


3. Triangular bases

preparation

triangular basis

application

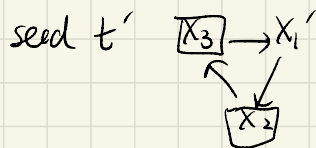
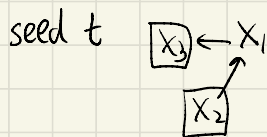


§ 1 Preparation

Example $G = SL_3(\mathbb{K})$, $N_- = \begin{pmatrix} 1 & & \\ x_1 & 1 & \\ x & x_1' & 1 \end{pmatrix}$, $x_3 := \det \begin{pmatrix} x_1 & 1 \\ x_2 & x_1' \end{pmatrix} = x_1 x_1' - x_2$

$\mathbb{K}[N_-] = \mathbb{K}[x_1, x_1', x_2]$ has dual PBW basis: $\{x_1^{c_1} x_2^{c_2} x_1'^{c_3} \mid c_i \geq 0\}$

dual canonical basis = cluster monomials



$I = \underbrace{\{1\}}_{I_{ut}} \cup \underbrace{\{2, 3\}}_{I_f}$, $d_i = 1$, $M^0(t) = \mathbb{Z}^3 = \bigoplus \mathbb{Z} f_i$, $\mathbb{K}[M^0(t)] = \mathbb{K}[x_1, x_2, x_3]$

$N_{uf}(t) = \mathbb{Z} = \mathbb{Z} e_1$

$\widetilde{B}(t) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$ is full rk. $Y_1 = x_2 x_3^{-1}$

$x_1' = \frac{1}{x_1} (x_3 + x_2) = x_1^{-1} x_3 (1 + Y_1)$ is pointed at $\deg -f_1 + f_3 \equiv -f_1 \pmod{\mathbb{Z} f}$

Denote $\mathcal{L}P(t) := \mathbb{K}[M^0(t)]$.

Def [Q.] A seed t is said to be injective-reachable, if \exists a permutation σ on I_{af} and a seed $t[1] = \tilde{\mu} t$ for some mutation sequence $\tilde{\mu}$, s.t.

$$\deg^t X_{\sigma_k}(t[1]) \equiv -f_k \pmod{\mathbb{Z}^{I_f}}.$$

(calculate in $\mathcal{L}P(t)$)

We can recursively define $t[d]$, $d \in \mathbb{Z}$ s.t. $t[d+1] = t[d][1]$.

Rem. $\circ \Rightarrow$ All $t' \in \Delta^+$ are injective-reachable

- $\circ I_k(t) := X_{\sigma_k}(t[1])$ injective cluster variable. $= CC(I_k)$
- \circ hold for almost all cluster alg from repr theory or Teichmüller theory
- $\circ [1]$ corresp. to the shift functor in cluster category
- $\circ [\pm 1]$ corresp. to left/right dual in monoidal category

From now on, assume seeds are injective reachable.

§2 Triangular basis [Q.17]

$k = \mathbb{Z}[v^{\pm}]$, $v = q^{\frac{1}{2}}$. • commutative product, \star twisted product: $X_i(t) \star X_j(t) = v^{\wedge_{ij}} X_i(t) \cdot X_j(t)$

Choose any $t = ((X_i), (b_{ij}))$. $\mathcal{L}^t(t)$ has bar-involution $\overline{q^\alpha x^m} = q^{-\alpha} x^m$.

Triangular basis L^t is a k -basis for \mathcal{A} , s.t.

① L^t contains all cluster monomials in t , $t[1]$.

② L^t is $M^0(t)$ -pointed: $L^t = \{L_m^t \mid m \in M^0(t)\}$, s.t. L_m^t are m -pointed in $k[M^0(t)]$

③ L_m^t are $(\bar{\cdot})$ -inv.

④ (triangularity) $\forall i, \exists \alpha$ s.t. $v^\alpha X_i \star L_m^t \in L_{f_i+m}^t + \sum_{m' <_t f_i+m} v^{-1} \mathbb{Z}[v^{-1}] L_{m'}^t$

Thm [Q.17]. B^\star for $U_q(n)$, $\{\text{simples of } U_q(\hat{g})\text{-mod}\}$ are L^t for initial seed t_0 .
(up to v^α , after localization at frozen variables)

PF: ① is known by T-system. ②, ③ known.

