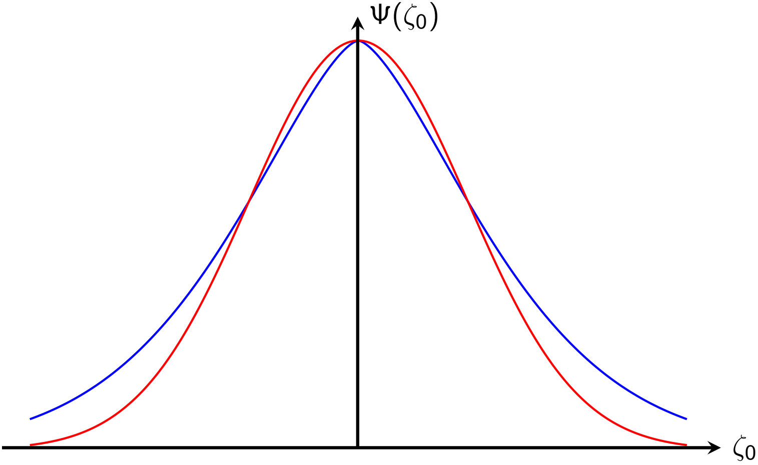
$$\Downarrow$$

$$\Psi[\zeta_0] \sim \exp\left[-\frac{\zeta_0^{3/2}}{\lambda^{1/4}P_\zeta^{3/4}}\right]$$



Approximation using ODE

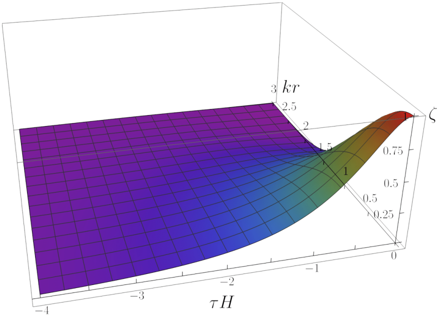
- $\Psi_G \sim \exp(-\zeta_0^2/2)$, $\Psi \sim \exp(-\zeta_0^{3/2}/2)$: Ψ is heavier than Ψ_G



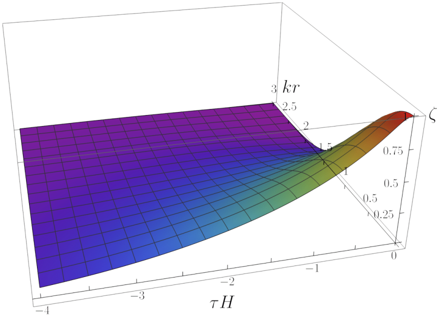
PDE with Gaussian profile

- The Gaussian profile at η_c: $\zeta(r) \sim \zeta_0 e^{-r^2}$

small λ



large λ

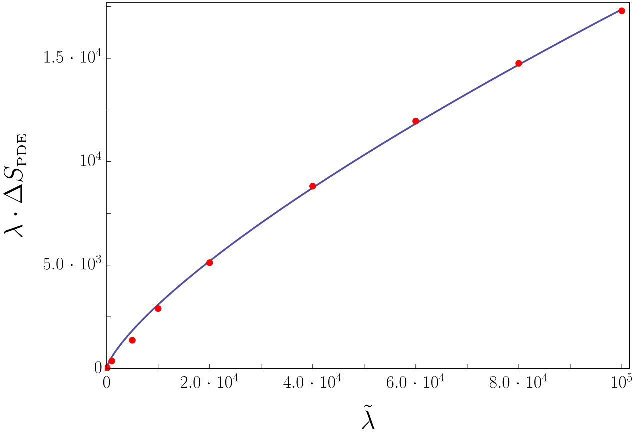


- For small λ, it reduces to perturbative result

PDE with Gaussian profile

- The on-shell action

$$\Delta S_{PDE} \sim \frac{1}{\lambda}\tilde{\lambda}^{3/4} \Rightarrow \Psi \sim \exp(-\zeta_0^{3/2}/\lambda^{1/4})$$



Future Directions: Inflation

- Explore PBH formation $\zeta_c \sim 1$, $\mathcal{P}[\zeta^0_{\boldsymbol{k}}] = |\Psi[\zeta_0]|^2$

$$\mathcal{P}[\zeta_c] = \mathcal{N}^{-1} \int \mathcal{D}[\zeta^0_{\boldsymbol{k}}]\, \mathcal{P}[\zeta^0_{\boldsymbol{k}}]\, \Theta(\hat{\zeta}[\zeta^0_{\boldsymbol{k}}] - \zeta_c)$$

$\hat{\zeta}[\zeta^0_{\boldsymbol{k}}] = \int_{\boldsymbol{k}} W(k) \zeta^0_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$

- Generalize to
  - Different interactions
  - Slow-roll inflation
  - Tensor mode γ_{ij}
- Connection to large number of legs limit (e.g. Badel et al. 20)
- Any implication for AdS/CFT ? Compute the exact Z for given source ?

Two fields in dS

• Idea: take one field to be on the background and compute the other non-perturbatively

$$S = \int d\eta d^3x \left[\frac{1}{2\eta^2 H^2}(\sigma'^2 - (\partial_i\sigma)^2) + \frac{1}{2\eta^2 H^2}(\chi'^2 - (\partial_i\chi)^2) - \frac{\lambda}{\eta^4 H^3}\chi\sigma^2 \right]$$

• For $k_\chi \ll k_\sigma$ we have

$$S_\sigma = \int d\eta d^3x \left[\frac{1}{2\eta^2 H^2}(\sigma'^2 - (\partial_i\sigma)^2) - \frac{\alpha H^2}{2\eta^4}\sigma^2 \right]$$

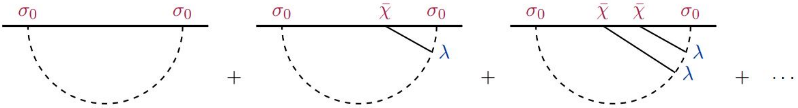
where $\alpha \equiv 2\lambda\bar{\chi}/H$. This is just a massive scalar field on dS whose power spectrum at late times is

$$\langle \sigma_{\boldsymbol{k}}\sigma_{-\boldsymbol{k}} \rangle' \simeq \frac{H^2}{2k^{3-\frac{2}{3}\alpha}} = \frac{H^2}{2k^{3-\frac{4}{3}\lambda\bar{\chi}/H}}$$

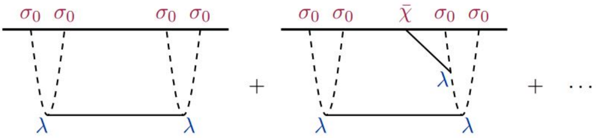
We have resummed all powers in $\lambda\bar{\chi}$

Two fields in dS

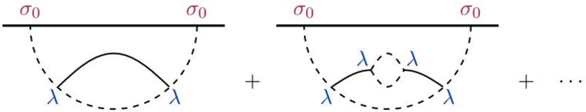
- Tree-level diagrams, enhanced by $\bar{\chi}$ and resummed



- Tree-level exchange diagrams, with fewer powers of $\bar{\chi}$



- Loop diagrams, subleading in λ



Work in progress: Spatial derivative coupling

- The spatial derivative interaction $(\partial_i\zeta)^4$

$$S = \int d^3x d\eta \frac{1}{P_\zeta}\left\{\frac{1}{2\eta^2}\left[\zeta'^2 - (\partial_i\zeta)^2\right] \pm \frac{\lambda}{4!}(\partial_i\zeta)^4\right\}$$

- All possible subtleties:

- The $+$ sign $\Rightarrow$ Gradient inst.

