Coulomb Branches of 3d Supersymmetric Gauge Theories

construction of new spaces

?

HIRAKU NAKAJIMA

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—— joint works with Braverman, Finkelberg

Motivated by theoretical physics — Seiberg, Witten, Hanany, Kapustin, ...

mid 90's ~

Coulomb branch = a branch of the moduli space of vacua in a supersymmetric gauge theory

Mathematical Approach — More recent 2015 ~ (It took ~ 20 years.)

We regard our result as :

a construction of new spaces.

$$\begin{tikzcd} & \text{3d SUSY gauge theory} \arrow[dr] & \\ (\underbrace{G}_{\text{cpt Lie group}}, \underbrace{N}_{\text{representation}}) \arrow[ur] \arrow[rr, Rightarrow] & & \text{Coulomb br.} \end{tikzcd}$$

— framework of algebraic geometry

— technique from geometric representation theory

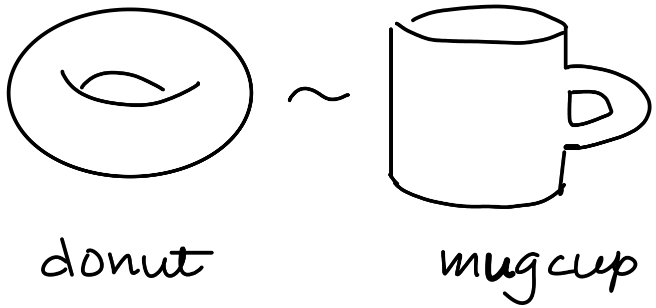
— idea from topological quantum field theory TQFT

A **space** is a set with some added structure.

I consider **geometric** ones.

Branches of Mathematics discussing geometric structures

- **differential geometry** (**Riemannian geometry**)
- **topology** → study properties preserved under continuous deformations
- **algebraic geometry**



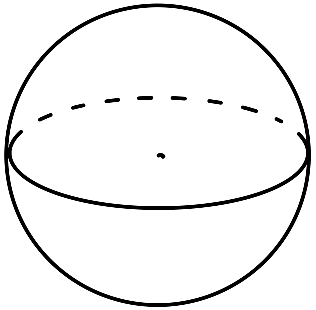
Classically study of zero of **polynomials**

Use algebraic techniques on **commutative algebras**

Algebraic Geometry

spaces vs commutative algebras
(the language of schemes)

Idea
Recall spaces $\longleftrightarrow$ equations by **coordinates**

Ex. sphere  $\longleftrightarrow$ $x^2+y^2+z^2=1$

x, y, z are functions on the sphere.
We can add and multiply functions.

$\longrightarrow$ Polynomials in x, y, z are functions on the sphere.

$\longrightarrow$ Functions form a commutative algebra.

$$\uparrow$$
$$fg = gf$$

We regard $x^2+y^2+z^2=1$ as an equality on functions.

On the other hand, we do not have such equality on 3-dimensional Euclidean space $\mathbb{R}^3 = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$.

$\longrightarrow$ Commutative algebras of all functions on sphere & $\mathbb{R}^3$ are different.

Many Mathematician use complex numbers $\mathbb{C}$
rather than real numbers $\mathbb{R}$

polynomial functions / holomorphic func's
rather than smooth/continuos functions.

Space X (in algebraic geometry) $\underset{\text{maximal ideals}}{\overset{\text{functions}}{\rightleftarrows}}$ a commutative algebra $\mathbb{C}[X]$

Technical Term:
an affine scheme $\xrightarrow{\text{gluing}}$ a more general **scheme**

