# 3. Triangular bases

preparation

triangular basis

application

## §1 Preparation

**Example** $G = SL_3(\Bbbk)$, $N_- = \begin{pmatrix} 1 & & \\ x_1 & 1 & \\ x & x_1' & 1 \end{pmatrix}$, $x_3 := \det\begin{pmatrix} x_1 & 1 \\ x_2 & x_1' \end{pmatrix} = x_1 x_1' - x_2$

$\Bbbk[N_-] = \Bbbk[x_1, x_1', x_2]$ has dual PBW basis: $\{ x_1^{c_1} x_2^{c_2} x_1'^{c_3} \mid c_i \geq 0 \}$

dual canonical basis = cluster monomials

seed $t$
$$\begin{tikzcd} \boxed{x_3} & x_1 \arrow[l] \\ \boxed{x_2} \arrow[ur] & \end{tikzcd}$$

seed $t'$
$$\begin{tikzcd} \boxed{x_3} \arrow[r] & x_1' \arrow[dl] \\ \boxed{x_2} \arrow[u] & \end{tikzcd}$$

$$\underset{I_{uf}}{I = \{1\}} \sqcup \underset{I_f}{\{2,3\}}, \quad d_i = 1. \quad M^\circ(t) = \mathbb{Z}^3 = \bigoplus \mathbb{Z} f_i, \quad \Bbbk[M^\circ(t)] = \Bbbk[X_1, X_2, X_3]$$

$$N_{uf}(t) = \mathbb{Z} = \mathbb{Z} e_1.$$

$$\tilde{B}(t) = \begin{array}{c c} & 1 \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \end{array}$$ is full rk. $\quad Y_1 = X_2 X_3^{-1}$

$x_1' = \frac{1}{x_1}(x_3 + x_2) = x_1^{-1} x_3 (1 + Y_1)$ is pointed at deg $-f_1 + f_3 \equiv -f_1 \bmod \mathbb{Z}^{I_f}$

Denote $\mathcal{LP}(t) := \mathbb{k}[M^\circ(t)]$.

Def [Q.] A seed $t$ is said to be injective-reachable, if $\exists$ a permutation $\sigma$ on $I_{uf}$ and a seed $t[1] = \overleftarrow{\mu} t$ for some mutation sequence $\overleftarrow{\mu}$ s.t.

$$\underbrace{\deg^t X_{\sigma k}(t[1])}_{\text{(calculate in } \mathcal{LP}(t))} \equiv -f_k \mod \mathbb{Z}^{I_f}.$$

We can recursively define $t[d]$, $d \in \mathbb{Z}$ s.t. $t[d+1] = t[d][1]$.

Rem.
- $\Longrightarrow$ All $t' \in \Delta^+$ are injective-reachable
- $I_k(t) := X_{\sigma k}(t[1])$ injective cluster variable. $= CC(I_k)$
- hold for almost all cluster alg from repr theory or Teichmuller theory
- $[1]$ corresp. to the shift functor in cluster category
- $[\pm 1]$ corresp to left/right dual in monoidal category

From now on, assume seeds are injective reachable.

## §2 Triangular basis [Q.17]

$\mathbb{k} = \mathbb{Z}[v^{\pm}]$, $v = q^{\frac{1}{2}}$. • commutative product, $*$ twisted product: $X_i(t) * X_j(t) = v^{\Lambda_{ij}} X_i(t) \cdot X_j(t)$

Choose any $t = ((X_i), (b_{ij}))$. $\mathcal{LP}(t)$ has bar-involution $\overline{q^{\alpha} X^m} = q^{-\alpha} X^m$.

Triangular basis $L^t$ is a $\mathbb{k}$-basis for $\mathcal{A}$, s.t.

① $L^t$ contains all cluster monomials in $t$, $t[1]$.

② $L^t$ is $M^{\circ}(t)$-pointed: $L^t = \{ L^t_m \mid m \in M^{\circ}(t) \}$, s.t. $L^t_m$ are $m$-pointed in $\mathbb{k}[M^{\circ}(t)]$

③ $L^t_m$ are $\overline{(\ )}$-inv.

④ (triangularity) $\forall i$. $\exists \alpha$ s.t. $v^{\alpha} X_i * L^t_m \in L^t_{f_i + m} + \sum_{m' <_t f_i + m} v^{-1}\mathbb{Z}[v^{-1}] L^t_{m'}$

Thm [Q.17]. $\mathcal{B}^*$ for $U_q(n)$, $\{[\text{simples of } U_q(\hat{g})\text{-mod}]\}$ are $L^{t_0}$ for initial seed $t_0$.
(up to $v^{\alpha}$, after localization at frozen variables)

PF: ① is known by T-system. ②, ③ known.