- The $-$ sign (healthy) $\Rightarrow$ the solution becomes complex for large λ

  - Study QM in p-space for $\hat{x}^2 + \hat{x}^4$, $\hat{x} \sim d/dp$, $\Psi(p) \sim e^{i\sigma(p)/\hbar}$

$$\frac{p^2}{2m} + V(-\sigma'(p)) = E\ , \quad \sigma(p_f) = \int^{p_f} dp\ V^{-1}\left(E - \frac{p^2}{2m}\right)$$

  - There are complex saddle points depending on p_f

Work in progress: Two fields model

- Two field model: $S = \int d\eta d^3x \left[\mathcal{L}_\sigma^0 + \mathcal{L}_\chi^0 - \frac{\lambda}{\Lambda^4} (\partial_i \sigma)^2 (\partial_i \chi)^2 \right]$
- Treat χ as a background for σ ($k_\chi \ll k_\sigma$)

$$\sigma_{\boldsymbol{k}}'' - \frac{2}{\eta}\sigma_{\boldsymbol{k}}' + (1 + \alpha\eta^2)k^2\sigma_{\boldsymbol{k}} = 0, \quad \alpha \equiv \frac{2\lambda(\partial_i\bar{\chi})^2 H^2}{\Lambda^4}$$

- The power spectrum of σ is

$$\langle \sigma_{\boldsymbol{k}} \sigma_{-\boldsymbol{k}} \rangle' = \frac{\pi}{8k^{3/2}\alpha^{3/4}} \frac{e^{-\pi k/(4\sqrt{\alpha})}}{\left| \Gamma\left(\frac{5}{4} + \frac{ik}{4\sqrt{\alpha}}\right) \right|^2}$$

$\Rightarrow$ Not analytic around $\alpha = 0$

Work in progress: Two fields model

- The result matches with PT for small α

$$\langle \sigma_{\boldsymbol{k}} \sigma_{-\boldsymbol{k}} \rangle' = \frac{H^2}{2k^3} \left(1 - \frac{5\lambda (\partial_i \bar{\chi})^2 H^2}{2\Lambda^4 k^2} \right)$$

- For large α

$$\langle \sigma_{\boldsymbol{k}} \sigma_{-\boldsymbol{k}} \rangle' \simeq \frac{H^2}{k^{3/2} \alpha^{3/4}}$$

- The Wavefunction of the Universe is

$$\Psi[\sigma_0] \sim \exp[-\alpha^{3/4} \sigma_0^2]$$

$\Rightarrow$ This is the WFU of σ in the large background of χ

Backup Dark Energy

Introduction

• Universe undergoes accelerating expansion $\implies$ Many models of dark energy (DE) e.g. Quintessence, $\mathcal{P}(X)$, (beyond) Horndeski, etc.

• Gravitational wave (GW) observations $\implies$ New test of GR and modified gravity theories

• Use GW propagation (LIGO/Virgo) to constrain those DE models

Dark enery models: Scalar-tensor theories

- One extra scalar field:

$$\mathcal{L} = R - \frac{1}{2}X - V(\phi) \qquad \text{Quintessence}$$

$$\mathcal{L} = f(\phi)R - \frac{1}{2}X - V(\phi) \qquad \text{Brans-Dicke}$$

$$\mathcal{L} = R - \mathcal{P}(\phi, X) \qquad \text{k-essence}$$

$X = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$

Scalar fluctuation: $\phi = \phi_0(t) + \pi(t, \boldsymbol{x})$ leads to a sound speed c_s

$$X^2 \supset \phi_0^2(t)\dot{\pi}^2 \quad \Longrightarrow \quad \mathcal{L}_\pi \sim \dot{\pi}^2 - c_s^2(\partial_i\pi)^2$$

Dark enery models: Scalar-tensor theories

- Most general scalar-tensor theories with 2nd order EoM: (Beyond) Horndeski

$$
\begin{aligned}
\mathcal{L}_2 =& G_2(\phi, X) \\
\mathcal{L}_3 =& G_3(\phi, X)\Box\phi \\
\mathcal{L}_4 =& G_4(\phi, X)R - 2G_{4,X}(\phi, X)[(\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}] \\
& - F_4(X, \phi)\epsilon^{\mu\nu\rho}{}_{\sigma}\epsilon^{\mu'\nu'\rho'\sigma}\phi_\mu\phi_{\mu'}\phi_{\nu\nu'}\phi_{\rho\rho'} \\
\mathcal{L}_5 =& G_5(\phi, X)G_{\mu\nu}\phi^{\mu\nu} + \frac{1}{3}G_{5,X}(\phi, X)[(\Box\phi)^3 - 3(\Box\phi)\phi_{\mu\nu}\phi^{\mu\nu} + 2\phi_{\mu\nu}\phi^{\sigma\mu}\phi^{\nu}{}_{\sigma}] \\
& - F_5(\phi, X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'}\phi_\mu\phi_{\mu'}\phi_{\nu\nu'}\phi_{\rho\rho'}\phi_{\sigma\sigma'}
\end{aligned}
$$

$$\phi_\mu \equiv \nabla_\mu\phi$$

Horndeski 74, Deffayet et al. 11,
Zumalacárregui and García-Bellido 14, Gleyzes et al. 14

Dark enery models

• The cosmological background $\phi_0(t)$ spontaneously breaks Lorentz invariance

• Interesting phenomena for tensor perturbation γ_{ij} from second derivatives

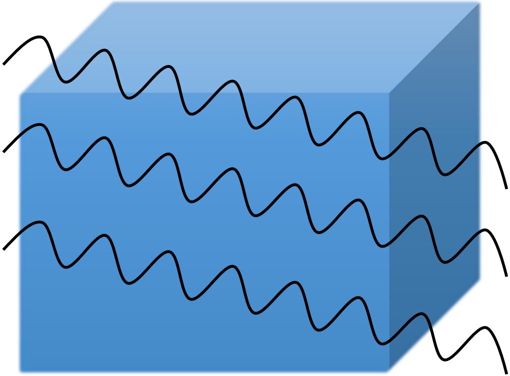
For example

$$(\nabla_\mu \nabla_\nu \phi)^2 \supset \dot{\phi}_0^2 \dot{\gamma}_{ij}^2$$

$$\mathcal{L}_\gamma \sim \dot{\gamma}_{ij}^2 - c_T^2 (\partial_l \gamma_{ij})^2$$

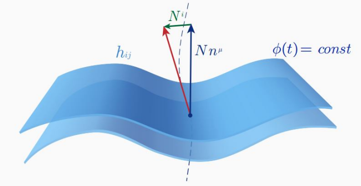
Dark enery models

Extra scalar field: Lorentz violating medium $\Rightarrow c_T \neq 1$



EFT of Dark Energy

- Efficient way to study a perturbation around fixed background
- Spontaneously break time diffeomorphism



$$ds^2 = -N^2 dt^2 + h_{ij}(N^i dt + dx^i)(N^j dt + dx^j)$$

$$S = \int d^4x \sqrt{-g}\; L[t; N, K^i_j, {}^{(3)}R, \ldots] \qquad g^{00} = -N^{-2}$$

- The action contains all possible invariances under 3d diffs

Cheung et al. 08,
Gubitosi et al. 12, and many others

EFT of Dark Energy

$$\begin{aligned} S^{\mathrm{EFT}} &= \int d^4x\sqrt{-g}\left[\frac{M_*^2}{2}f(t)^{(4)}R - \Lambda(t) - c(t)g^{00}\right. \\ &+ \frac{m_2^2(t)}{2}(\delta g^{00})^2 - \frac{m_3^3(t)}{2}\delta K\delta g^{00} - m_4^2(t)\delta\mathcal{K}_2 + \frac{\tilde{m}_4^2(t)}{2}\delta g^{00\,(3)}R \\ &- \left.\frac{m_5^2(t)}{2}\delta g^{00}\delta\mathcal{K}_2 - \frac{m_6(t)}{3}\delta\mathcal{K}_3 - \tilde{m}_6(t)\delta g^{00}\delta\mathcal{G}_2 - \frac{m_7(t)}{3}\delta g^{00}\delta\mathcal{K}_3\right] \end{aligned}$$

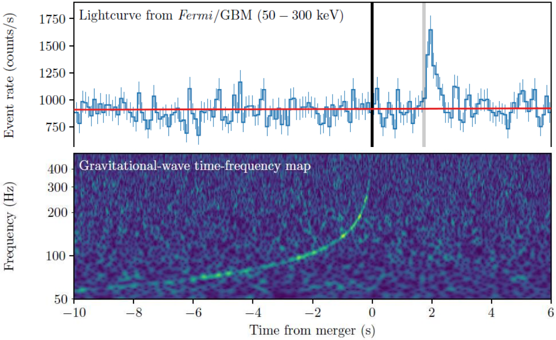
This term changes the speed of GWs

$$\delta\mathcal{K}_2 = \delta K^2 - \delta K^\mu_\nu\delta K^\nu_\mu \supset \dot{\gamma}_{ij}^2, \qquad \delta K^\mu_\nu = K^\mu_\nu - H\delta^\mu_\nu$$

$$\delta\mathcal{G}_2 = \delta K^{\mu\,(3)}_\nu R^\nu_\mu - \delta K^{(3)}R/2$$

$$\delta\mathcal{K}_3 = \delta K^3 - 3\delta K\delta K^\mu_\nu\delta K^\nu_\mu + 2\delta K^\nu_\mu\delta K^\mu_\rho\delta K^\rho_\nu$$

