3d Supersymmetric Theories on an Elliptic Curve

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Introduction

Introduction

- Consider a supersymmetric quantum mechanics with target X .
- The supersymmetric ground states are de Rham cohomology.

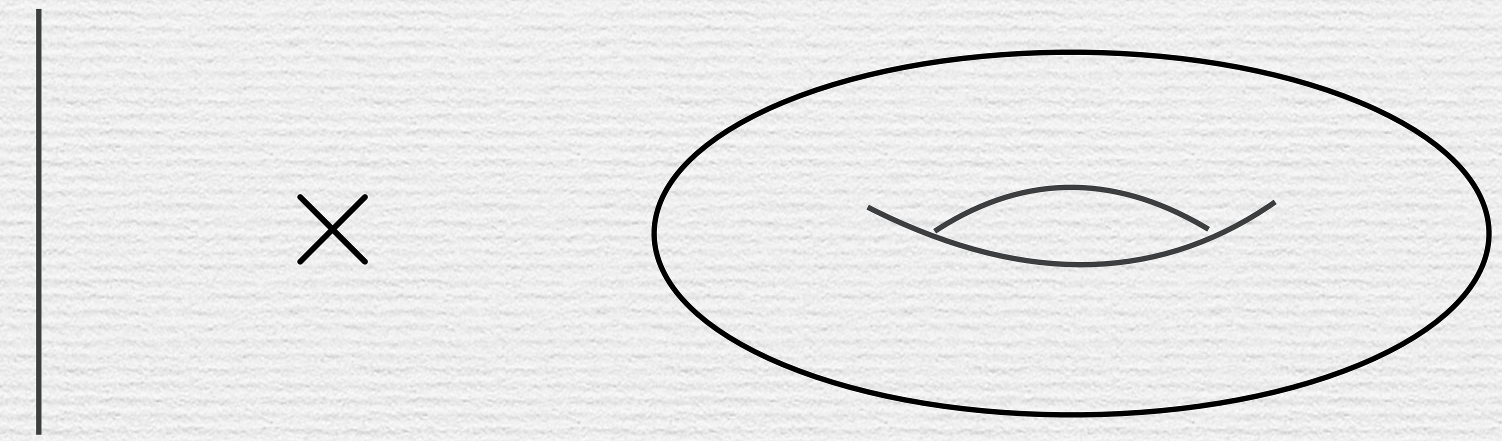
Witten '82

$$H^\bullet(X, \mathbb{C})$$

- Generalisations to 2d / 3d related to K-theory / elliptic cohomology
- The generalisation is more interesting in the equivariant case when X has symmetries.

Introduction

- What physical setup realises equivariant elliptic cohomology?
- Study 3d supersymmetric theories on an elliptic curve.



- Study supersymmetric ground states of this system.
- Physical construction of elliptic stable envelopes and R-matrices.