④ is deduced from the triangularity between $\mathcal{B}^*$ and dual PBW / simples and standard mods.

Lemma 1. $L^t$ is unique if it exists.

Obstruction: dual PBW basis $\Rightarrow$ $B^\alpha$. but $A$ does NOT have dual PBW basis

① Construct distinguished funcs:

Recall $\deg^t X_i = f_i$, $\deg^t I_k = -f_k \bmod \mathbb{Z}^{I_f}$.

$\Rightarrow \forall m \in M^\circ(t)$, $\exists\, m' \in \mathbb{N}^{I_{uf}} \oplus \mathbb{Z}^{I_f}$, $m'' \in \mathbb{N}^{I_{uf}}$, s.t. $\deg^t(X^{m'} * I^{m''}) = m$.

The distinguished func $I^t_m$ is the unique $m$-pointed func of form

$I^t_m = v^\alpha X^{m'} * I^{m''}$, for some $\alpha \in \mathbb{Z}$.

② Formal completion $\widehat{\mathcal{LP}(t)} := \mathcal{LP}(t) \bigotimes_{\Bbbk[y_k,\, k \in I_{uf}]} \Bbbk[[y_k,\, k \in I_{uf}]]$

Lemma. [Q19, DM19] $I^t := \{ I^t_m \mid \forall m \}$ is a topological basis for $\mathcal{LP}(t)$.

( $\forall z \in \widehat{\mathcal{LP}(t)}$, $z = \sum_m b_m I^t_m$ unique inf. decomposition, $b_m \in \Bbbk$.

well-defined: $\{ m \mid b_m \neq 0 \}$ is $\prec_t$-bounded from above )

③ By Lusztig's lemma for Kazhdan-Lusztig type basis, $\exists ! \, \tilde{L}^t \subset \widehat{LP}(t)$ s.t.

$\tilde{L}^t$ is $M^\circ(t)$-pointed, bar-inv. and $I^t_m \in L^t_m + \sum_{m' \prec_t m} v^{-1}\mathbb{Z}[v^{-1}] L^t_{m'}$.

Proof of Lem 1. If $L^t$ exist, then $L^t = \tilde{L}^t \Rightarrow$ unique.

Def. A $\mathbb{k}$-basis $L$ of $\mathcal{A}$ is said to be the common triangular basis, if $L$ is $L^t$ for any $t \in \Delta^+$.

- By defn, the common triangulation $L$ contains all cluster monomials.

Proposition. $L$ is naturall parametrized by the tropical pts. (next talk)

PF: This property was required in the original def of [Q.19],

But can be deduced by using [Q.19]

General strategy for find $L$:

① Start at a well-known basis $B$, show $B = L^{t_0}$. (not difficult)

② Try to show $B$ contains cluster monomials for enough (finite) many seeds. Then general criterion [Q.17, 20] $\Rightarrow B = L^t \; \forall t$. i.e. $B = L$.
(usually use T-system + some effort)

# §3 Application

## §3.1 Dual canonical basis

[FZ] expects that cluster alg provide a framework to study "dual canonical basis" for $\mathbb{C}[X]$, where $X = G$ semi-simple, $G/N$, etc. In particular, such basis should contain all cluster monomials.

$N_-^w = N_- \cap BwB$ unipotent cell. Its $q$-coord ring $\mathcal{O}_q[N_-^w]$ is a $q$.cl.alg. [GLS][Goodearl-Yakimov]

Conj. [FZ][Kim10] For $\mathcal{O}_q[N_-^w]$, $B^*$ contains all cluster monomials.

PF: [Q17] ADE; some $w$ for symm KM. Use quiver varieties and $L$.
[KKKO18] Symmetric Kac-Moody. Use KLR-alg.
[Q20] Any Kac-Moody. Use tropical properties and $L$.

$B^*$ no longer positive

Thm. Up to $q^\alpha$, $B^*$ agree wt $L$ for $\mathcal{O}_q[N_-^w]$.
[Q17][Q20].

Rem. $L$ can be viewed as a generalization of $B^*$ for $\mathcal{A}$.

## § 3.2 Monoidal categorification

a monoidal category $(\mathcal{C}, \otimes)$. $\{[\text{simple obj}]\}$ form a basis of $K_0(\mathcal{C})$

- [Hernandez-Leclerc 10] Introduce level-$N$ category $\mathcal{C}_N \subset (\text{mod } U_q(\hat{\mathfrak{g}}))$

  $K_0(\mathcal{C}_N)$ is a cluster alg $\mathcal{A}$

  Conjecture: $\mathcal{C}_N$ categorifies $\mathcal{A} \Leftrightarrow$ all cluster monomials are simples.