GW170817 = GRB170817A



$$|c_T^2 - 1| \lesssim 10^{-15}$$

LIGO/Virgo + Fermi/GBM + INTEGRAL 17

EFT of DE after GW170817 & GRB170817A

- The speed of GWs can be expressed as

$$c_T^2 = 1 - \frac{2m_4^2}{M_*^2 f + 2m_4^2}$$

- $c_T^2 = 1 \Rightarrow m_4^2 = 0$
- The EFT action becomes

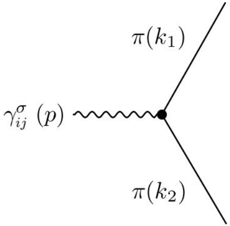
$$\begin{aligned} \mathcal{L}^{\rm EFT}_{c_T=1} &= \frac{M_{\rm Pl}^2}{2} f(t)^{(4)}R - \Lambda(t) - c(t) g^{00} + \frac{m_2^2(t)}{2}(\delta g^{00})^2 - \frac{m_3^3(t)}{2}\delta K \delta g^{00} \\ &+ \frac{\tilde{m}_4^2(t)}{2}\delta g^{00}\left({}^{(3)}R - \delta\mathcal{K}_2\right) \end{aligned}$$

Creminelli and Vernizzi 17,
Ezquiaga and Zumalacárregui 17,
Baker et al. 17, Sakstein and Jain 17

Pertubative decay of GWs due to $\tilde{m}_4^2$-term

• Spontaneous breaking of Lorentz allows the decay

• $\tilde{m}_4^2$: $\delta g^{00}({}^{(3)}R - \delta\mathcal{K}_2) \Leftrightarrow$ Beyond Horndeski $(F_4 \& F_5)$



• The interaction term:

$$S_{\gamma\pi\pi} = \frac{1}{\Lambda_\star^3}\int d^4x \ddot{\gamma}_{ij}\partial_i\pi\partial_j\pi \quad , \quad \Lambda_\star^3 \simeq \sqrt{2}\frac{\alpha}{\alpha_{\rm H}}\Lambda_3^3$$

$\alpha_H = 2\tilde{m}_4^2/M_{\rm Pl}^2$, $\Lambda_3 = (M_{\rm Pl}H_0^2)^{1/3}$

• The perturbative decay rate: $\Gamma_{\gamma\to\pi\pi} \simeq \left(\frac{\alpha_{\rm H}}{\Lambda_3^3}\right)^2 \frac{\omega^7(1-c_s^2)^2}{480\pi c_s^7}$

Creminelli et al. 18

Constraint from no pert. decay

- At LIGO/Virgo, take $\omega \sim \Lambda_3,\ \Lambda_3 \sim 10^{-13}\ \mathrm{eV}$
- Compare the decay rate with the cosmological distances $\sim H_0^{-1}$

$$\frac{\Gamma_{\gamma\to\pi\pi}}{H_0} \sim 10^{20}\alpha_{\mathrm{H}}^2 \frac{(1-c_s^2)^2}{480\pi c_s^7} \lesssim 1$$

$H_0 \sim 10^{-33}$ eV

$\alpha_{\mathrm{H}} \lesssim 10^{-10} \implies$ beyond Horndeski is highly constrained

- What if a large occupation number of GWs is taken into account $\implies$ non-perturbative effect, resonant π-production ?

EFT of DE with constraints

Recap most general EFT action with $c_T^2 = 1$: $S = S_0 + S_{m_3} + S_{\tilde{m}_4}$

$$S_0 = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}\,{}^{(4)}R - \lambda(t) - c(t)g^{00} + \frac{m_2^4(t)}{2}(\delta g^{00})^2\right]$$

$$S_{m_3} = -\int d^4x\sqrt{-g}\,\frac{m_3^3(t)}{2}\delta K\delta g^{00}$$ Cubic Horndeski

$$S_{\tilde{m}_4} = \int d^4x\sqrt{-g}\,\frac{\tilde{m}_4^2(t)}{2}\delta g^{00}\left({}^{(3)}R + \delta K_\mu^\nu\delta K_\nu^\mu - \delta K^2\right)$$

Quartic beyond Horndeski

$\delta g^{00} = 1 + g^{00}, \delta K_\nu^\mu = K_\nu^\mu - H\delta_\nu^\mu$

Classification of instabilities

- $\mathcal{L}_0 + \mathcal{L}_{\tilde{m}_4} = \frac{1}{2}\dot{\gamma}_{ij}^2 - \frac{1}{2}(\partial_k \gamma_{ij})^2 + \frac{1}{2}\dot{\pi}^2 - \frac{c_s^2}{2}(\partial_i \pi)^2 + \frac{1}{\Lambda_\star^3}\ddot{\gamma}_{ij}\partial_i\pi\partial_j\pi$
- Treat GW as a classical background: $\gamma_{ij} = M_{\rm Pl} h_0^+ \sin(\omega(t-z))\epsilon_{ij}^+$
- The EoM of π reads

$$\ddot{\pi} - c_s^2 \nabla^2 \pi - c_s^2 \beta \sin[\omega(t-z)](\partial_x^2 - \partial_y^2)\pi = 0$$

where

$$\beta \equiv \frac{2\omega^2 M_{\rm Pl} h_0^+}{c_s^2 |\Lambda_\star^3|} \quad , \quad \Lambda_\star^3 \simeq \frac{\Lambda_3^3}{\alpha_{\rm H} c_s^2}$$

$\beta < 1$: Resonant instability

$\beta > 1$: Gradient instability

Resonant decay of GWs

- EoM of π for $\tilde{m}_4^2$-operator

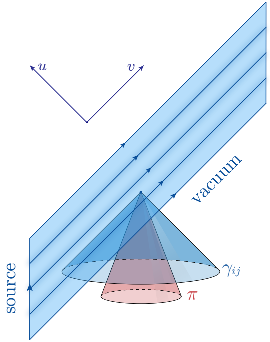
$$\ddot{\pi} - c_s^2 \partial^2 \pi - \beta \sin(\omega(t-z))(\partial_x^2 - \partial_y^2)\pi = 0$$

- Light-cone coord.

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\tau^2} + [A - 2q\cos(2\tau)]f = 0$$

$$\pi(u, \tilde{\boldsymbol{x}}) \sim \int e^{i\tilde{\boldsymbol{p}}\cdot\tilde{\boldsymbol{x}}} f_{\tilde{p}}(u)\hat{a}_{\tilde{\boldsymbol{p}}} + \text{h.c.}$$

- $f_p \sim e^{\mu_k \tau}$, $\mu \sim \beta < 1$ (Narrow resonance)



Resonant decay of GWs

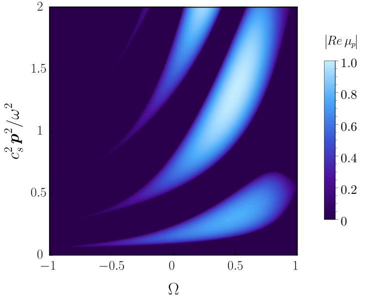
- In Fourier space f_p satisfies the Mathieu eq.

$$\frac{d^2 f}{d\tau^2} + [A - 2q\cos(2\tau)]f = 0$$

- $f_p \sim e^{\mu_k \tau}$

- the exponent $\mu \sim \beta$ for $\beta < 1$ (Narrow resonance)

- Need ~ 700 cycles to reach $\rho_\pi \sim \rho_\gamma$



$$\tau = \frac{\omega u}{2} \quad , \quad \Omega = p_z/|\boldsymbol{p}|$$

$$A = 4\frac{c_s^2\,\boldsymbol{p}^2}{\omega^2}\frac{(1-c_s\Omega)^2}{(1-c_s^2)^2}$$

$$q = 2\beta\frac{c_s^2\,\boldsymbol{p}^2}{\omega^2}\frac{(1-\Omega^2)\cos(2\phi)}{1-c_s^2}$$

Modifications of GWs signal

For $\tilde{m}_4^2$-operator: $\ddot{\gamma}_{ij}\partial_i\pi\partial_j\pi$

- EoM of γ

$$\ddot{\gamma}_{ij} - \nabla^2\gamma_{ij} - \frac{2}{\Lambda_\star^3}\Lambda_{ij,kl}\partial_t^2\left(\partial_k\pi\partial_l\pi\right) = 0$$

- Write $\gamma_{ij} \equiv \bar{\gamma}_{ij} + \Delta\gamma_{ij}$

$$\Delta\gamma_{ij}(u,v) = -\frac{v}{4\Lambda_\star^3}\partial_u J_{ij}(u) \quad , \quad J_{ij}(u) \equiv \Lambda_{ij,kl}\partial_k\pi\partial_l\pi$$

- Expand $\pi \sim f_p \sim e^{\mu\tau}$, use the saddle-point approx. $(\tau \gg 1)$

$$\Delta\gamma(u,v)_{ij} \simeq -\frac{v}{4\Lambda_\star^3}\frac{(1-c_s^2)^2}{c_s^5\sqrt{\beta}}\frac{\omega^{7/2}}{(8u\pi)^{3/2}}\exp\left(\frac{\beta}{4}\omega u\right)\epsilon_{ij}^+$$