Aganagic-Okounkov '16

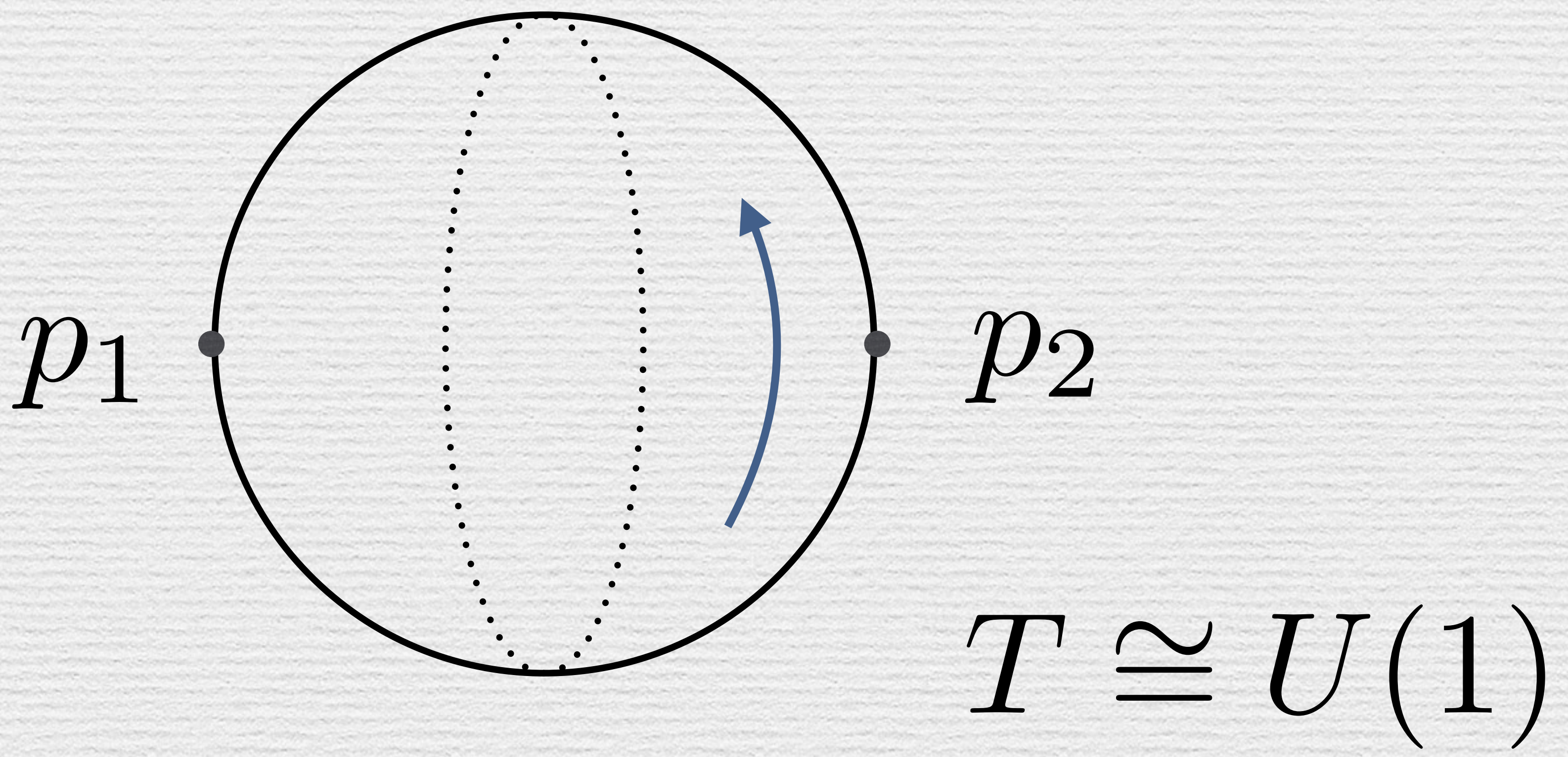
Part 1:Background

Supersymmetry

- A three-dimensional QFT with at least $\mathcal{N} = 2$ supersymmetry.
- Example: choose a smooth Kahler manifold X .
  - A supersymmetric sigma model to X .
  - A supersymmetric gauge theory with Higgs branch X.
- I will assume X is compact to simplify the presentation.

Flavour Symmetry

- I will assume there is an abelian symmetry T .
- This acts by Hamiltonian isometries of X .
- I will assume isolated fixed points $X^T = \{p_\alpha\} \quad \alpha = 1, \ldots, N$.
- Example: $X = \mathbb{CP}^1$

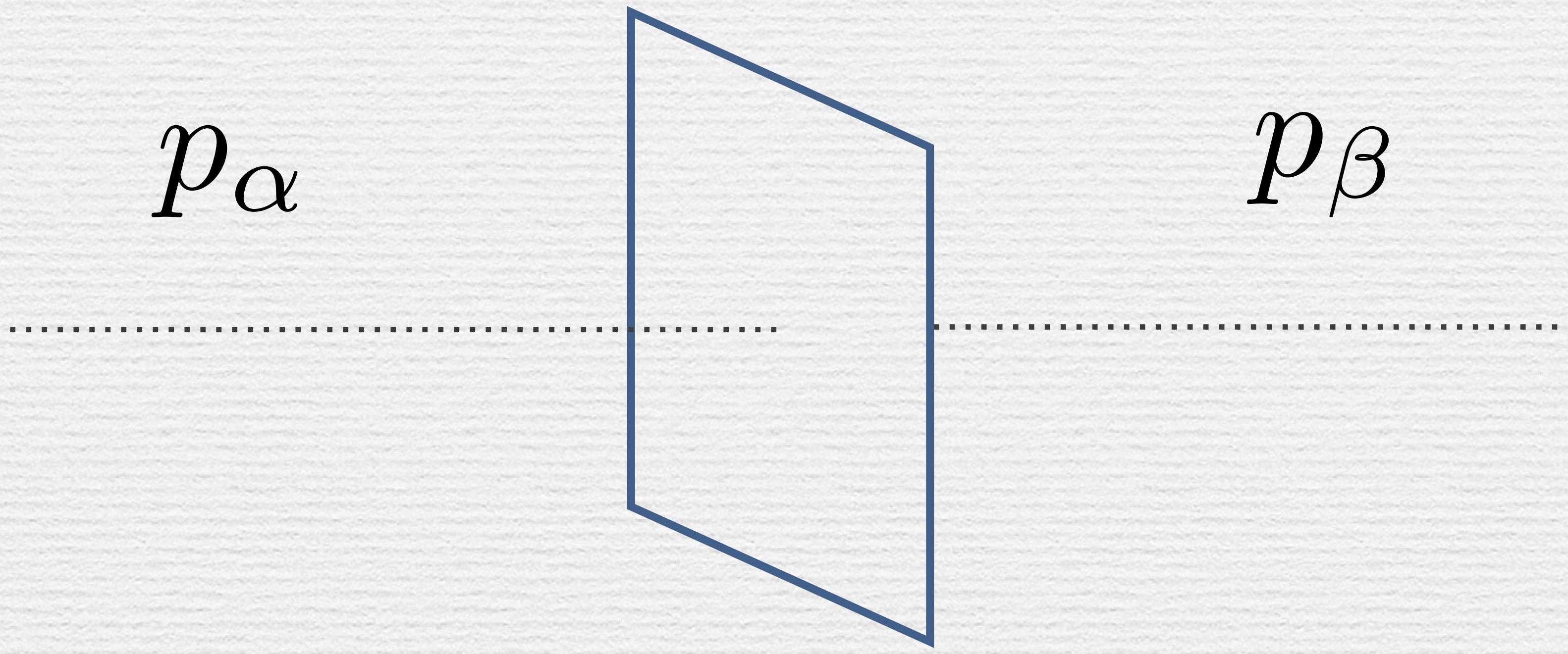


Massive Vacua

- Introduce real mass parameters $m \in \mathfrak{t} := \mathrm{Lie}(T)$.
- This generates a real super-potential $h : X \to \mathbb{R}$.
- It is the moment map for IPS $T_m \subset T$ generated by m .
- For generic mass parameters $\mathrm{Crit}(h) = \{p_\alpha\}$.
- The critical points are isolated massive vacua.