$K_v(\mathcal{C}_N)$: $v$-deformed (quantized), via quiver varieties.

Thm [Q 17]. $\{[\text{Simples}]\}$ give the triangular basis for $K_v(\mathcal{C}_N)$. $\Longrightarrow$ HL conj is true.

Rem. For (almost) all cluster alg known to admit monoidal cat, $\{[\text{simples}]\} = L$.

(quantum aff. alg, KLR-alg, CohPerv(aff Grassm.))

[Cautis-Williams]

$\rightsquigarrow$ the existence of $L$ suggests a possible monoidal cat.

# 4. Tropical points

upper cluster algebras, cluster varieties

tropical pts parametrization

bases parametrized by tropical pts.

## §1 Cluster varieties

Recall base ring $\Bbbk = \mathbb{Z}$ (or $\mathbb{Z}[q^{\pm\frac{1}{2}}]$). Seeds $\Delta^+ = \Delta^+_{t_0}$. Identify $\mathcal{F}(\mu_k t) \xrightarrow[\simeq]{\text{mutation } \mu_k^*} \mathcal{F}(t)$.

$$\begin{array}{ccc} \mathcal{F}(\mu_k t) & \xrightarrow{\mu_k^*} & \mathcal{F}(t) \\ \cup & & \cup \\ \Bbbk[M^\circ(\mu_k t)] & & \Bbbk[M^\circ(t)] \end{array}$$

Upper cluster alg $\mathcal{U} := \bigcap_{t \in \Delta^+} \Bbbk[M^\circ(t)]$.

Cluster $\mathcal{A}$-variety $\mathcal{A}$:

- Define $N^\circ(t) = \oplus \mathbb{Z} f_i^*$ dual of $M^\circ(t) = \oplus \mathbb{Z} f_i$.
- $T_{N^\circ(t)} = \operatorname{Spec} \Bbbk[M^\circ(t)]$. (For $\Bbbk = \mathbb{C}$, $T_{N^\circ(t)} = N^\circ(t) \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq (\mathbb{C}^*)^I$ split alg torus )
- $\mu_k^*$ corresp to birational map $\mu_k : T_{N^\circ(t)} \dashrightarrow T_{N^\circ(\mu_k t)}$
- Cluster $\mathcal{A}$-variety $\mathcal{A} = \bigcup_{t \in \Delta^+} T_{N^\circ(t)}$. glued by mutation birational maps
- $\mathcal{U} = \Bbbk[\mathcal{A}]$ coord. ring

Recall $Y_j = \prod X_i^{b_{ij}}$

Lemma. $\mu_k^*(Y_i') = \begin{cases} Y_i Y_k^{[b_{ki}]_+} (1+Y_k)^{-b_{ki}} & i \neq k \\ Y_k^{-1} & i = k \end{cases}$

Cluster $\mathcal{X}$-variety $\mathcal{X}$ constructed using Y-variables. (omit details)

## § 2. Tropical pts parametrization

$t^\vee$: Langlands dual seed of $t$, s.t. $b_{ij}(t^\vee) = -b_{ji}(t)$,

tropical semifield $\mathbb{Z}^T = (\mathbb{Z}, \max(\,,\,), +)$

Fock-Goncharov conjecture:

$\mathcal{U}(t_0) = \mathbb{k}[\mathcal{A}(t_0)]$ has a $\mathbb{k}$-basis parametrized by the tropical pts of its "dual" $\mathcal{X}(t_0^\vee)$.

### § 2.1 tropical pts

$\forall t' = \mu_k t$, we have the tropical transformation $\phi_{t',t}: M^\circ(t) \xrightarrow{\sim} M^\circ(t')$

s.t. $\forall g = (g_i) \in M^\circ(t)$ its image $g' = (g_i') \in M^\circ(t')$ is

$$g_i' = \begin{cases} -g_k & i = k \\ g_i + [b_{ik}]_+[g_k]_+ - [-b_{ik}]_+[-g_k]_+ & i \neq k \end{cases}$$

Lem. $\phi_{t',t}$ is the tropicalization of mutation of $Y$-variables for $t^\vee$.

Cor. The composition of these maps give a well-defined

$$\phi_{t',t}: M^\circ(t) \simeq M^\circ(t'),\ \forall t, t' \in \Delta^+.$$

Def. The set of tropical pts $\mathcal{M}^\circ$ consists of the equivalent classes $[g]$ in $\bigsqcup_{t \in \Delta^+} M^\circ(t)$, where $[g] = [g']$ if $g' = \phi_{t',t}\, g$.

