p-complete arc-descent for finite projective modules over perfectoid rings

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Outline

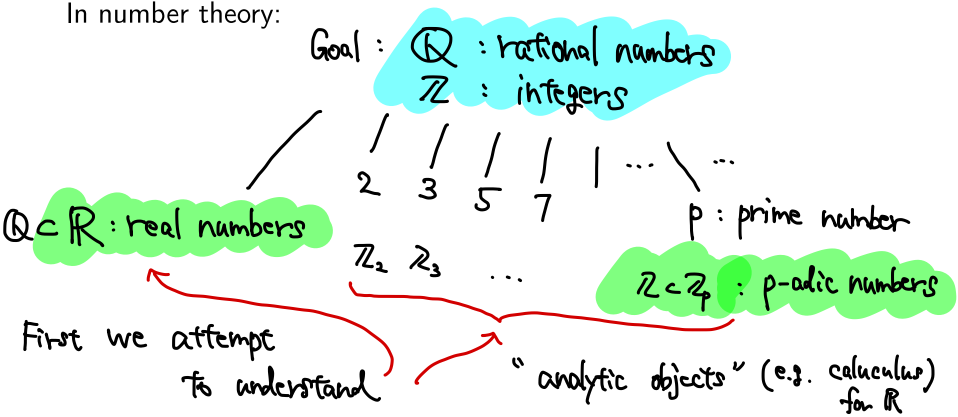
1. Introduction
2. Classification of p-divisible groups
3. p-complete arc-descent for finite projective modules over perfectoid rings

Introduction

Slogan for today's talk

There are a lot of "reduction (descent) arguments" in mathematics.

In number theory:



Introduction

Here $\mathbb{Z}_p \supset \mathbb{Z}$ is the commutative ring obtained by adjoining to $\mathbb{Z}$ some formal power series, such as

$$1 + p + p^2 + \ldots + p^n + \ldots .$$

Formally,

$$\mathbb{Z}_p := \{\sum_{n \geq 0}^{\infty} a_n p^n \mid a_n \in \{0, 1, 2, \ldots, p-1\}\}.$$

Example: Hasse-Minkowski theorem

Let $f(X, Y) := aX^2 + bY^2 - 1$ with nonzero $a, b \in \mathbb{Q}$. The following are equivalent.

1. $f(X, Y) = 0$ has solutions in $\mathbb{Q}$.
2. $f(X, Y) = 0$ has solutions in $\mathbb{R}$ and $\mathbb{Z}_p[1/p]$ for all prime numbers p.

Introduction

Remark

- $f(X, Y) = 0$ has solutions in $\mathbb{R} \Leftrightarrow a > 0$ or $b > 0$.
- $\exists$ explicit criterion "in terms of $\mathbb{Z}/p\mathbb{Z}$" for the existence of solutions to $f(X, Y) = 0$ in $\mathbb{Z}_p[1/p]$. Here

$$\mathbb{Z}/p\mathbb{Z} := \{0, 1, 2, \ldots, p-1\}$$

is the set of integers modulo p.

- As above, sometimes, we can reduce certain problems for $\mathbb{Z}_p$ to questions about $\mathbb{Z}/p\mathbb{Z}$.
- This is one of the motivations for studying algebraic geometry in positive characteristic (in the world where $p = 0$).
- Advantage: commutative rings R over $\mathbb{Z}/p\mathbb{Z}$ have the following "symmetry".

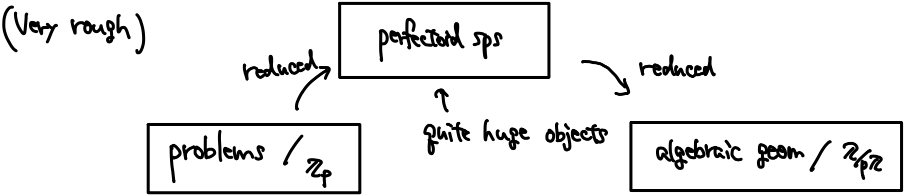
$\varphi\colon R \to R$, $x \mapsto x^p$ defines a ring homomorphism, called Frobenius.

Introduction: Perfectoid rings

- Focus on $\mathbb{Z}_p$.
- Strong tool: analytic spaces (rigid spaces) over $\mathbb{Z}_p$.

Perfectoid

In 2011, Scholze introduced the notion of a perfectoid space, which presented a new framework to reduce certain problems for $\mathbb{Z}_p$ to corresponding problems in positive characteristic.



Remark The notion of a perfectoid space plays an important role in mathematics, e.g. Langlands program. (However it is not so easy to handle. It is based on many p-adic techniques.)

