# 1. Topological models

Skein algebra        bases

$\cap$

Cluster algebra :

- related to exact WKB analysis and physics :

  see Gaiotto - Moore - Neitzke , 0907.3987 (Fock - Goncharov coord.)

  Iwaki - Nakanishi , 1401.7094

- survey. Bernhard keller   www.newton.ac.uk/seminar/34042

- some slides wt references on my homepage

## § 1 Skein algebra

base ring $\mathbb{K} = \mathbb{Z}$

$\Sigma = (S, M)$ $\quad S$ : topological surface $\quad M$ : marked pts

s.t. each connected component of $\partial S$ contains $\geq 1$ marked pt

curve $\gamma$ : ending at $M$, or closed

diagram (multicurve) $D = \cup \gamma_i$ :

considered up to isotopy fixing $M$, crossing

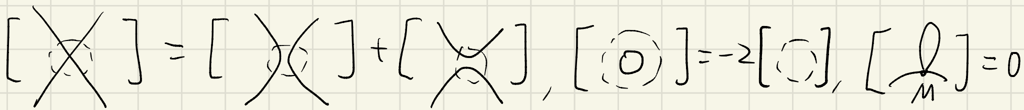
$[D]$ : isotopy class. Sometimes write $D$.

"simple" : no crossing, no contractible component

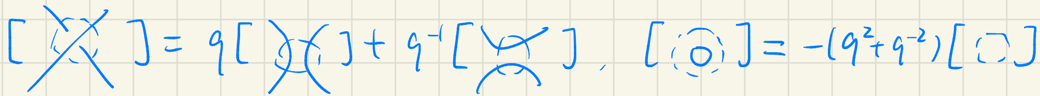
Skein alg $\quad \mathrm{SK}(\Sigma) := \bigoplus_{\forall [D]} \mathbb{K}[D] \Big/$ kauffman Skein relations

multiplication $\quad [D_1] \cdot [D_2] = [D_1 \cup D_2]$

$[\phi] = 1$

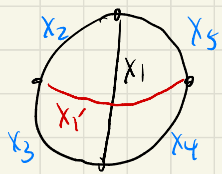



Remark: quantization $\mathbb{K} = \mathbb{Z}[q^{\pm}]$.

Remark: Skein relation ⟵ knot theory. Teichmüller theory

Example 4-gon




Interesting curves $X_1, X_1'$ : internal (unfrozen)

$X_2 \sim X_5$ : boundary (frozen)

$Sk(\Sigma) = \mathbb{K}[X_1, X_1', X_2 \sim X_5] / X_1 X_1' = X_3 X_5 + X_2 X_4$

arc := simple curve ending at M (cluster variables)

triangulation $\Delta$ = max collection of diff arcs (cluster)

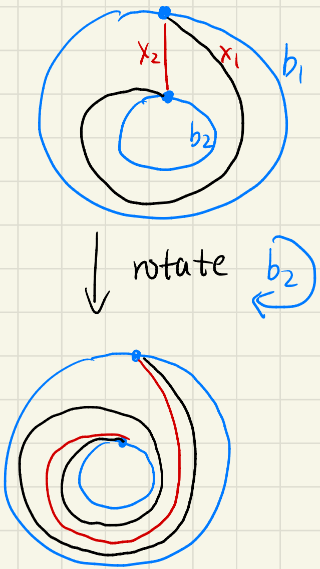
cluster monomial = monomials of $[\gamma]$. $\gamma \in$ same $\Delta$. eg. $X_1 X_2^2$, $X_1' X_2^2$

$\Longrightarrow$ they form a $\mathbb{K}$-basis for $Sk$(4-gon)

# §2 Bases for $SK(\Sigma)$

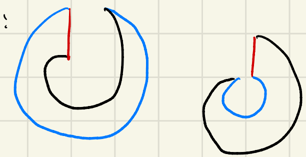
Focus on annulus example ( generalization is easy )

$\Sigma$




Initial $\Delta = \{x_1, x_2\} \cup \{b_1, b_2\}$

2 triangles :



Repeat. we see $\Sigma$ has inf. many $\Delta$.