## §2.2 Parametrization [Q19.]

Def Assume $Z \in \Bbbk[M^\circ(t)]$ is $g$-pointed, (i.e. $Z \in x^g \cdot (1 + \sum_{n>0} \Bbbk\, y^n)$)

We say $Z$ is $[g]$-pointed, or parametrized by $[g]$, if $\forall t' \in \Delta^+$, $Z$ is $\phi_{t',t}\, g$-pointed when viewed in $\Bbbk[M^\circ(t')]$.

- A subset $S \subset U$ is called $\mathcal{M}^\circ$-pointed, or parametrized by $\mathcal{M}^\circ$ if $S = \{S_{[g]} \mid [g] \in \mathcal{M}^\circ\}$ s.t. $S_{[g]}$ are $[g]$-pointed.

Rem. The generic basis is parametrized by $\mathcal{M}^\circ$ [Plamondon]

theta [Carl-Pumpeda-Siebert] [GHKK]

triangular [Q.] definition/property.

Assume injective-reachable from now on.

technique: work at both $t, t[1]$ to obtain boundedness: $\prec_{t[1]} \xleftrightarrow{\text{mutation}} \succ_t$. lowest-deg $\leftrightarrow$ highest deg.

Lemma: Assume $Z = X^g(1+\Sigma c_n Y^n) \in \mathcal{U}$ is $[g]$-pointed. Then

$$Z = X^g(1+\sum_{0<n<d} c_n Y^n + Y^d) \text{ for some } d \in \mathbb{N}^{\mathrm{Inf}}$$ determined by $g$

Rem. $Z$ has the $\prec_t$-minimal deg term: $\mathrm{Codeg}^t Z := g + \tilde{B} n$. determined by $g$.

☆ Lemma. If $Z \in \mathcal{U}$ and a cluster monomial $M$ are both pointed at some $[g]$, then $Z = M$.

Rem. To require the degree of a pointed element transforms under $\phi_{t',t}$ is a strong restriction.

We use such tropical property to show triangular basis contains cluster monomials, without using positivity.

## §3 All bases parametrized by $\mathcal{M}^\circ$. [Q.19] (sketch)

**Def.** We say $[g'] \leq_t [g]$ in $\mathcal{M}^\circ$ if $g' \leq_t g$ in $\mathcal{M}^\circ(t)$

$\Longrightarrow$ many dominance orders on $\mathcal{M}^\circ$.

**Def.** $\forall [g] \in \mathcal{M}^\circ$, define the deformation factor

$$\mathcal{M}^\circ_{<_{\Delta^+}[g]} := \{ [g'] \mid [g'] <_t [g], \forall t \in \Delta^+ \}$$

**Thm** [Q.19] Denote $\mathcal{S} = \{$ subset $S \subset \mathcal{U}$ parametrized by $\mathcal{M}^\circ \}$. $\mathbb{k} = \mathbb{Z}$ or $\mathbb{Z}[v^\pm]$

(1) All $\mathcal{M}^\circ_{<_{\Delta^+}[g]}$ are finite.

(2) Choose any $Z \in \mathcal{S}$, we have a bijection

$$\prod_{\mathcal{M}^\circ} \mathbb{k}^{\mathcal{M}^\circ_{<_{\Delta^+}[g]}} \xrightarrow{\sim} \mathcal{S}$$

$$(b_{[g][g']})_{[g'] <_{\Delta^+}[g]} \longmapsto S = \{ S_{[g]} := Z_{[g]} + \sum b_{[g][g']} Z_{[g']}$$

(3) All $S \in \mathcal{S}$ are bases of $\mathcal{U}$. except ($\mathbb{k} = \mathbb{Z}[v^\pm]$, $d_i$ unequal) is open

Application: generic cluster characters are $\mathcal{M}^0$-pointed [Plamondon]

$\Longrightarrow$ They form a basis (generic basis)

Rem. By ☆ lemma, $\mathcal{M}_{<_{\Delta^+[g]}} = \emptyset$ if $Z_{[g]}$ is a cluster monomial.

Example. $Q \quad 2 \rightrightarrows 1$ (annulas example)

bangle basis element $\mathbb{L}_{(n,-n)} = [L]^n$

For any $\mathcal{M}^0$-pointed basis $S$, $S_{(n,-n)}$ is of form

$$S_{(n,-n)} = [L]^n + b_{n-2}[L]^{n-2} + b_{n-4}[L]^{n-4} + \cdots + b_i [L]^i$$

$i = 0$ or $1$

$$\mathcal{M}^0_{<_{\Delta^+}[(n,-n)]} = \{ (n-2, -(n-2)), \dots \ (i,-i) \}$$