④ is deduced from the triangularity between B^\star and dual PBW / simples and standard mods.

Lemma 1. L^t is unique if it exists.

Obstruction: dual PBW basis $\Rightarrow B^*$ but A does NOT have dual PBW basis

(1) Construct distinguished funcs:

Recall $\deg^t X_i = f_i$, $\deg^t I_k = -f_k \bmod \mathbb{Z}^t$.

$\Rightarrow \forall m \in M^0(t)$, $\exists m' \in \mathbb{N}^{\mathbb{I}_t} \oplus \mathbb{Z}^{\mathbb{I}_t}$, $m'' \in \mathbb{N}^{\mathbb{I}_t}$, s.t. $\deg^t(X^{m'} * I^{m''}) = m$.

The distinguished func I_m^t is the unique m -pointed func of form

$I_m^t = u^\alpha X^{m'} * I^{m''}$, for some $\alpha \in \mathbb{Z}$.

(2) Formal completion $\widehat{\mathcal{L}P(t)} := \mathcal{L}P(t) \hat{\otimes} \mathbb{K}[[X_k, k \in \mathbb{I}_t]]$
 $\mathbb{K}[[X_k, k \in \mathbb{I}_t]]$

Lemma. [Q19, DM19] $I^t := \{I_m^t \mid \forall m\}$ is a topological basis for $\mathcal{L}P(t)$.

($\forall z \in \widehat{\mathcal{L}P(t)}$, $z = \sum_m b_m I_m^t$ unique inf. decomposition, $b_m \in \mathbb{K}$.
well-defined: $\{m \mid b_m \neq 0\}$ is \prec_t -bounded from above)

③ By Lusztig's lemma for Kazhdan-Lusztig type basis, $\exists ! \tilde{L}^t \subset \hat{L}^t(t)$ s.t.

\tilde{L}^t is $M^0(t)$ -pointed, bar-inv, and $I_m^t \in L_m^t + \sum_{m' < m} v^+ Z[w^{-1}] L_{m'}^t$.

Proof of Lem 1. If L^t exist, then $L^t = \tilde{L}^t \Rightarrow$ unique.

Def. A \mathbb{K} -basis L of A is said to be the **common triangular basis**, if L is L^t for any $t \in \Delta^+$.

• By defn, the common triangulation L contains all cluster monomials.

Proposition. L is naturally parametrized by the tropical pts. (next talk)

PF: This property was required in the original def of [Q.19],

But can be deduced by using [Q.19]

General strategy for find L :

① Start w/ a well-known basis B , show $B = L^{t_0}$, (not difficult)

② Try to show B contains cluster monomials for enough many seeds. Then general criterion [Q.17.26] $\Rightarrow B = L^t \forall t$ i.e. $B = L$.
(usually use T-system + some effort)

§3 Application

§3.1 Dual canonical basis

[FZ] expects that cluster alg provide a framework to study "dual canonical basis" for $\mathbb{C}[X]$, where $X = G$ semi-simple, G/N , etc. In particular, such basis should contain all cluster monomials.

$N_-^w = N \cap B w B$ unipotent cell. Its q -coord ring $\mathcal{O}_q[N_-^w]$ is a q -cl. alg. [GLS] [Goudeaul-Yakimov]

Conj. [FZ] [kim10] For $\mathcal{O}_q[N_-^w]$, B^* contains all cluster monomials.

PF: [Q17] ADE; some w for symm KM. Use quiver varieties and L .
[KKKO18] Symmetric Kac-Moody. Use KLR-alg.
[Q20] Any Kac-Moody. Use tropical properties and L .
 B^* no longer positive \rightarrow

Thm. Up to $q^{\mathbb{Z}}$, B^* agree w L for $\mathcal{O}_q[N_-^w]$.
[Q17][Q20].

Rem. L can be viewed as a generalization of B^* for \mathcal{A} .

§ 3.2 Monoidal categorification

a monoidal category (\mathcal{C}, \otimes) . $\{[simple\ obj]\}$ form a basis of $K_0(\mathcal{C})$

- [Hernandez-Leclerc 10] Introduce level- N category $\mathcal{C}_N \subset (\text{mod } U_q(\hat{\mathfrak{g}}))$

$K_0(\mathcal{C}_N)$ is a cluster alg \mathcal{A}

Conjecture: \mathcal{C}_N categorifies $\mathcal{A} \Leftrightarrow$ all cluster monomials are simples.