Observational signature for $\tilde{m}_4$

- Modification of GWs signal: $\Delta\gamma_{ij} \sim -A\exp(\beta\omega u/4)\epsilon^{+}_{ij}$
- Effect of G_4 (Quartic Galileon), $\Lambda_c^6 \sim \Lambda_3^6/(\alpha_{\rm H} c_s^4)$ for $m_3^3 = 0$

$$G_4 = \frac{1}{\Lambda_c^6}(\partial\pi)^2[(\Box\pi)^2 - \pi_{\mu\nu}\pi^{\mu\nu}] \sim \frac{1}{\Lambda_\star^3}\ddot{\gamma}_{ij}\partial_i\pi\partial_j\pi$$

We obtain

$$\frac{\Delta\gamma}{\bar{\gamma}} \lesssim (\beta N_{\rm cyc})^{3/2}(rH_0)^2 \equiv \left(\frac{\Delta\gamma}{\bar{\gamma}}\right)_{NL}$$

Observational signature for $\tilde{m}_4$

- For a binary system (M_c, f, r): $h_0^+ \sim 10^{-3}/(f N_{\rm cyc} r)$
- Sizeable effect in GW waveform requires $\exp(\beta\omega u/4) \sim \mathcal{O}(10^2)$

$$\frac{\Delta\gamma}{\bar{\gamma}} > 0.1 \quad \Rightarrow \quad \alpha_{\rm H} \gtrsim 10^{-17} \cdot rH_0 \cdot \frac{\Lambda_3}{2\pi f} \alpha c_s^2$$

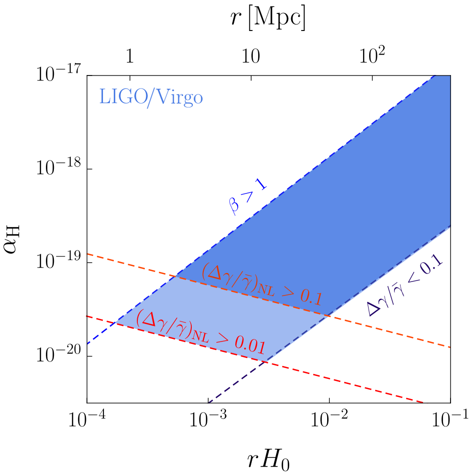
- Our calculation is valid when $\beta < 1$

$$\alpha_{\rm H} \lesssim \frac{H_0}{f} \cdot N_{\rm cyc} \cdot rH_0 \quad , \quad N_{\rm cyc} \sim (GM_c f)^{-5/3}$$

- To neglect effect of NL, demands $(\Delta\gamma/\bar{\gamma})_{NL} > 0.1$

$$\alpha_{\rm H} \gtrsim \frac{H_0}{f} (rH_0)^{1/3}$$

Observational signature for $\tilde{m}_4$



$f = 30$ Hz, $\mathrm{M_c} = 1.2\mathrm{M}_\odot$
GW170817,
40 Mpc ($\mathrm{rH_0} \sim 5 \cdot 10^{-3}$)

perturbative bound: $\alpha_{\rm H} \lesssim 10^{-10}$

What about the resonant effect from m_3^3-term ?

- One can run the same procedure with $m_3^3 \delta K \delta g^{00}$

$$m_3^3 \delta K \delta g^{00} \supset \frac{1}{\Lambda^2} \dot{\gamma}_{ij} \partial_i \pi \partial_j \pi \ , \quad \Lambda^2 \simeq -\frac{\alpha \Lambda_2^2}{\sqrt{2} \alpha_{\rm B}}$$

$\alpha_{\rm B} \equiv -m_3^3/(2M_{\rm Pl}^2 H)$, $\beta = 2\omega M_{\rm Pl} h_0^+/(c_s^2 |\Lambda^2|)$, $\Lambda_2 \sim 10^{-3}\ {\rm eV}$

- Once the resonance happens ($\beta < 1$), the cubic self-interaction quickly becomes important

$$G_3 \sim \frac{1}{\Lambda_{\rm B}^3} \Box \pi \, (\partial_i \pi)^2 \ , \quad \Lambda_{\rm B}^3 \sim \alpha_{\rm B}^{-1} \Lambda_3^3$$

- No sizable effect on GWs signal: $\Delta\gamma/\bar{\gamma} \ll 1 \Rightarrow$ Need to study $\beta > 1$

Gradient/Ghost instabilities ($\beta > 1$)

- Let's consider

$$\begin{aligned}\mathcal{L}_\pi &= \frac{1}{2}\left[\dot{\pi}^2 - c_s^2(\partial_i\pi)^2\right] + \frac{1}{\Lambda^2}\dot{\gamma}_{ij}\partial_i\pi\partial_j\pi + \dots \\ &= \frac{1}{2}\dot{\pi}^2 - \frac{c_s^2}{2}(1-\beta)(\partial_i\pi)^2 + \text{NL self-couplings} + \text{Source terms}\end{aligned}$$

Generally, this leads to the gradient instability of π.

- Can the non-linearity quench the instability ?

- Study the stability at NL level with the bg. of π induced by GWs

Classical stability conditions

- Consider a generic Lagrangian for π

$$\mathcal{L}_{\pi} \xrightarrow{\pi=\hat{\pi}+\delta\pi} \mathcal{L}_{\delta\pi} = Z^{\mu\nu}(x)\,\partial_{\mu}\delta\pi\partial_{\nu}\delta\pi$$

- Free of instability $\Rightarrow$ Conditions on $Z^{\mu\nu}$
- Absence of ghost: $Z^{00} > 0$
- Absence of gradient: $Z^{0i}Z^{0j} - Z^{ij}Z^{00}$ positive definite
- Cubic Galileon w/o GWs: no ghost/gradient inst. for non-relativistic source (Nicolis and Rattazzi 04)

Instability in the presence of GWs

- The Lagrangian for π now is

$$\mathcal{L} = -\frac{1}{2}\bar{\eta}^{\mu\nu}\partial_\mu\pi\partial_\nu\pi - \frac{1}{\Lambda_{\rm B}^3}\Box\pi(\partial\pi)^2 + \frac{1}{\Lambda^2}\dot{\gamma}_{ij}\partial_i\pi\partial_j\pi + \frac{\Lambda_{\rm B}^3}{2\Lambda^4}\pi\dot{\gamma}_{ij}^2$$

$\bar{\eta}_{\mu\nu} \equiv \mathrm{diag}(-1, c_s^2, c_s^2, c_s^2)$, The parameter $\beta \sim \dot{\gamma}_{ij}/\Lambda^2 > 1$

- $\pi = \hat{\pi} + \delta\pi$. The kinetic matrix $Z^{\mu\nu}$ for $\delta\pi$ is

$$Z^{\mu\nu} \equiv -\frac{1}{2}\bar{\eta}^{\mu\nu} - 2\left(\mathcal{K}^{\mu\nu} - \eta^{\mu\nu}\mathcal{K}\right) + \frac{\dot{\gamma}_{\mu\nu}}{\Lambda^2}\,, \quad \mathcal{K}_{\mu\nu} = -\frac{1}{\Lambda_{\rm B}^3}\partial_\mu\partial_\nu\hat{\pi}$$

- The EoM for $\hat{\pi}$ is

$$\bar{\Box}\hat{\pi} - \frac{2}{\Lambda_{\rm B}^3}\left[(\partial_\mu\partial_\nu\hat{\pi})^2 - \Box\hat{\pi}^2\right] - \frac{2}{\Lambda^2}\dot{\gamma}_{\mu\nu}\partial^\mu\partial^\nu\hat{\pi} - \frac{\Lambda_{\rm B}^3}{2\Lambda^4}\dot{\gamma}_{\mu\nu}^2 = 0$$

Instability in the presence of GWs with $c_s < 1$

- Assume that $\gamma_{\mu\nu} = \gamma_{\mu\nu}(u)$
- One can solve the EoM for $\hat{\pi}$ analytically

$$\hat{\pi}''(u) = -\frac{\Lambda_{\mathrm{B}}^3 \dot{\gamma}_{\mu\nu}^2}{2(1-c_s^2)\Lambda^4}$$