Coefficient ring $R = \mathbb{K}[\text{ frozen variables } b_1, b_2 ]$

$SK(\Sigma)$ is an $R$-alg.

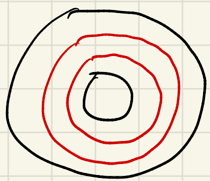
## § 2.1 Bangle

bangle = simple diagram (no self crossing, no contractible component)

internal bangle : no boundary arcs

For annulus, internal bangles = $\{$unfrozen cluster monomials$\}$ e.g. $x_1 x_2^2$

$\cup \{ [L]^n, n \geq 1 \}$



$[L \cup L] = [L]^2$

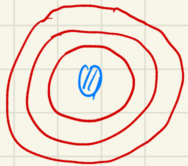
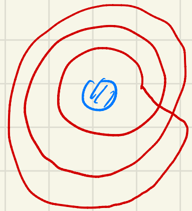
By Skein relation, any diagram $[D] \in \mathrm{Span}_R \{[\text{internal bangle}]\}$

Thm [Musiker-Schiffler-Williams]

$\{[\text{internal bangle}]\}$ is an $R$-basis of $Sk(\Sigma)$.

(Equivalently, $\{[\text{bangle}]\}$ is a $\Bbbk$-basis.)

## §2.2 Bracelet

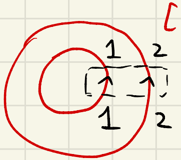
example  LULUL $\Longrightarrow$  $\mathrm{Brac}^3(L)$

closed loop
wt min-crossing

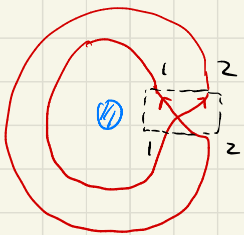
Thm [MSW] $Sk(\Sigma)$ has the $R$-basis:

$\{[\text{internal bracelet}]\} = \{\text{unfrozen cluster monomials}\} \cup \{[\mathrm{Brac}^n(L)] \; n \geq 1\}$

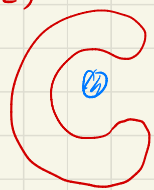
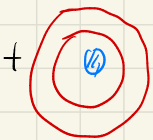
## §2.3 Band

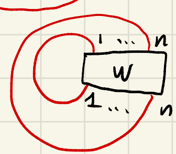



$[L]^2$




$\mathrm{brac}^2(L) = -2 + [L]^2$

$=$  $+$ 

$\mathrm{Band}^n(L) := \dfrac{1}{|G_m|} \sum_w$ 


Eg. $\mathrm{Band}^2(L) = \frac{1}{2}(L^2 + (-2 + L^2))$

$= L^2 - 1$

So $Sk(\Sigma)$ has an $R$-basis: $\{[\text{internal band}]\} = \{\text{cluster monomials}\}$

$\cup \{[\mathrm{Band}^n(L)]\}$

1st kind Chebyshev polynomial $T_n(x)$ : $T_n(x+x^{-1}) = x^n + x^{-n}$

2nd $U_n(x)$ : $U_n(x+x^{-1}) = x^n + x^{n-2} + \cdots + x^{-n}$

Then $[\mathrm{Brac}^n(L)] = T_n([L])$, $[\mathrm{Band}^n(L)] = U_n([L])$

Comments

| $SK(\Sigma)$ | | cluster alg | | Lie theory |
|---|---|---|---|---|
| [MSW] bangle basis | [G-Labardini-S.] $\subset$ | [Dupont] generic basis | [Q.] [GLS] $\supset$ | [Lusztig] dual semi-canonical basis |
| [FG][MSW] bracelet basis | [Mandel-Q.] $\subset$ | [GHKK] theta basis | $\supset$ ? | Mirković-Vilonen basis |
| [Thurston] band basis | $\subset$ ? | [Q.] triangular basis | $\supset$ [Q.] [KKKO] [KKOP] | [Lusztig] [Kashiwara] dual canonical basis, {simples} (upper global basis) |
| $\downarrow$ parametrized by | | $\downarrow$ | | $\downarrow$ |
| shear coordinates | | g-vectors (tropical pts) | | many rich structures, eg. crystals |