Domain Walls

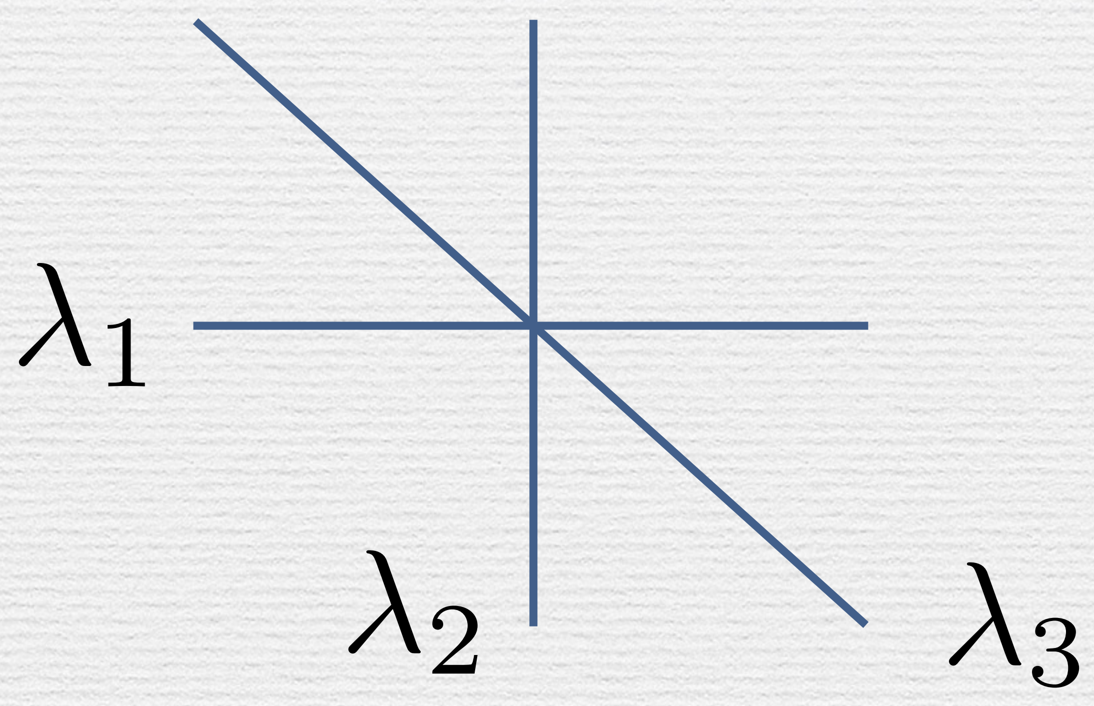
- There are BPS domain walls interpolating between vacua $p_\alpha \to p_\beta$.



- They correspond to gradient flow for h with tension $|h_\alpha - h_\beta|$.
- Define linear hyperplanes in $\mathfrak{t}$ where $|h_\alpha - h_\beta| = 0$.
- Hyperplanes are loci where $\mathrm{Crit}(h) \neq X^T$ and a moduli space containing p_α, p_β opens up.

Hyperplane Arrangement

- This forms a hyperplane arrangement in $\mathfrak{t} \cong \mathbb{R}^{\mathrm{rk}(T)}$.