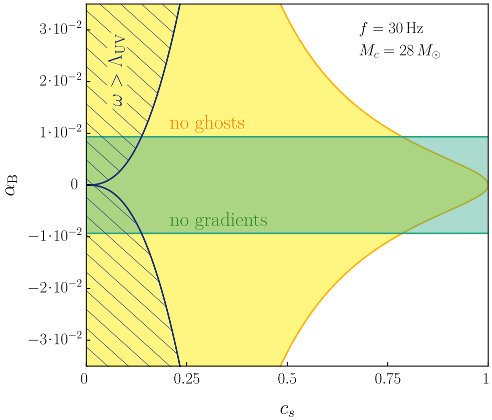
- The components of $Z^{\mu\nu}$ are

$$Z^{00} = \frac{1}{2} + 2\frac{\hat{\pi}''(u)}{\Lambda_{\mathrm{B}}^3}\,, \quad Z^{03} = Z^{30} = 2\frac{\hat{\pi}''(u)}{\Lambda_{\mathrm{B}}^3}\,, \quad Z^{33} = -\frac{1}{2}c_s^2 + 2\frac{\hat{\pi}''(u)}{\Lambda_{\mathrm{B}}^3}$$

$$Z^{11} = -\frac{1}{2}c_s^2 + \frac{\dot{\gamma}^{11}}{\Lambda^2}\,, \quad Z^{22} = -\frac{1}{2}c_s^2 + \frac{\dot{\gamma}^{22}}{\Lambda^2}$$

Phenomenological consequences

- Free of gradient int.: $Z^{11}, Z^{22} < 0 \Rightarrow \beta < 1$
- Free of ghost int.: $Z^{00} > 0 \Rightarrow \beta^2 < (1 - c_s^2)c_s^{-4}$



Free of instabilities: $|\alpha_{\mathrm{B}}| \lesssim 10^{-2}$

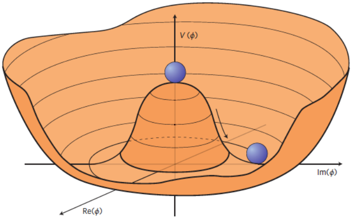
Fate of the instability

• The instabiliy occurs: the fluctuation grows at rate of the cutoff

• What happens next to the instability relies on the UV completion, so does the fate of $\gamma_{\mu\nu}$

• $\mathcal{L}_{\text{IR}} = \mathcal{P}(X)$ with constant X background $\Rightarrow$ Ghost + gradient inst.

• $\mathcal{L}_{UV} = -|\partial\phi|^2 - \lambda(|\phi|^2 - v^2)^2$
$\phi = \phi_0 e^{i\pi}$, $\langle\phi_0\rangle = v^2 - \frac{X}{2\lambda}$,
$X = (\partial\pi)^2$



Ellis, et al. 15

All the modes are stable in the UV theory

Conclusion

- Perturbative/Resonant decays of GWs $\Rightarrow$ a strong bound on quartic beyond Horndeski (α_{H})

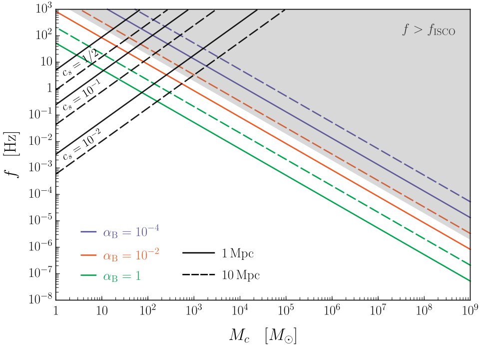
- Ghost/Gradient instabilities of π in GWs bg. $\Rightarrow$ a bound on Cubic Horndeski (α_{B})

-The surviving scalar-tensor theory: $g_{\mu\nu} \to C(\phi, X) g_{\mu\nu}$

$$\mathcal{L} = G_2(\phi, X) + C(\phi, X) R + \frac{6 C_{,X}(\phi, X)^2}{C(\phi, X)} \phi^{;\mu} \phi_{;\mu\nu} \phi_{;\lambda} \phi^{;\nu\lambda}$$

- Fate of instability relies on the UV completion

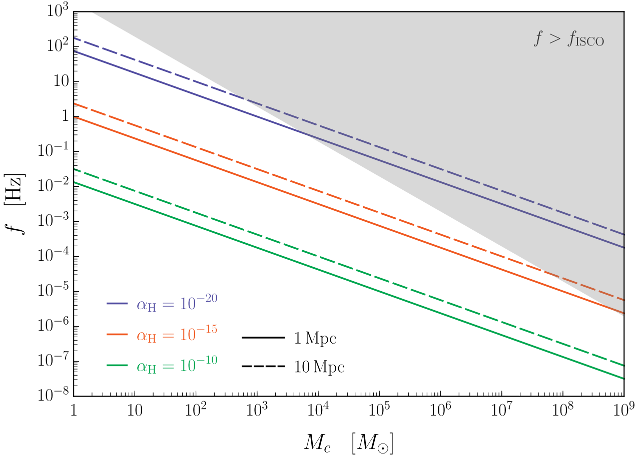
Bound on α_B: instability



Independently of M_c the instability is triggered close to the ISCO for $\alpha_B \gtrsim 10^{-2}$

- Gradient-instability lines, $\beta = 1$, for different value of α_B as a function of M_c of the binary system
- The black lines indicate frequencies $\omega > \Lambda_{UV}$

Bound on α_{H}: instability



- Gradient-instability lines, $\beta = 1$, for different value of α_{H} as a function of M_c of the binary system
- $\alpha_{\mathrm{H}} \gtrsim 10^{-20}$ triggers Gradient-instability

Bound on α_{H}: instability

- The cubic and quartic self-couplings are negligible compared to $\gamma\pi\pi$ term
- Cubic self-interaction

$$\frac{1}{\Lambda_\star^3}\ddot{\gamma}_{ij}\partial_i\pi\partial_j\pi \sim \frac{(\partial\pi)^2}{\Lambda_\star^3}\partial^2\pi \;\Rightarrow\; \frac{\partial^2\hat{\pi}}{\ddot{\gamma}} \sim \beta\frac{H}{\omega} \ll 1$$

- Quartic self-interaction

$$\frac{(\partial\pi)^2}{\Lambda_c^6}[(\Box\pi)^2 - (\partial_\mu\partial_\nu\pi)^2] \sim \frac{(\partial\pi)^2}{\Lambda_\star^3}\partial^2\pi \;\Rightarrow\; \frac{(\partial^2\hat{\pi})^2\Lambda_\star^3}{\ddot{\gamma}\Lambda_c^6} \sim \beta^2 h_0^+ \ll 1$$

Note $\Lambda_\star \simeq \alpha_{\mathrm{H}}^{-1/3}\alpha^{1/3}\Lambda_3$ and $\Lambda_c \simeq \alpha_{\mathrm{H}}^{-1/6}\alpha^{1/3}\Lambda_3$

- No stability argument of quartic and quintic Galileons

Stability in the absence of GWs

Here the Lagrangian is

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\pi\partial_\nu\pi - \frac{1}{\Lambda_{\rm B}^3}\Box\pi(\partial\pi)^2 + \frac{\pi T}{2M_{\rm Pl}}$$

- $\pi = \hat{\pi} + \delta\pi$, the EoM for $\hat{\pi}$ is

$$\mathcal{K} + 2\left(\mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu} - \mathcal{K}^2\right) = \frac{T}{2M_{\rm Pl}\Lambda_{\rm B}^3}\,, \quad \mathcal{K}_{\mu\nu} = -\frac{1}{\Lambda_{\rm B}^3}\partial_\mu\partial_\nu\hat{\pi}$$

- The kinetic matrix reads: $\mathcal{L}_{\delta\pi} = Z^{\mu\nu}\partial_\mu\delta\pi\partial_\nu\delta\pi$

$$Z^{\mu\nu} \equiv -\frac{1}{2}\eta^{\mu\nu} - 2\left(\mathcal{K}^{\mu\nu} - \eta^{\mu\nu}\mathcal{K}\right)$$

Nicolis and Rattazzi 04

Stability in the absence of GWs

- In terms of $Z_{\mu\nu}$ the EoM of $\hat{\pi}$ becomes

$$\frac{1}{3}Z^2 - (Z_{\mu\nu})^2 = \frac{1}{3} - \frac{T}{M_{\rm Pl}\Lambda_{\rm B}^3}$$

- For the non-relativistic source $v \ll 1$, the matrix $Z^{\mu\nu}$ is diagonalizable with a Lorentz boost, so that $Z^{\mu}_{\nu} = \mathrm{diag}(z_0, z_1, z_2, z_3)$ and $T \simeq -\rho \leq 0$
- Consider the plane $z_0 = 0$ in z_i-space

$$-\frac{1}{3}\left[(z_1 - z_2)^2 + (z_1 - z_3)^2 + (z_2 - z_3)^2\right] = \frac{1}{3} + \frac{\rho}{M_{\rm Pl}\Lambda_{\rm B}^3}$$