# 2. Cluster algebras

Fix $\Bbbk = \mathbb{Z}$ for simplicity. Unfrozen & frozen vertices $I = I_{uf} \sqcup I_f$

symmetrizers $d_i \in \mathbb{Z}_{>0}$, $i \in I$

## §1 Seed

a seed $t = ((X_i)_{i\in I}, (b_{ij})_{i,j\in I})$

- cluster variables $X_i$: indeterminate (unfrozen/frozen)
- cluster monomial $X^m = \prod X_i^{m_i}$, $\forall m = (m_i) \geq 0$
- $(b_{ij})$ is skew-symmetric: $b_{ij} d_j = -b_{ji} d_i$

  When skew-symmetric, associate quiver $\tilde{Q}$: $b_{ij} = \#(i \to j) - \#(j \to i)$

Lattice $M^\circ(t) := \mathbb{Z}^I = \bigoplus_i \mathbb{Z} f_i$, $f_i$ unit vectors

Group ring $\Bbbk[M^\circ(t)] := \bigoplus_{m \in M^\circ(t)} \Bbbk X^m \simeq$ Laurent poly ring $\Bbbk[X_i^{\pm}, i \in I]$

$X^{f_i} \mapsto X_i$

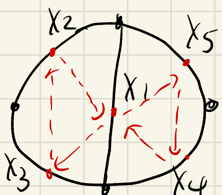
- $\mathcal{F}(t)$: its fraction field

- Y-variables $Y_j := \prod X_i^{b_{ij}} = X^{\mathrm{col}_k}$, $\mathrm{col}_k = k$-th col of $(b_{ij})$

Lattice $N(t) := \bigoplus \mathbb{Z}^I = \bigoplus \mathbb{Z} e_i$. unit vect. $e_i$

$$N_{uf}(t) = \bigoplus_{k \in I_{uf}} \mathbb{Z} e_k \,, \quad N_{uf}^{\geq 0}(t) = \bigoplus \mathbb{N} e_k$$

Example

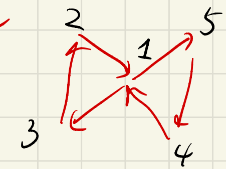



$\triangle \longrightarrow$ seed $t_\triangle$

$I = \{1\} \sqcup \{2 \sim 5\}$, $d_i = 1$

arcs     cluster variables

"dual graph"    $\tilde{Q}$




$$(b_{ij}) = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & * & \\ -1 & & 0 & & \\ 1 & * & & 0 & \\ -1 & & & & 0 \end{pmatrix}, \quad Y_1 = X_2 X_3^{-1} X_4 X_5^{-1}$$

$$X_1' = \frac{1}{X_1}(X_3 X_5 + X_2 X_4) = \frac{X_3 X_5}{X_1} \underbrace{(1 + Y_1)}_{\text{poly in } Y_k}$$

To study many nice structures, such as bases, quantization, need:

Full rank assumption:

$\mathrm{Col}_k$, $k \in I_{uf}$, are linearly independent. Equivalently, $\tilde{B} := (b_{ik})_{k \in I_{uf}}$ is of full rank.

Inspired by repr. theory!

## Dominance order [Q.17]

$\forall\, m, m' \in M^{\circ}(t)$, define $m' \preceq_t m$, if $\exists\, n \in N_{uf}^{\geq 0}(t)$ s.t. $m' = m + \tilde{B} n$.

(m dominated by $m'$)

i.e. $X^{m'} = X^m Y^n$

## Pointedness [Q.17]

- We say $Z \in \Bbbk[M^{\circ}(t)]$ is $m$-pointed or pointed at $m$, if

$$Z = X^m \cdot \Big(1 + \sum_{0 < n \in N_{uf}(t)} c_n Y^n\Big), \quad c_n \in \Bbbk.$$

Define $\deg^t Z = m$.

- We can call $F := 1 + \sum c_n y^n$ its $F$-polynomial, $m$ its (extended) $g$-vector.