$K_v(\mathcal{C}_N)$: v -deformed (quantized), via quiver varieties.

Thm [Q 17]. $\{[simples]\}$ give the triangular basis for $K_v(\mathcal{C}_N)_*$. \Rightarrow HL conj is true.

Rem. For (almost) all cluster alg known to admit monoidal cat, $\{[simples]\} = L$.
(quantum aff. alg, KLR-alg, CohPer v (aff Grassm.))
[Cautis-Williams]

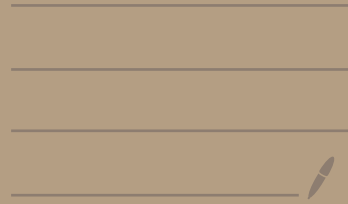
\leadsto the existence of L suggests a possible monoidal cat.

4. Tropical points

upper cluster algebras, cluster varieties

tropical pts parametrization

bases parametrized by tropical pts.



§ 1 Cluster varieties

Recall base ring $k = \mathbb{Z}$ (or $\mathbb{Z}[q^{\pm \frac{1}{2}}]$). Seeds $\Delta^+ = \Delta_{t_0}^+$. Identify $f(\mu_k t) \xrightarrow{\text{mutation } \mu_k^*} f(t)$.

$$\bigcup_{t \in \Delta^+} k[M^\circ(\mu_k t)] \xrightarrow{\text{mutation } \mu_k^*} \bigcup_{t \in \Delta^+} k[M^\circ(t)]$$

Upper cluster alg $\mathcal{U} := \bigcap_{t \in \Delta^+} k[M^\circ(t)]$.

Cluster A-variety A :

- Define $N^\circ(t) = \bigoplus \mathbb{Z} f_i^*$ dual of $M^\circ(t) = \bigoplus \mathbb{Z} f_i$.
- $T_{N^\circ(t)} = \text{Spec } k[M^\circ(t)]$. (For $k = \mathbb{C}$, $T_{N^\circ(t)} = N^\circ(t) \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq (\mathbb{C}^*)^I$ split alg torus)
- μ_k^* corresp to birational map $\mu_k: T_{N^\circ(t)} \dashrightarrow T_{N^\circ(\mu_k t)}$
- Cluster A-variety $A = \bigcup_{t \in \Delta^+} T_{N^\circ(t)}$ glued by mutation birational maps
- $\mathcal{U} = k[A]$ coord. ring

Recall $Y_j = \prod X_i^{b_{ij}}$

Lemma. $\mu_k^*(Y_i') = \begin{cases} Y_i Y_k^{[b_{ki}]}_+ (1 + Y_k)^{-b_{ki}} & i \neq k \\ Y_k^{-1} & i = k \end{cases}$

Cluster X -variety \mathcal{X} constructed using Y -variables. (omit details)

§ 2. Tropical pts parametrization

t^\vee : Langlands dual seed of t , s.t. $b_{ij}(t^\vee) = -b_{ji}(t)$,

tropical semifield $\mathbb{Z}^T = (\mathbb{Z}, \max(\cdot, \cdot), +)$

Fock-Goncharov conjecture:

$U(t_0) = \mathbb{K}[A(t_0)]$ has a \mathbb{K} -basis parametrized by the tropical pts of its "dual" $\mathcal{X}(t_0^\vee)$.

§ 2.1 tropical pts

$\forall t' = \mu_K t$, we have the tropical transformation $\phi_{t', t}: M^\circ(t) \xrightarrow{\sim} M^\circ(t')$

s.t. $\forall g = (g_i) \in M^\circ(t)$ its image $g' = (g'_i) \in M^\circ(t')$ is

$$g'_i = \begin{cases} -g_K & i = K \\ g_i + [b_{iK}]_+ [g_K]_+ - [-b_{iK}]_+ [-g_K]_+ & i \neq K \end{cases}$$

Lem. $\phi_{t',t}$ is the tropicalization of mutation of y -variables for t^v .

Cor. The composition of these maps give a well-defined

$$\phi_{t',t}: M^0(t) \rightarrow M^0(t'), \forall t, t' \in \Delta^t.$$

Def. The set of tropical pts M^0 consists of the equivalent classes $[g]$ in $\bigsqcup_{t \in \Delta^t} M^0(t)$, where $[g] = [g']$ if $g' = \phi_{t',t} g$.