$\Rightarrow$ A solution crossing the plane doesn't exist

$\Rightarrow$ The initial stable solution ($Z^{\mu\nu} = -\eta^{\mu\nu}/2$) remains stable everywhere

Summary of the results

EFT of DE operator	$\frac{1}{2}\tilde{m}_4^2\,\delta g^{00}\left({}^{(3)}R + \delta K^\nu_\mu \delta K^\mu_\nu - \delta K^2\right)$	$m_3^3\,\delta g^{00}\delta K$
GLPV theories with $c_T = 1$ $\mathcal{L} = G_2 + G_3\Box\phi + B_4 R - \frac{4B_{4,X}}{X}(\phi^{;\mu}\phi^{;\nu}\phi_{;\mu\nu}\Box\phi - \phi^{;\mu}\phi_{;\mu\nu}\phi_{;\lambda}\phi^{;\lambda\nu})$	$-\frac{2XB_{4,X}}{B_4}$	$\frac{2XB_{4,X}}{B_4} + \frac{\dot{\phi}XG_{3,X}}{2HB_4}$
Dimensionless function α_i	α_{H}	α_{B}
Perturbative decay ($\Gamma_{\gamma\to\pi\pi}/H_0 > 1$)	$\lvert\alpha_{\mathrm{H}}\rvert \gtrsim 10^{-10}$	Irrelevant ($\lvert\alpha_{\mathrm{B}}\rvert \gtrsim 10^{10}$)
Narrow resonance ($\beta < 1,\ \beta\omega u > 1$)	$3\times 10^{-20} \lesssim \lvert\alpha_{\mathrm{H}}\rvert \lesssim 10^{-17}$ with LIGO-Virgo $10^{-16} \lesssim \lvert\alpha_{\mathrm{H}}\rvert \lesssim 10^{-10}$ with LISA	Not applicable (large non-linearities)
Instability ($\beta > 1,\ \beta\omega > 1$)	$\lvert\alpha_{\mathrm{H}}\rvert \gtrsim 10^{-20}$	$\lvert\alpha_{\mathrm{B}}\rvert \gtrsim 10^{-2}$

The surviving scalar-tensor theory: $g_{\mu\nu} \to C(\phi, X)g_{\mu\nu}$

$$\mathcal{L} = G_2(\phi, X) + C(\phi, X)R + \frac{6C_{,X}(\phi, X)^2}{C(\phi, X)}\phi^{;\mu}\phi_{;\mu\nu}\phi_{;\lambda}\phi^{;\nu\lambda}$$

Turn on both m_3 and $\tilde{m}_4$ with $c_T = 1$

- m_3 gives $\dot{\gamma}_{ij}\partial_i\pi\partial_j\pi$, while $\tilde{m}_4$ gives both $\dot{\gamma}_{ij}\partial_i\pi\partial_j\pi$ and $\ddot{\gamma}_{ij}\partial_i\pi\partial_j\pi$
- One tunes $\alpha_{\rm B}$ and $\alpha_{\rm H}$ such that $\dot{\gamma}_{ij}\partial_i\pi\partial_j\pi$ is absent
- Narrow resonance: the bound on $\alpha_{\rm H}$ remains the same (non-linearities are negligible) and inconclusive for $\alpha_{\rm B}$
- π-Instability: No general stability argument for quartic Galileon even w/o GWs. One expects the operator $\ddot{\gamma}_{ij}\partial_i\pi\partial_j\pi$ will lead to the instability anyway $\Rightarrow$ Bound on $\alpha_{\rm H}$

Generic frame with $c_T \neq 1$

• We then put a bound on a combination of m_4, $\tilde{m}_4$ and m_5 (the coupling of $\ddot{\gamma}_{ij}\partial_i\pi\partial_j\pi$)

$$\hat{\alpha}_{\rm H} = \frac{1+\alpha_{\rm H} - c_T^2(1+\alpha_{\rm V})}{1+\alpha_{\rm H} + c_T^2(1+\alpha_{\rm V})}$$

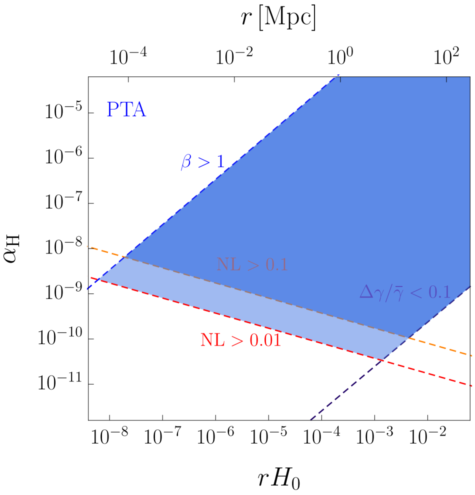
where $\alpha_{\rm V} \equiv -2m_5^2/M^2$, $\alpha_{\rm H} \equiv 2(\tilde{m}_4^2 - m_4^2)/M^2$, $c_T^2 \equiv 1 - 2m_4^2/M^2$ and hat quantities are with $\hat{c}_T^2 = 1$.

• This happens when we don't have bound on c_T^2 such as PTA ($f \sim 10^{-8}$ $\rm Hz$ and $M_{\rm c} \sim 10^6$ $\rm M_\odot$)

$\Rightarrow$ Pert. bound not applicable $\hat{\alpha}_{\rm H} < 10^{-10}(\Lambda_3/E)^3$, $E \sim 10^{-11}\Lambda_3$

$\Rightarrow$ Resonant bound $10^{-9} \lesssim \hat{\alpha}_{\rm H} \lesssim 10^{-5}$ is excluded

Resonant bound: PTA



$f = 10^{-8}$ Hz, $\mathrm{M_c} = 10^6\ \mathrm{M_\odot}$

PTA, 25 Mpc ($\mathrm{rH_0} \sim 10^{-3}$)

Perturbative decay of GWs due to m_3^3-term

- m_3^3: $\delta K \delta g^{00} \Leftrightarrow$ Cubic Horndeski G_3
- The interaction term:

$$S_{\gamma\pi\pi} = \frac{1}{\Lambda^2} \int d^4x \dot{\gamma}_{ij} \partial_i \pi \partial_j \pi \quad , \quad \Lambda^2 = -\frac{\alpha}{\sqrt{2}\alpha_{\mathrm{B}}} \Lambda_2^2$$

$\alpha_{\mathrm{B}} \equiv -m_3^3/2M_{\mathrm{Pl}}^2 H$

- The perturbative decay rate

$$\Gamma_{\gamma\to\pi\pi} \simeq \left(\frac{\alpha_B}{\Lambda_2^2}\right)^2 \frac{\omega^5(1-c_s^2)^2}{480\pi c_s^7}$$

$\Gamma/H_0 \lesssim 1 \Rightarrow |\alpha_{\mathrm{B}}| \lesssim 10^{10}$, $\Lambda_2 \sim 10^{-3}$ eV

Vainshtein effect on the instability

• Suppose $\hat{\pi}$ is generated by a non-relativistic astrophysical object. The object possibly gives a large kinetic matrix Z for $\delta\pi$ and healthy (shown by Nicolis & Rattazzi) within r_V ($\sim$ kpc). One sees that within this region the coupling $\delta\pi T$ is suppressed and GR is recovered at small scales (non-linear).

• Can this happen to the instability induced by GWs ? Suppose again $\hat{\pi}$ is sourced by an astrophysical object. $\delta\pi$ seems to acquire a large Z. The parameter β of $\gamma\pi\pi$ seems to get suppressed due to a large Z and the instability might be stopped by this screening mechanism. But this is not the case for the GWs traveling over the cosmo. distances ($\gg$ the typical r_V) since at large distances one expects the linear perturbation theory is recovered, so that the Vainshtein mechanism is negligible. Hence, the argument of having large Z to suppress the instability is not applicable in the presence of GWs traveling over cosmo. distances and the instability still remains active.