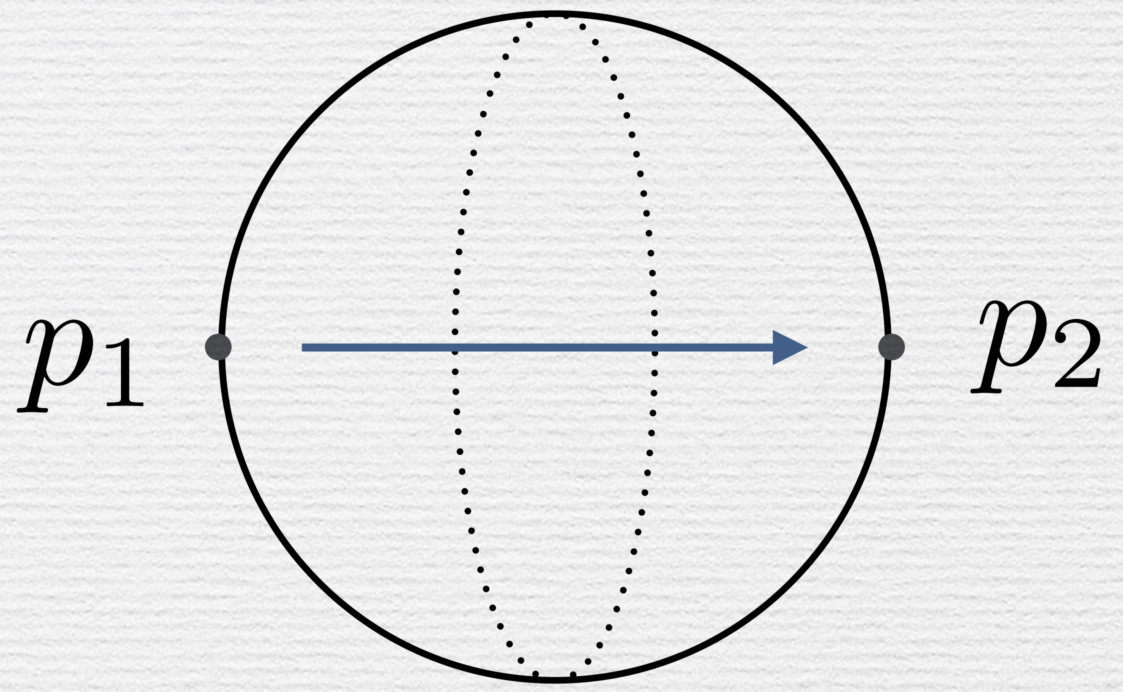


- Each hyperplane is labelled by a T-weight λ such that

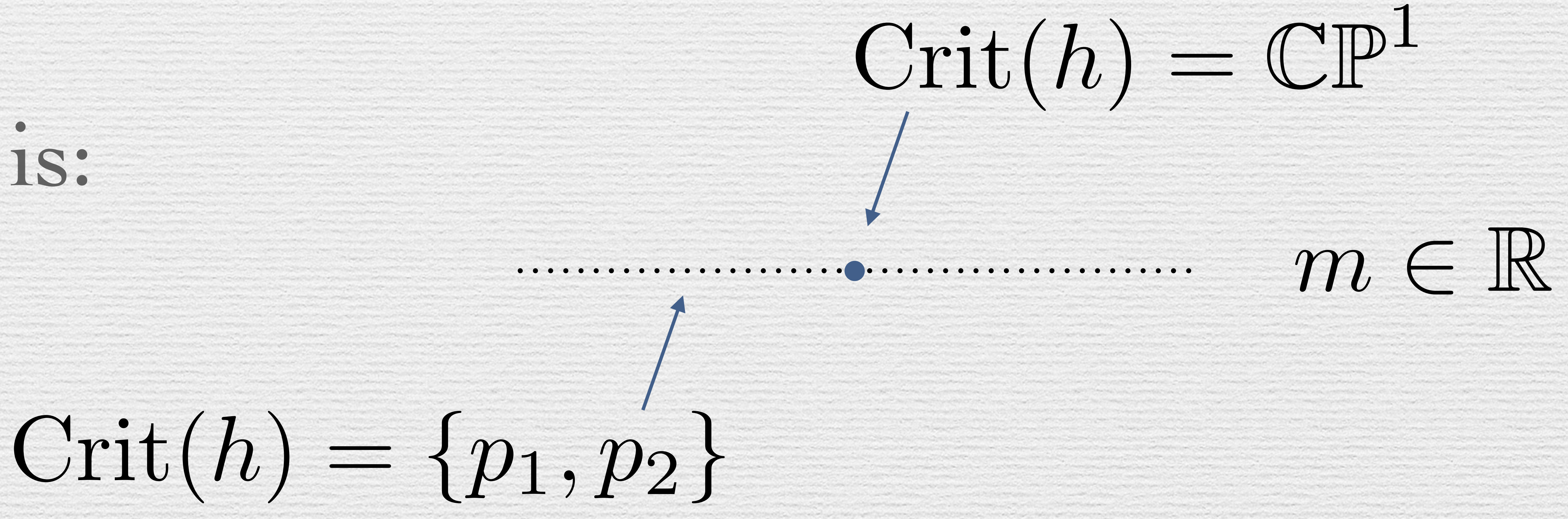
$$h_\alpha - h_\beta \propto \langle \lambda, m \rangle$$

- It is a common tangent weight $\lambda \in T_\alpha X$ and $\lambda \in -T_\beta X$.