We can construct **many** examples of spaces X from **various** equations

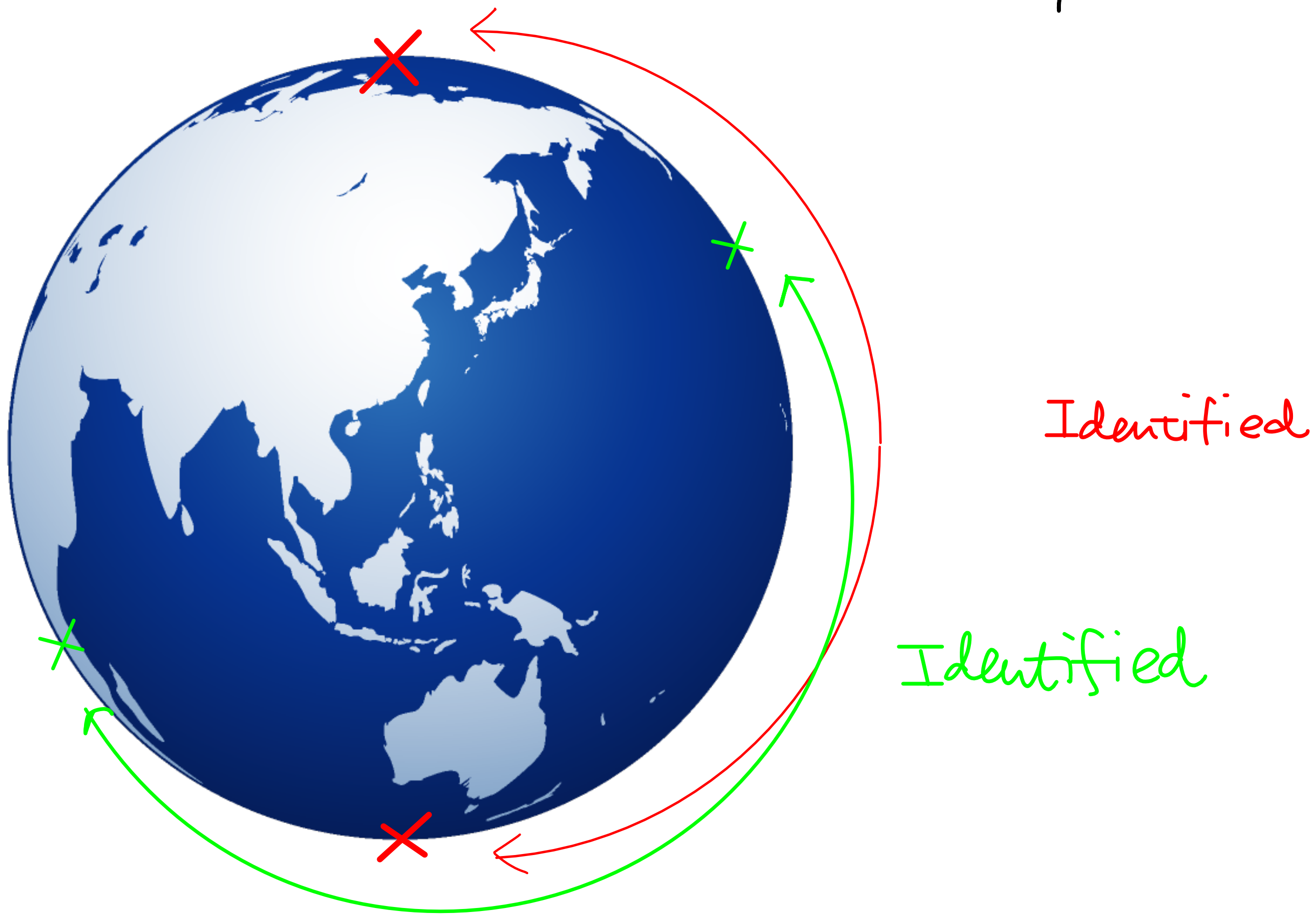
But that is <u>not</u> the only way:

<u>Example</u> "quotient" $X /\!\!/ G \rightleftarrows \mathbb{C}[X]^G$ algebra of invariants

$\longrightarrow$ geometric invariant theory
a method to construct new varieties

Example of a quotient space: (Real) Projective Plane $\mathbb{RP}^2$

↑ not complex



$\mathbb{RP}^2 = \text{sphere}/\mathbb{Z}/_2$ $\quad (x, y, z) \mapsto (-x, -y, -z)$

x^2, xy, etc are functions on $\mathbb{RP}^2$

Quotient spaces naturally appear in gauge theories, as we have **gauge symmetry**.

→ The group is much, much larger.
e.g. Map(space-time, $SU(2)$)

Geometric invariant theory (discussing quotients by e.g. $SL(2,\mathbb{C})$) is useful in algebro-geometric approaches in gauge theory.

⟶ quiver varieties
(spaces introduced in '90s)

Algebraic approach is powerful to

extend the notion of **spaces**.

e.g. — treat more general fields than $\mathbb{R}, \mathbb{C}$.

$\longrightarrow$ useful for number theory

— replace commutative algebras

by $\left(\begin{matrix}\text{differential} \\ \text{graded}\end{matrix}\right)$ noncommutative algebras

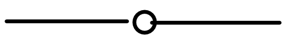
$\hookrightarrow$ spaces without points

Topology study properties preserved under continuous deformations

Question : How to prove


line

and

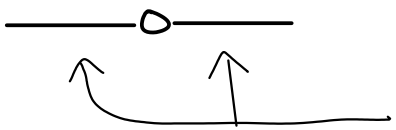

line − point

are not transformed by continuous deformation ?