Instability of plane wave $\hat{\pi}$

- Let $\hat{\pi} = Af(u)$ (consider u-direction)
- $\Box\hat{\pi}(u) = -4\partial_u\partial_v\hat{\pi}(u) = 0$
- $\partial_t^2\hat{\pi}(u) = Af''(u)$, $\partial_z^2\hat{\pi}(u) = Af''(u)$, $\partial_t\partial_z\hat{\pi}(u) = -Af''(u)$
- Without GWs: $Z^{\mu\nu} \equiv -\frac{1}{2}\eta^{\mu\nu} - 2\left(\mathcal{K}^{\mu\nu} - \eta^{\mu\nu}\mathcal{K}\right)$, we have

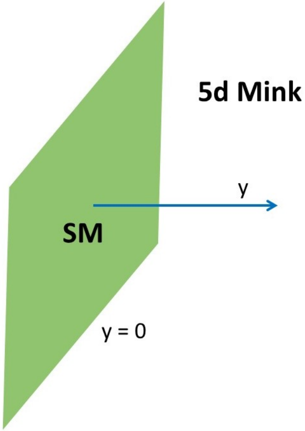
$$Z^{00} = \frac{1}{2} + 2\frac{Af''(u)}{\Lambda_{\rm B}^3}\,, \quad Z^{33} = -\frac{1}{2} + 2\frac{Af''(u)}{\Lambda_{\rm B}^3}\,, \quad Z^{03} = 2\frac{Af''(u)}{\Lambda_{\rm B}^3}$$

$Z^{11} = Z^{22} = -1/2$

- Ghost: $Z^{00} < 0 \Rightarrow Af'' < -\Lambda_{\rm B}^3/4$.
- No gradient: $Z^{11}, Z^{22} < 0$ and $(Z^{03})^2 - Z^{33}Z^{00} = 1/4 > 0$
- Non-diagonalizable $Z^{\mu\nu}$: $2|Z^{03}| = |Z^{00} + Z^{33}|$ (avoid the theorem)

DGP - Self-accelerating Universe

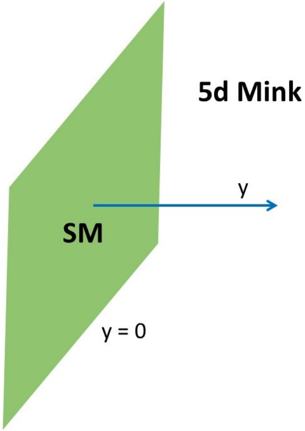
$\kappa = M_5^3/M_{\rm Pl}^2$, $r_c = 1/\kappa \sim H^{-1}$



- At decoupling limit of DGP $\Rightarrow$ $(\partial\pi)^2\Box\pi/\Lambda^3$, $\Lambda^3 = M_{\rm Pl}\kappa^2$

DGP - Self-accelerating Universe

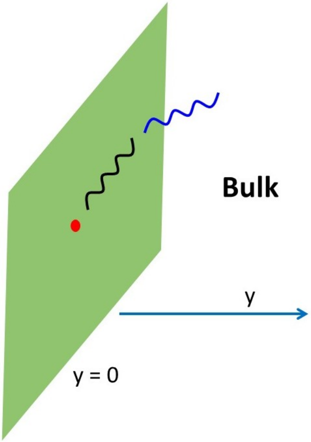
$\kappa = M_5^3/M_{\rm Pl}^2$, $r_c = 1/\kappa \sim H^{-1}$



- At decoupling limit of DGP $\Rightarrow$ $(\partial\pi)^2\Box\pi/\Lambda^3$, $\Lambda^3 = M_{\rm Pl}\kappa^2$
- Self-accelerating universe: $H = 2M_5^3/M_{\rm Pl}^2$
- The brane bending mode π becomes ghost in self-accelerating branch
- π-instability w/ GWs ? (in progress)

Dvali, Gabadadze and Porrati 2000,
Luty et al. 2003 and many others

DGP - Infrared transparency



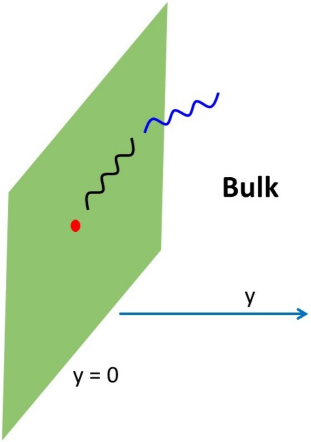
- A source $J(x, y) \sim \delta(y)\delta^{(3)}(x)e^{i\omega t}$
- For $r \ll \omega r_c^2$: 4d behaviour

$$G(\omega, r) \sim \frac{e^{-i\omega r}}{r}$$

- For $r \gg \omega r_c^2$: 5d behaviour

$$G(\omega, r) \sim \frac{r_c\sqrt{\omega}e^{-i\omega r}}{r^{3/2}}$$

DGP - Infrared transparency



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- Further applications ? (in progress)

- Similar effect for localized gauge field (Dvali, Gabadadze, and Shifman 2000)

Backup Inflation

EFT of Inflation: Large field limit

• Take $M_2 = M_3 = 0$. The operators $\pi^{N>4}$ are suppressed by $g_{\rm NL}$. $\pi \to -\pi$ is an approx. symmetry when $g_{\rm NL} >> 1$. The operators with odd power will then be suppressed by $g_{\rm NL}$.

• Loop corrections to $M_2(\delta g^{00})^2$ and $M_3(\delta g^{00})^3$ also are suppressed by $g_{\rm NL}$ since their leading terms are odd in π.

• What about $(\delta g^{00})^n$?

- For n odd, these will be suppressed by approx. symmetry

- For n even, no suppression $\Rightarrow$ consider all of them or the loop integral can be cut at $\Lambda < \Lambda_U$. At least they are down by $(\Lambda/\Lambda_U)^{\#}$. Otherwise UV completion is needed

• $\delta g^{00} = 1 + g^{00} \to -2\dot{\pi} + (\partial_\mu \pi)^2$ under $t \to t + \pi$

EFT of Inflation: Large field limit

- $\mathcal{L}_\zeta$ from $(\delta g^{00})^4$: $(\zeta = -H\pi)$

$$S_\zeta = \int d^4x\sqrt{-g}\,\frac{|\dot{H}|M_{\rm Pl}^2}{H^2}\left[(\partial_\mu\zeta)^2 + g_{\rm NL}\frac{1}{H^2}\dot{\zeta}^4 + g_{\rm NL}\frac{1}{H^3}\dot{\zeta}^3(\partial_\mu\zeta)^2 + \dots\right]$$

- Comparison with $\mathcal{L}_2$:

$$\frac{\mathcal{L}_4}{\mathcal{L}_2} \sim g_{\rm NL}\zeta^2 \sim 1\ , \quad \frac{\mathcal{L}_5}{\mathcal{L}_2} \sim g_{\rm NL}\zeta^3 = g_{\rm NL}\zeta^2\zeta \ll 1$$

for $g_{\rm NL} \gg 1$ (Exp. $g_{\rm NL} \ll 10^6$).

- $\mathcal{L}_5$ becomes important $(g_{\rm NL}\zeta^3 \sim 1)$ when $g_{\rm NL}\zeta^2 \gtrsim g_{\rm NL}^{1/3}$
- If $\zeta \sim 1 \Rightarrow$ all the terms inside each $(\delta g^{00})^n$ are important, e.g. $\mathcal{L}_5/\mathcal{L}_4 \sim \zeta$.

Connection to Stochastic approach

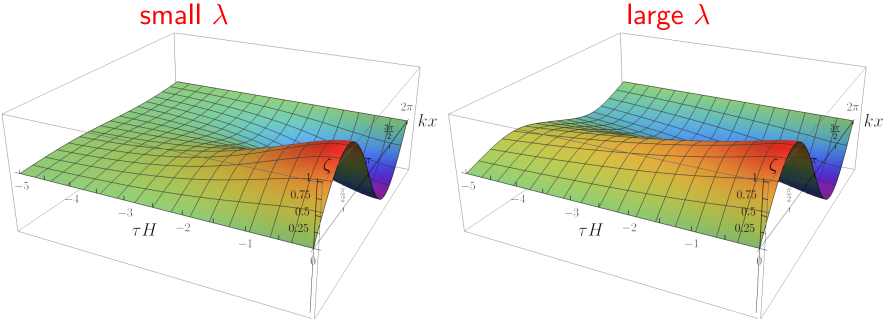
• They study the evolution of ζ_L outside horizon, compute its PDF using Fokker-Planck eq and treating ζ_S as a quantum noise.