Example



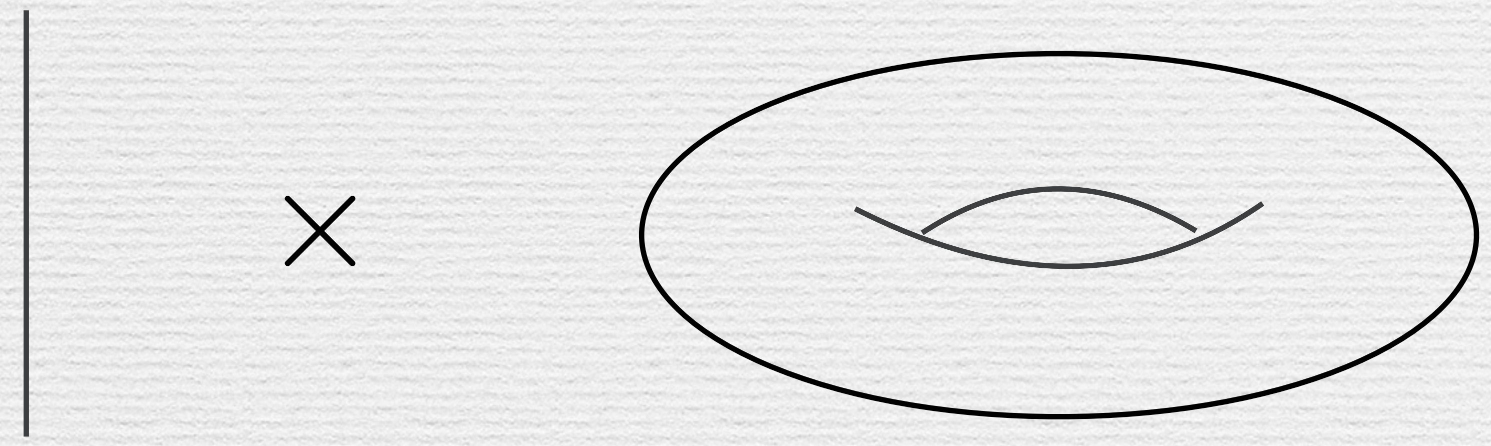
- Consider again $X = \mathbb{CP}^1$.
- There are two massive vacua p_1, p_2 .
- Domain walls $p_1 \to p_2$ have tension $h_1 - h_2 \sim \zeta m$.
- Here ζ is the Kahler parameter of $\mathbb{CP}^1$.
- The hyperplane arrangement is:

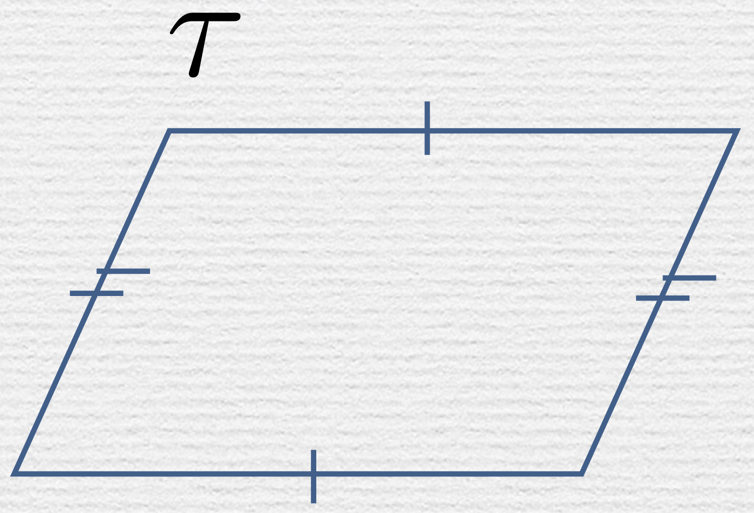


Part II: Ground States on an Elliptic Curve

The Setup

- Place the theory on the real line times an elliptic curve E_τ .





$$\mathrm{Im}(\tau) > 0$$

- Ramond-Ramond boundary conditions for fermions.
- This preserves a quantum mechanics with 4 supercharges.

Flat Connections

- We have already introduced mass parameters $m \in \mathfrak{t} \cong \mathbb{R}^{\mathrm{rk}\,T}$.
- We can introduce a background flat connection on E_τ ,

$$a \sim a + \mu + \tau\nu$$

- They parametrise are $\mathrm{rk}(T)$ -dimensional torus $E_T \cong (E_\tau)^{\mathrm{rk}(T)}$.
- The total moduli space of parameters m, a is

$$M_T \cong (\mathbb{R} \times E_\tau)^{\mathrm{rk}(T)}$$

Supersymmetric Ground States

- We are interested in supersymmetric ground states.

- For generic m, a , they are in 1-1 correspondence with fixed points.

$$\Psi_\alpha \qquad \alpha = 1, \ldots, N$$

- What is the general dependence on the parameters m, a ?

The Berry Connection

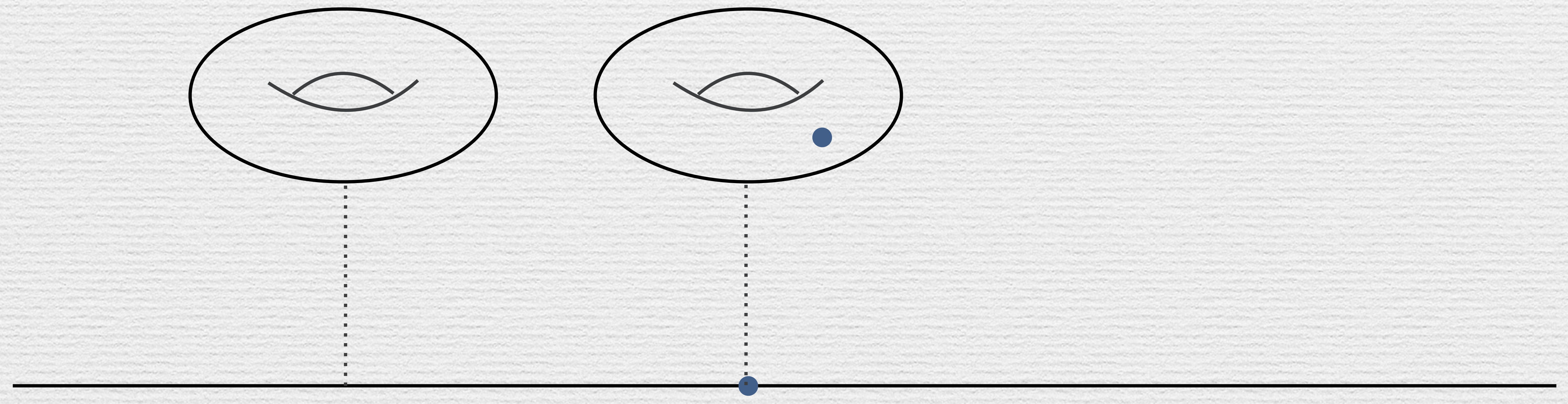
- The dependence is controlled by a supersymmetric Berry Connection:
  - A connection on a principle $\mathrm{SU}(N)$ bundle P .
  - A $\mathfrak{t}^*$-valued section ϕ of $\mathrm{Ad}(P)$.
  - They solve a set of generalised Bogomolny equations on