## § 2 Cluster algebras

$[a]_+ := \max(a, 0)$, $\quad [(a_i)]_+ := ([a_i]_+)$

$\forall k \in I_{uf}$, mutation $\mu_k$ gives a new seed $t' = \mu_k t = ((X_i'), (b_{ij}'))$

- $b_{ij}' = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} + [b_{ik}]_+ [b_{kj}]_+ - [-b_{ik}]_+ [-b_{kj}]_+ & \text{else} \end{cases}$

- Identity $\mathcal{F}(t') \xrightarrow[\sim]{\mu_k^*} \mathcal{F}(t)$

$$X_i' \longmapsto \begin{cases} X_i & i \neq k \\ X_k^{-1} \left( \prod X_i^{[b_{ik}]_+} + \prod X_j^{[-b_{jk}]_+} \right) & i = k \end{cases}$$

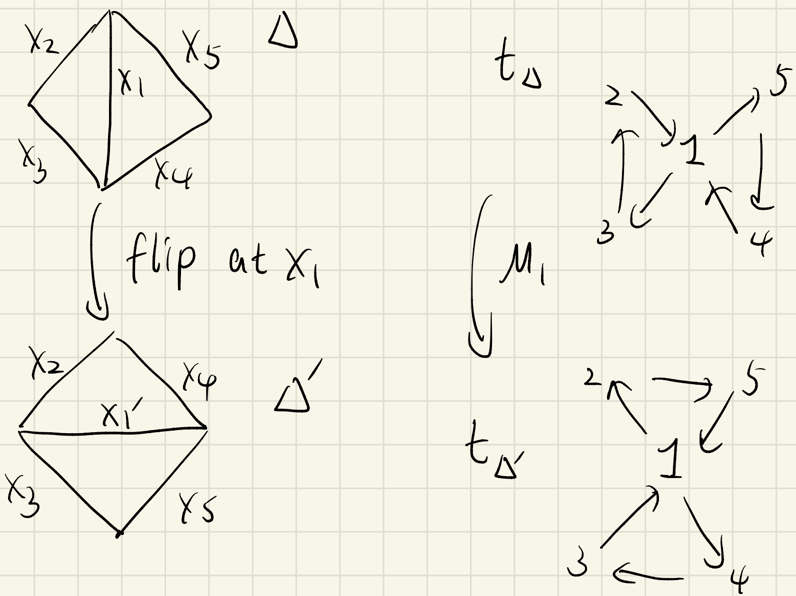
Lemma $\mu_k(\mu_k t) = t$

Choose any initial seed $t_0$, $\Delta^+ := \Delta^+_{t_0} := \{$ seeds obtained from $t_0$ by mutations $\}$

Partially compactified cluster algebra $\overline{\mathcal{A}} := \mathbb{k}[X_i(t)]_{\forall i,\, \forall t \in \Delta^t}$

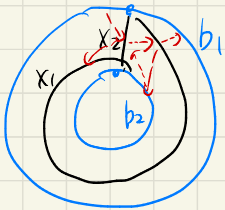
(localized) cluster algebra $\mathcal{A} := \overline{\mathcal{A}}[X_i^{-1}]_{i \in I_f}$ (frozen variables never mutate)

Example




Thm [Muller] $\overline{\mathcal{A}}(t_\Delta) \subset Sk(\Sigma)$. "$=$" if $\geq 2$ marked pts on each connected comp. of $\partial S$.

## §3 Example of bases




$$t_\Delta: \begin{tikzcd} & b_1 \arrow[dl] & \\ X_2 \arrow[rr, shift left] \arrow[rr, shift right] & & X_1 \arrow[ul] \arrow[dl] \\ & b_2 \arrow[ul] & \end{tikzcd}.$$