• Instead, our ζ freezes outside horizon (single field inflation). We study non-perturbative effects at horizon crossing (derivative interaction), which are fully quantum-mechanical. Therefore, our non-perturbative result has no direct connection to the stochastic approach

• Some recently papers address this issue (connection between stochastic approach and standard PT), e.g. Cruces and Germani 21, Green et al. 21, Starobinsky et al. 21

PDE with sinusoidal profile

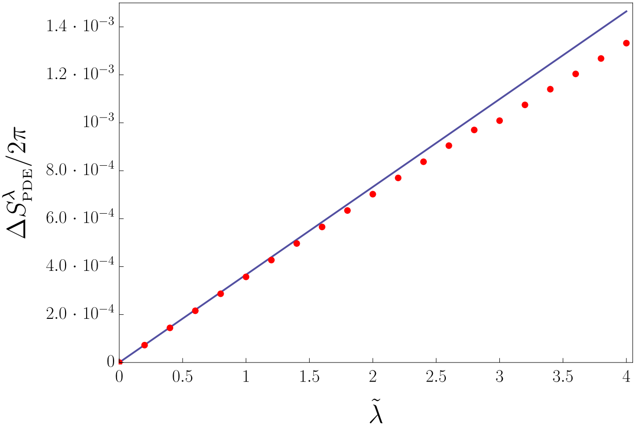
- The Gaussian profile at η_c: $\zeta(r) \sim \zeta_0 \sin(kx)$



- This can be easily checked with perturbation theory

Perturbative check with PDE sine wave

- Perturbative 4-pt: $iS'_{\rm int} = 3\zeta_0^2\tilde{\lambda}k^3/(8192P_\zeta)$
- Numerical: $\Delta S_{PDE}^{\tilde{\lambda}} = -(\Delta S_{PDE} - \Delta S_{PDE}^0)$



Non-linearity $\neq$ Breaking down of EFT

• It is not generally true that when the non-linearities become important the EFT we are considering necessarily breaks down

• Take GR in which all the non-linear terms are controlled by diff-invariance but the EFT (GR) is still valid as long as ∂/Λ is small

• It's the same spirit as one considers the Vainshtein mechanism where there is a regime which is dominated by non-linear term, but the EFT is still valid

• The issue of instabilities has to be taken care of separately. We are not saying that all the solutions to the non-linear EoM are healthy (also it depends on the background we are expanding around). The presence of instabilities might signal the need of the UV completion.

Non-linearity $\neq$ Breaking down of EFT

• Take $X + X^2$ in which the UV completion is known, but it does not mean that once the non-linearity becomes important the IR theory breaks down

• One can also take the DBI action and work out all the non-linear terms of DBI around $\phi_0(t)$. Again the EFT action is valid even though the non-linearities become important

• Of course the question whether the UV completion exists or not is interesting on its own, but it does not really mean that the EFT breaks down once the non-linear terms are important

Non-linearity $\neq$ Breaking down of EFT

- UV completion of $X + X^2$
- $\mathcal{L}_{\rm IR} = \mathcal{P}(X)$ with constant X background $\Rightarrow$ Ghost + gradient inst.

The non-linear terms are contained in X^2

- $\mathcal{L}_{UV} = -|\partial\phi|^2 - \lambda(|\phi|^2 - v^2)^2$ $\phi = \phi_0 e^{i\pi}$, $\langle\phi_0\rangle = v^2 - \frac{X}{2\lambda}$, $X = -(\partial\pi)^2$
- Around $\phi_0(t)$, $X + \beta X^2$ yields

$$S_E = i\int d\eta d^3x \frac{1}{P_\zeta}\left\{\frac{1}{2\eta^2}[\zeta'^2 + (\partial_i\zeta)^2] + \frac{\lambda}{4!}(\partial_i\zeta)^4 + \frac{\lambda c_s^2}{6\eta}\zeta'(\partial_i\zeta)^2 + \frac{\lambda c_s^2}{12}\zeta'^2(\partial_i\zeta)^2 + \frac{\lambda c_s^4}{6\eta}\zeta'^3 + \frac{\lambda c_s^4}{4!}\zeta'^4\right\}$$

No suppression due to small c_s^2 since $c_s^2 = (1+\beta\dot\phi_0^2)/(1+3\beta\dot\phi_0^2) \in (1/3, 1)$

- The suppression happens for $-X + \beta X^2$ for small $c_s^2 \in (0, 1/3)$

Analytic Continuation

• The Euclidean rotation $\eta \to iz$ can be easily shown in perturbation theory - order by order of the solution given the source is analytic

$$\zeta(\eta, \boldsymbol{k}) = K(\eta, \boldsymbol{k})\zeta^0_{\boldsymbol{k}} + \int_{-\infty(1-i\epsilon)}^{\eta_c} \mathrm{d}\eta' \, G(\eta, \eta'; \boldsymbol{k}) \frac{\delta S_{int}}{\delta\zeta(\eta', \boldsymbol{k})}$$

$K(\eta, \boldsymbol{k})$ is bulk-boundary propagator

$$K(\eta, \boldsymbol{k}) = \frac{(1 - ik\eta)}{(1 - ik\eta_c)} e^{ik(\eta - \eta_c)}$$

Analytic Continuation

The bulk-bulk propagator

$$G(\eta,\eta';\boldsymbol{k}) = \frac{-iH^2}{2k^3}\left[\phi_+(\eta)\phi_-(\eta') - \frac{\phi_-(\eta_c)}{\phi_+(\eta_c)}\phi_+(\eta')\phi_+(\eta)\right], |\eta| > |\eta'|$$
$$= \frac{-iH^2}{2k^3}\left[\phi_+(\eta')\phi_-(\eta) - \frac{\phi_-(\eta_c)}{\phi_+(\eta_c)}\phi_+(\eta')\phi_+(\eta)\right] \ , |\eta| < |\eta'|$$

$$\phi_-(\eta) \equiv (1+ik\eta)e^{-ik\eta} \ , \quad \phi_+(\eta) \equiv (1-ik\eta)e^{ik\eta}$$

Euclidean Path-Integral

• Is this the only real solution in Euclidean space ?

- If yes, the Picard-Lefschetz thimbles $\Rightarrow$ it is the only saddle that contributes to path integral

- If not, there are contributions from complex saddles and one needs to worry about the Stokes phenomenon (the jump in asymptotic behaviour $\Rightarrow$ other saddles can dominate)

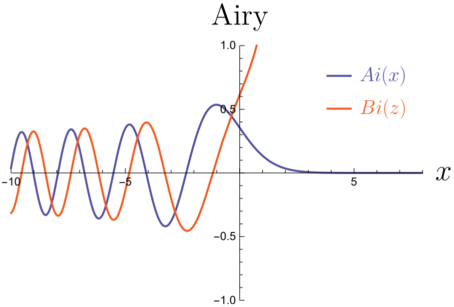
• In QM with quartic potential, there is only one real solution (Serone, Spada, and Villadoro 17)

Stokes phenomenon of Airy function

$$Ai(x) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\ e^{\frac{i}{3}t^3+ixt}$$

- For $x \in \mathcal{R}^+$, two imaginary saddles: $\pm i\sqrt{|x|} \Rightarrow$ Oscillatory
- For $x \in \mathcal{R}^-$, two real saddles: $\pm\sqrt{|x|} \Rightarrow$ Decaying and growing (neglect the growing behaviour)

- Changing from negative to positive the integral is dominated by different saddles (Stokes phenomenon)



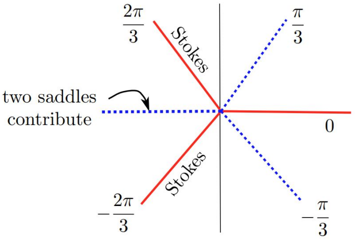
Stokes phenomenon of Airy function

- For complex Airy function

$$Ai(z) \sim z^{-1/4} \exp\left(-\frac{2}{3} z^{2/3}\right), \quad Bi(z) \sim z^{-1/4} \exp\left(\frac{2}{3} z^{2/3}\right)$$

- Stokes lines: $Im(z^{2/3}) = 0 \Rightarrow \arg(z) = 0, \pm 2\pi/3$
- Anti-Stokes lines: $Re(z^{2/3}) = 0 \Rightarrow \arg(z) = \pm\pi/3, \pi$

- $Ai(z)$ is subdominant in $-\pi/3 < \arg(z) < \pi/3$, dominant otherwise
- $Bi(z)$ is dominant in $-\pi/3 < \arg(z) < \pi/3$, subdominant otherwise



(Mariño, Pasquetti, and Putrov 10)

Analytic Continuation beyond PT

Real time Path-Integral

• Not well-defined because of huge oscillatory behaviour

• Need to give $i\epsilon$ to have a well-defined integral

- How many saddle points are there ? All of them contribute to path-integral ?
- Are they analytic ? If yes, the full rotation to Euclidean can be done

• For real time instanton, the on-shell action with $i\epsilon$ is the same as the on-shell Euclidean action (Cherman and Unsal 14)

• For real time quantum tunneling, the solution with $i\epsilon$ admits poles and zeros in complex t-plane (Turok 14)

Analytic Continuation beyond PT

- From dS to Euclidean AdS

$$ds^2 = \frac{1}{H^2\eta^2}(-d\eta^2 + d\boldsymbol{x}^2)$$

Perform $\eta \to iz$ and $H \to i/L$

$$ds^2 = \frac{L^2}{z^2}(dz^2 + d\boldsymbol{x}^2)$$

- It has been shown that the functional integral can be analytically continued from dS to EAdS once restricted on the analytic functions (Harlow and Stanford 11)