$$(\mathbb{R} \times E_\tau)^{\mathrm{rk}\, T}$$

Cecotti-Gaiotto-Vafa '13

Example

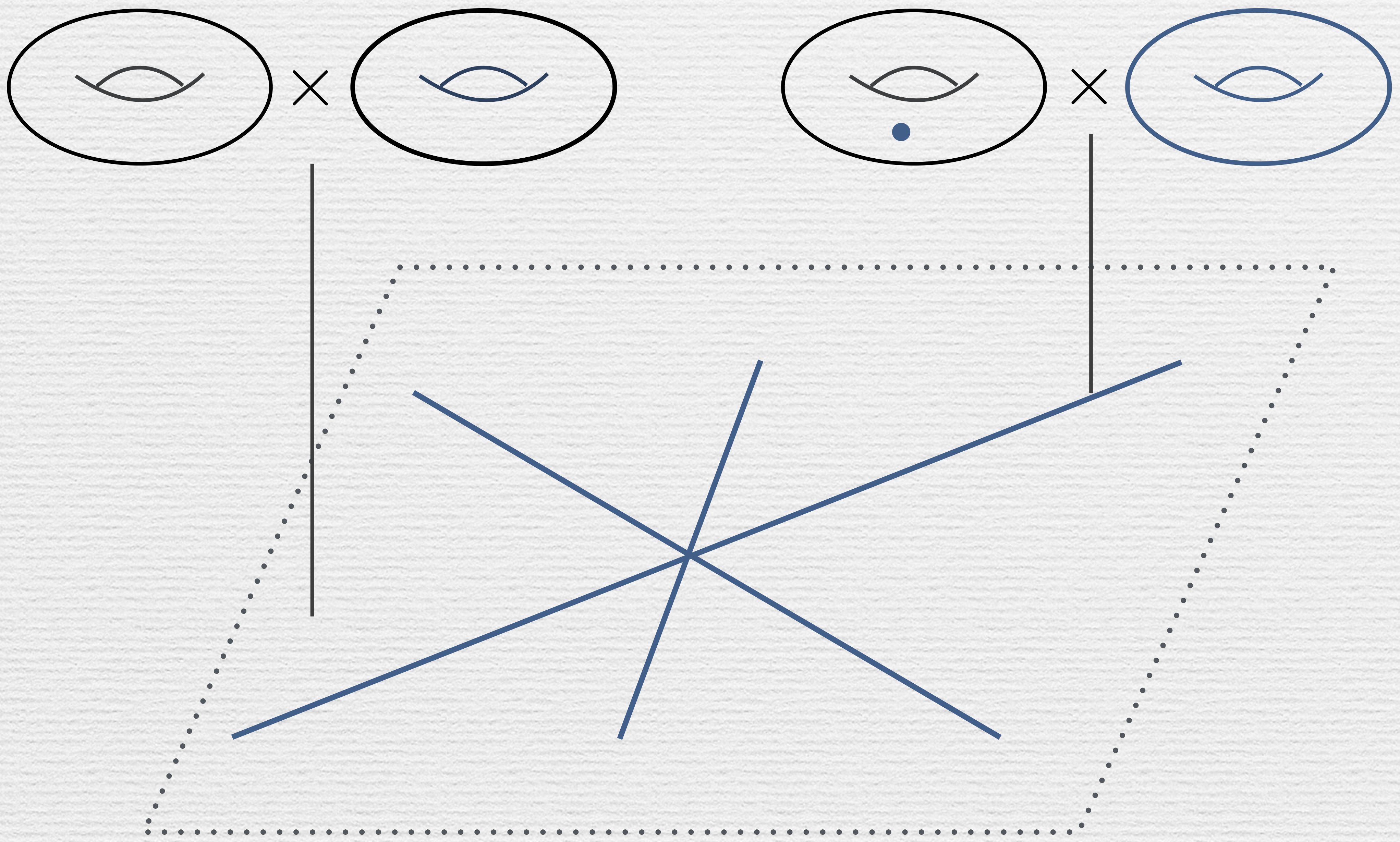
- Consider again $X = \mathbb{CP}^1$.
- There are two supersymmetric ground states Ψ_1, Ψ_2 .
- The Berry connection is a smooth $\mathrm{SU}(2)$ monopole on $\mathbb{R} \times E_\tau$.