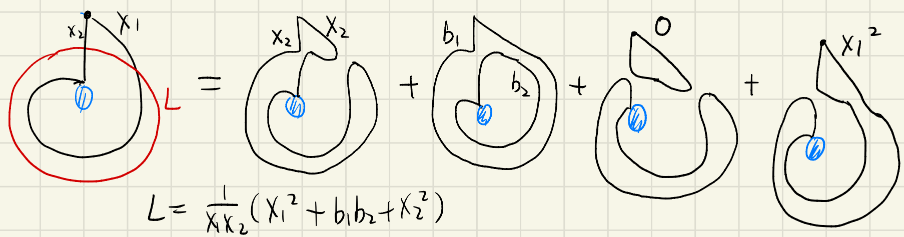
$$Y_1 = \frac{X_2^2}{b_1 b_2}$$

$$Y_2 = \frac{b_1 b_2}{X_1^2}$$

full subquiver on Iut

$$Q: 2 \rightrightarrows 1$$

Compute $L$:




$$L = \frac{1}{X_1 X_2}(X_1^2 + b_1 b_2 + X_2^2)$$

$$= X_1 X_2^{-1}(1 + Y_2 + Y_1 Y_2)$$

$Q^{op}: 1 \rightrightarrows 2$. $\underset{\text{abelian}}{\mathrm{Rep}(Q^{op})} = \{ \mathbb{C}^{d_1} \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} \mathbb{C}^{d_2} \} = \{\text{functor from } Q^{op} \text{ to } \mathbb{C}\text{-vect.sp}\}$

$\forall$ fixed $d = (d_1, d_2)$, $\mathrm{Rep}(Q^{op}, d) = \mathrm{Mat}_{d_2 \times d_1} \times \mathrm{Mat}_{d_2 \times d_1}$

Ind. obj, up to iso, are $V^{(r,r+1)}: \mathbb{C}^r \underset{\binom{0}{I}}{\overset{\binom{I}{0}}{\rightrightarrows}} \mathbb{C}^{r+1}$, $V^{(r+1,r)}: \mathbb{C}^{r+1} \underset{(0|I)}{\overset{(I|0)}{\rightrightarrows}} \mathbb{C}^r$

many indec. $\mathbb{C}^r \rightrightarrows \mathbb{C}^r$.

Ind injectives $I_1: \mathbb{C} \rightrightarrows 0$, $I_2$ $\mathbb{C}^2 \rightrightarrows \mathbb{C}$. $I^g := I^{\oplus g_1} \oplus I^{\oplus g_2}$ $\forall g = (g_1, g_2) \geq 0$

$\forall$ repr $V$, $\forall n \in \mathbb{N}^2$, $Gr_n V := \{$subsps. $\mathbb{C}^{n_1} \hookrightarrow \mathbb{C}^{d_1}$, $\mathbb{C}^{n_2} \hookrightarrow \mathbb{C}^{d_2}$ closed under $\alpha_1(V), \alpha_2(V)\}$

quiver Grassmannian

$$F_V := \sum_{n \geq 0} \chi(Gr_n V)\, Y^n$$

$\exists g_V \in \mathbb{Z}^2$, s.t. we have min. resolution $V \hookrightarrow I^{[-g_V]_+} \to I^{[g_V]_+}$

$CC(V) := X^{g_V} \cdot F_V$

eg. $CC(\mathbb{C} \underset{1}{\overset{0}{\rightrightarrows}} \mathbb{C}) = [L]$, $\quad CC(\mathbb{C} \underset{1}{\overset{0}{\rightrightarrows}} \mathbb{C} \oplus \mathbb{C} \underset{1}{\overset{1}{\rightrightarrows}} \mathbb{C}) = [L]^2$

$CC(\mathbb{C}^2 \underset{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}{\overset{\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)}{\rightrightarrows}} \mathbb{C}^2) = [Band^2(L)]$

$\forall g \in \mathbb{Z}^2$, on an open dense subset of $\mathrm{Hom}(I^{[-g]_+}, I^{[g]_+}) = \{\varphi\}$.

$CC(\ker\varphi)$ takes a const value $\mathbb{L}_g$, called generic cluster character.

Thm [Q.19] For "many" cluster alg $\mathcal{A}$, $\{\mathbb{L}_g \mid g \in \mathbb{Z}^{I_{uf}}\}$ is a $\mathbb{k}[x_i^{\pm}]_{i \in I_f}$-basis, called the generic basis.

Thm [Geiss-Labardini-Schröer 20] [bangle] $\in$ generic basis.