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Classification of p-divisible groups: Motivation

- The theory of Shimura varieties lies at the heart of the Langlands program.
- Shimura varieties (over $\mathbb{C}$) are roughly the moduli spaces of abelian varieties (complex tori $\mathbb{C}^n/\Lambda$).

Example: moduli $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ of elliptic curves $\mathbb{C}/\Lambda$, where $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.

- A: abelian variety
  $\rightarrow$ the p-power torsion part $A[p^\infty]$ is a p-divisible group.
- classification of p-divisible groups over $\mathbb{Z}_p$ (more generally $\mathcal{O}_K$)
  $\Rightarrow$ good models of Shimura varieties over $\mathbb{Z}_p$, used to compute etale cohomology.

Remark

Shimura varieties over $\mathbb{Z}_p$ have applications to the Tate conjecture for K3 surfaces.

Classification of p-divisible groups

Definition: p-divisible group

Let S be a scheme. A p-divisible group over S is an inductive system $\{(G_n, \iota_n \colon G_n \to G_{n+1})\}_{n\in\mathbb{N}}$ of finite locally free commutative group schemes G_n over S such that:

1. ι_n induces $G_n \cong G_{n+1}[p^n]$.
2. The rank of G_n is p^{nh} where h (function) is independent of n.

Example: $\{A[p^n]\}_n$ for an abelian scheme A.

Classification over $\mathbb{Z}_p$ (more generally $\mathcal{O}_K$) (Kisin, Lau)

$\exists$ equivalence of categories

$$\{p\text{-divisible groups over } \mathbb{Z}_p\} \cong \{\text{Breuil-Kisin modules over } \mathbb{Z}_p[[T]]\}$$

where a Breuil-Kisin module over $\mathbb{Z}_p[[T]]$ is a finite free module M over $\mathbb{Z}_p[[T]]$ with Frobenius $\varphi \colon M \to M$. (Linear algebraic object)

A new approach to the classification over $\mathbb{Z}_p$

- A geometric valuation ring V is a p-complete valuation ring of rank ≤ 1 with algebraically closed fraction field.
- Example: $\mathcal{O}_{\mathbb{C}_p}$ for $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}_p}}$.
- $\{$perfectoid rings$\} \supset \{$geometric valuation rings$\}$

Classification over geometric valuation rings

$\Downarrow$

Classification over perfectoid rings

$\Downarrow$

Classification over $\mathbb{Z}_p$ (or $\mathcal{O}_K$)

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p-complete arc-descent

- A ring homomorphism $A \to B$ is a p-complete arc-cover if for any $A \to V$ with geometric valuation ring V there exists

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ V & \dashrightarrow & W \end{array}$$

  where $V \to W$ is an extension of geometric valuation rings.
- Example: Faithfully flat maps.
- {perfectoid rings} $\supset$ {geometric valuation rings} "forms a basis" for the p-complete arc-topology.

Definition: perfect ring

A commutative ring R over $\mathbb{Z}/p\mathbb{Z}$ is perfect if $\varphi\colon R \to R$, $x \mapsto x^p$ is bijective. We have {perfectoid rings} $\supset$ {perfect rings}.

p-complete arc-descent

Fact (Bhatt-Scholze, Bhatt-Mathew)

Let $Y = \mathrm{Spec}(B) \to X = \mathrm{Spec}(A)$ be a p-complete arc-cover of perfect rings over $\mathbb{Z}/p\mathbb{Z}$. Then $\exists$ equivalence of categories

$$\{\text{vector bundles over } X\}$$
$$\cong \{(\mathcal{V}, \sigma) \,|\, \mathcal{V} : \text{vector bundle over } Y \text{ and } \sigma : \text{descent datum}\}.$$

A descent datum on $\mathcal{V}$ is an isomorphism $\sigma\colon p_1^*\mathcal{V} \cong p_2^*\mathcal{V}$ satisfying a certain cocycle condition, where $p_1, p_2\colon Y \times_X Y \to Y$ are the projections.

Theorem (Ito)

The same statement holds for perfectoid rings.

p-complete arc-descent

Key

For the proof of Theorem, we need ∞-categories and higher algebras ($\mathbb{E}_\infty$-rings).

Remark : $\{\mathbb{E}_\infty\text{-rings}\} \supset \{\text{commutative rings}\}$

$\{\infty\text{-categories}\} \supset \{\text{categories}\}$

Descent for perfect rings

$\Downarrow$

Descent for $A/^{\mathbb{L}}p$

$\Downarrow$

Descent for perfectoid rings

Thank you very much for your attention!