$m \in \mathbb{R}$

General Structure

- Smooth monopoles centred on codimension-3 loci in $(\mathbb{R} \times E_\tau)^{\mathrm{rk}\, T}$.



$$\lambda \cdot m = 0$$

$$\lambda \cdot a \in \mathbb{Z} + \tau\mathbb{Z}$$

- They project onto the hyperplane arrangement in $\mathfrak{t} \cong \mathbb{R}^{\mathrm{rk}\, T}$.

Algebraic Picture

- The Bogomolny equations include $[D_m + \phi, D_{\bar{a}}] = 0$.
- A rank N holomorphic vector bundle $\mathcal{E}$ on each fiber

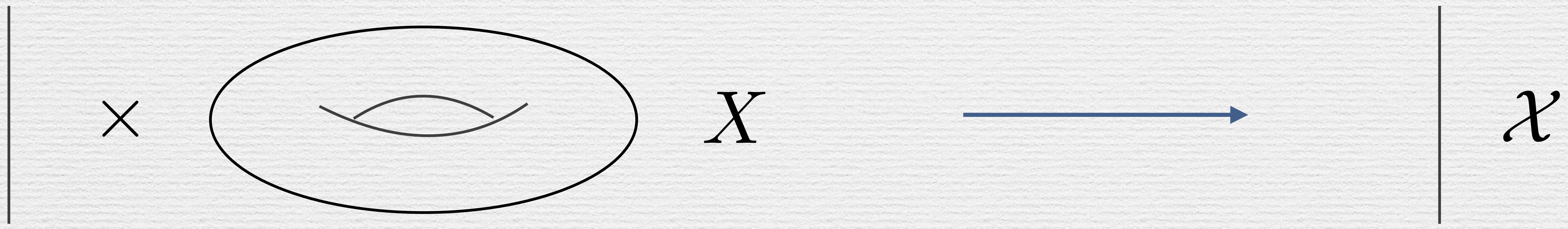
$$\{m\} \times E_T \cong (E_\tau)^{\mathrm{rk}(T)}$$

- The holomorphic bundle $\mathcal{E}$ is piecewise constant in m .
- This suggests a more algebraic approach.

Part III: Algebraic Description

Supersymmetric QM

- Reduce three-dimensional theory on E_τ to obtain a supersymmetric quantum mechanics on $\mathbb{R}$.



- Quantum mechanics has infinite-dimensional target $\mathcal{X} = \mathrm{Map}(E_\tau, X)$
- Coupled to background vectormultiplets for the $S^1 \times S^1$ and T actions on $\mathcal{X}$.

Supersymmetric Ground States

- Alternative presentation of supersymmetric ground states.
- They are representatives of cohomology classes of the supercharge

Witten '82

$$Q = e^{-h}(d + \iota_V)e^h$$

- Acts on forms on $\mathcal{X}$ where
  - h is the moment map for the 1PS T_m-action on $\mathcal{X}$.
  - V is a complex vector field generating $S^1 \times S^1$ action and the T action with parameter a.

Localisation

- Supersymmetric localisation leads to a simpler description.
- Supersymmetric ground state are representatives of cohomology classes of the equivariant differential $d + \iota_V$ on $\mathrm{Crit}(h) \subset \mathcal{X}$.
- The outcome depends on a face of the hyperplane arrangement.
- I will focus on the origin $m = 0$.

Zero Mass

- In this case, $\mathrm{Crit}(h) = \mathcal{X}$.
- For generic flat connection a , the fixed points of the vector field V are constant maps $E_\tau \to p_\alpha$.
- A basis of supersymmetric ground states from equivariant Euler class of tangent space to constant maps in $\mathcal{X}$:

$$\Psi_\alpha \sim \prod_{\lambda \in T_\alpha X} \prod_{n,m \in \mathbb{Z}} (n + m\tau + \lambda \cdot a) \sim \prod_{\lambda \in T_\alpha X} \frac{\vartheta_1(\lambda \cdot a, \tau)}{\eta(\tau)}$$

Spectral Data

- They are sections of holomorphic line bundles $L_1, \ldots, L_N$ on E_T .
- Combine into section of holomorphic line bundle on

$$\mathrm{E}_T(X) := \bigsqcup_{\alpha=1}^{N} E_T^{(\alpha)} / \Delta$$

  - Identical copies of E_T associated to each vacuum α .
  - Pairs $E_T^{(\alpha)}$ and $E_T^{(\beta)}$ are identified at loci $\lambda \cdot a \in \mathbb{Z} + \tau\mathbb{Z}$.
  - Here λ labels the hyperplane where domain walls $p_\alpha \to p_\beta$ are tensionless.

Example

- Consider again $X = \mathbb{CP}^1$.

- Supersymmetric ground states at $m = 0$:

$$\Psi_1 \sim \frac{\vartheta_1(2a,\tau)}{\eta(\tau)} \qquad \Psi_2 \sim \frac{\vartheta_1(-2a,\tau)}{\eta(\tau)}$$

- They glue to section of holomorphic line bundle on

$$E_T(X) = E_T^{(1)} \sqcup E_T^{(2)} / \Delta$$

where Δ identifies the two copies at $a \in \mathbb{Z} + \tau\mathbb{Z}$.