§ 2.2 Parametrization [Q19.]

Def Assume $Z \in \mathbb{K}[M^0(t)]$ is g -pointed, (i.e. $Z \in X^g \cdot (1 + \sum_{n \geq 1} \mathbb{K} y^n)$)

We say Z is $[g]$ -pointed, or parametrized by $[g]$, if $\forall t' \in \Delta^t$,

Z is $\phi_{t',t} g$ -pointed when viewed in $\mathbb{K}[M^0(t')]$.

◦ A subset $S \subset U$ is called M^0 -pointed, or parametrized by M^0 if $S = \{S_{[g]} \mid [g] \in M^0 \text{ s.t. } S_{[g]} \text{ are } [g]\text{-pointed}\}.$

Rem. The generic basis is parametrized by M^0 [Plamondon]

theta
triangular

[CarL-Pumpela-Siebert] [GHKK]

[Q.] definition/property.

Assume injective-reachable from now on.

technique: work w/ both $t, t[1]$ to obtain boundedness: $\begin{matrix} & \text{mutation} \\ & \longleftrightarrow \\ \leftarrow_{t[1]} & & \rightarrow_t \end{matrix}$ lowest-deg \leftrightarrow highest deg.

Lemma: Assume $Z = X^g(1 + \sum_{0 < n < d} c_n Y^n) \in \mathcal{U}$ is $[g]$ -pointed. Then

$$Z = X^g(1 + \sum_{0 < n < d} c_n Y^n + Y^d) \text{ for some } d \in \mathbb{N}^{\text{Int}} \text{ determined by } g$$

Rem. Z has the \mathcal{L}_t -minimal deg term: $\text{Codeg}_t^+ Z := g + \tilde{B}n$ determined by g .

★ Lemma. If $Z \in \mathcal{U}$ and a cluster monomial M are both pointed at some $[g]$, then $Z = M$.

Rem. To require the degree of a pointed element transforms under $\phi_{t,t}$ is a strong restriction.

We use such tropical property to show triangular basis contains cluster monomials, without using positivity.

§3 All bases parametrized by M^0 . [Q.19] (sketch)

Def. We say $[g'] \leq_t [g]$ in M^0 if $g' \leq_t g$ in $M^0(t)$

\Rightarrow many dominance orders on M^0 .

Def. $\forall [g] \in M^0$, define the deformation factor

$$M^0_{<_{\Delta^+}[g]} := \{ [g'] \mid [g'] <_t [g], \forall t \in \Delta^+ \}$$

Thm [Q.19] Denote $\mathcal{S} = \{ \text{subset } S \subset \mathcal{U} \text{ parametrized by } M^0 \}$. $k = \mathbb{Z}$
or $\mathbb{Z}[u^{\pm}]$

(1) All $M^0_{<_{\Delta^+}[g]}$ are finite.

(2) Choose any $Z \in \mathcal{S}$, we have a bijection

$$\prod_{M^0} k^{M^0_{<_{\Delta^+}[g]}} \xrightarrow{\sim} \mathcal{S}$$

$$(b_{[g][g']})_{[g'] \in M^0_{<_{\Delta^+}[g]}} \longmapsto S = \{ S_{[g]} := Z_{[g]} + \sum b_{[g][g']} Z_{[g']} \}$$

(3) All $S \in \mathcal{S}$ are bases of \mathcal{U} . except ($k = \mathbb{Z}[u^{\pm}]$, d_i unequal) is open

Application: generic cluster characters are M^0 -pointed [Plamondon]

\Rightarrow They form a basis (generic basis)

Rem. By \star lemma, $M_{<_{\Delta+[Eg]} = \emptyset}$ if $Z_{[Eg]}$ is a cluster monomial.

Example. Q $2 \Rightarrow 1$ (annulus example)

bangle basis element $\mathbb{L}_{(n,-n)} = [L]^n$

For any M^0 -pointed basis S , $S_{(n,-n)}$ is of form

$$S_{(n,-n)} = [L]^n + b_{n-2} [L]^{n-2} + b_{n-4} [L]^{n-4} + \dots + b_i [L]^i$$

$i=0 \text{ or } 1$

$$M_{<_{\Delta+[Eg]}^{(n,-n)}} = \left\{ (n-2, -(n-2)), \dots, (i, -i) \right\}$$