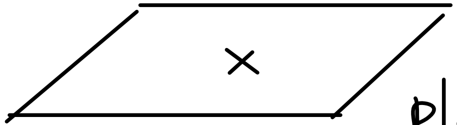
Answer : We can consider all continuous functions.
But more simply consider

the space of locally constant functions.

e.g. $df = 0$, independent of continuous deformation



constant can be different.

How about  plane vs  plane minus point ?

<u>Hint</u> : Residue theorem

$$\int_{C=\{|z|=1\}} \frac{dz}{z} = 2\pi i$$

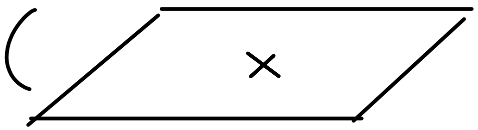
If we replace $\frac{1}{z}$ by $f(z)$: holomorphic function defined on $\mathbb{C}$, it changes to 0.

We should consider differential forms instead of functions.

$\longrightarrow$ This leads to **cohomology** $H^*(X)$ of a space X

graded $\mathbb{C}$-vector space

e.g. $H^0(X)$ = space of locally constant functions

$H^1($  $)$ = 1-dimensional vector space

$H^*(X)$ is unchanged under continuous deformation.

$\therefore$ $H^*(X) \neq H^*(Y) \Longrightarrow X$ is not deformed to Y.

Cohomology and homology (the "dual" notion) were

— introduced in the 1st half of 20th century in topology

— used also in other branches of mathematics.

Let us start to explain our definition of Coulomb branches.

Coulomb branch $\mathcal{M}_C(G,N)$ is an (affine) scheme.

Therefore we instead define

a commutative algebra $\mathbb{C}[\mathcal{M}_C(G,N)]$.

We construct $\mathbb{C}[\mathcal{M}_C(G,N)]$ by technique of geometric representation theory.

Representation theory usually study

noncommutative algebras.

Prototype of the construction: group ring $\mathbb{C}[G]$ of a finite group G

$\mathbb{C}[G] = \{ \varphi : G \to \mathbb{C} \}$ ~ This is commutative, but we consider different multiplication group ring $\mathbb{C}[G]$ of a finite group G

$$\mathbb{C}[G] = \{ \varphi : G \to \mathbb{C} \}$$

$\varphi = \sum a_g g$, $\psi = \sum b_h h$

$\varphi * \psi = \sum a_g b_h \, gh$

$$(\varphi * \psi)(x) \overset{\text{def.}}{=} \sum_{y \in G} \varphi(y)\psi(y^{-1}x)$$

convolution product

<u>Remark</u> : $\mathbb{C}[G]$ is commutative $\Longleftrightarrow$ G : abelian

We use the same idea for $G \rightsquigarrow$ a certain topological space $\mathcal{R}$

$\mathbb{C}[G] \rightsquigarrow$ its homology group

The space $\mathcal{R}$ has a similar property as a $\underbrace{\text{finite group } G}_{\text{commutative}}$

so that one can define the $\underbrace{\text{convolution}}_{\text{commutative}}$ product.

Which topological space $\mathcal{R}$?

Ans. moduli space , arising from the gauge theory for S^2

3d TQFT : 3-manifold $M^3 \rightsquigarrow$ number $Z(M^3)$

2-manifold $\Sigma^2 \rightsquigarrow$ vector space $Z(\Sigma^2)$

3-manifold M with bdry ∂M $\rightsquigarrow Z(M) \in Z(\partial M)$

+ composition of cobordisms $\rightsquigarrow$ composition of linear maps etc.

Then  $D \setminus D_1 \cup D_2 \rightsquigarrow Z(S^2) \otimes Z(S^2) \longrightarrow Z(S^2)$

commutative multiplication

The TQFT is not yet rigorously defined, but

$Z(S^2)$ + commutative multiplication — realized by homology of moduli spaces and convolution.

Merit Coulomb branches and moduli spaces are connected. $\longrightarrow$ 3d mirror symmetry / symplectic duality

cf. Usual mirror symmetry X vs $X^\vee$

counting curves vs period integrals

Open Problem Hyperkähler metrics on Coulomb branches?

In fact, it is more interesting in 4d gauge theories

— should be related to instanton counting, as Seiberg-Witten curves are involved in HyperKähler metrics.