General Picture

- On a general face of the hyperplane arrangement containing m .
- Supersymmetric ground states transform as section of a line bundle on the equivariant elliptic cohomology variety:

$$\mathrm{Ell}_T(X^{T_m})$$

- Is this a type of spectral data for the supersymmetric Berry connection?

Part IV: Boundary Conditions

Boundary Conditions

- Boundary condition preserving at least $\mathcal{N} = (0, 2)$ supersymmetry.

- Assume it preserves the flavour symmetry T.
- It may have 't Hooft anomalies for T.
- It may or may not be compatible with mass parameters m .

Boundary Amplitudes

- Now consider boundary amplitudes:

- They behave like $\mathcal{N} = (0, 2)$ elliptic genera.
- They may be computed exactly by supersymmetric localisation.

Sugiyama-Yoshida '20

- Quasi-periodicities fixed by boundary 't Hooft anomalies.

Elliptic Cohomology Classes

- Suppose the boundary condition is compatible with mass m .
- The boundary amplitudes glue to a section of a line bundle on.

$$\mathrm{Ell}_T(X^{T_m})$$

- Boundary conditions produce classes in equivariant elliptic cohomology.

N = 4 Supersymmetry

- Special interest in N = 4 supersymmetry, broken to N = 2.
- Interesting classes of boundary conditions B_α labelled by vacua α .

MB-Dimofte-Gaiotto-Hilburn '16

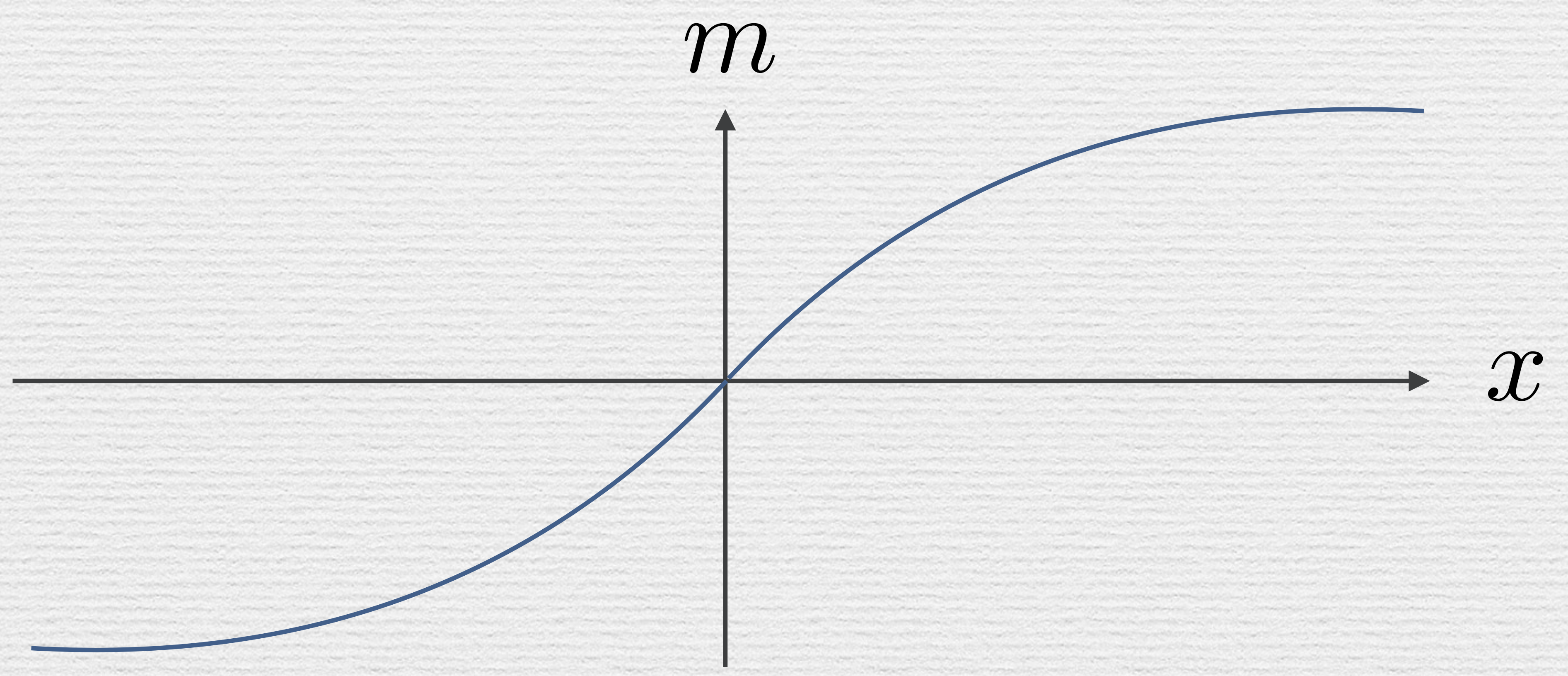
- We constructed boundary conditions corresponding to
    - Attracting sets.
    - Stable envelopes.

Aganagic-Okounkov '16

- They are exchanged under 3d mirror symmetry.

Janus Interfaces

- We also studied correlation functions of Janus interfaces.
- This is a position dependent mass $m(x)$ along $\mathbb{R}$ that interpolates between two faces of the hyperplane arrangement at $x \to \pm\infty$.



- This reproduces the construction of elliptic R-matrices.

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Thank you!