Fermionizing Yangians via central extensions of preprojective algebras

Ben Davison
University of Edinburgh

Quivers and their representations

Quivers

Q will always denote a finite quiver, i.e. a pair of finite sets Q_1 and Q_0 and morphisms $s, t \colon Q_1 \to Q_0$ between them.

Path algebra

Given a field K, we denote by KQ the *path algebra* of Q over K. It has a basis given by paths in the quiver, including paths e_i of length zero at each $i \in Q_0$. Multiplication of paths is given by $p \cdot q = pq$ if q ends at start-point of p, and $p \cdot q = 0$ otherwise.

Given a finite-dimensional KQ-module ρ, we denote by $(\dim_K(e_i \cdot \rho))_{i \in Q_0} \in \mathbb{N}^{Q_0}$ the *dimension vector* of ρ. A d-dimensional KQ-module ρ is given by a tuple of K-vector spaces $\rho_i := e_i \cdot \rho$ of dimension d_i, and a linear map $\rho_{s(a)} \to \rho_{t(a)}$ for each arrow $a \in Q_1$.

Stacks of modules

For $\mathrm{d} \in \mathbb{N}^{Q_0}$ we denote by $\mathfrak{M}_{\mathrm{d}}(Q)$ the stack of d-dimensional $\mathbb{C}Q$-modules. We can describe this stack explicitly:

- We define $\mathbb{A}_{Q,\mathrm{d}} := \prod_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{\mathrm{d}_{s(a)}}, \mathbb{C}^{\mathrm{d}_{t(a)}})$
- We define the gauge group $\mathrm{GL}_{\mathrm{d}} := \prod_{i \in Q_0} \mathrm{GL}_{\mathrm{d}_i}(\mathbb{C})$
- Then $\mathfrak{M}_{\mathrm{d}}(Q) \cong \mathbb{A}_{Q,\mathrm{d}} / \mathrm{GL}_{\mathrm{d}}$.

We will start by studying the cohomology of this stack:

$$\mathrm{H}(\mathfrak{M}_{\mathrm{d}}(Q), \mathbb{Q}) \cong \mathrm{H}_{\mathrm{GL}_{\mathrm{d}}}(\mathbb{A}_{Q,\mathrm{d}}, \mathbb{Q})$$

Because $\mathbb{A}_{Q,\mathrm{d}}$ is equivariantly contractible, there is an isomorphism

$$\mathrm{H}_{\mathrm{GL}_{\mathrm{d}}}(\mathbb{A}_{Q,\mathrm{d}}, \mathbb{Q}) \cong \mathrm{H}_{\mathrm{GL}_{\mathrm{d}}}(\mathrm{pt}, \mathbb{Q}) \cong \mathbb{Q}[x_{i,n} | i \in Q_0, 1 \leq n \leq \mathrm{d}_i]^{\mathfrak{S}_{\mathrm{d}}}$$

where the symmetric group $\mathfrak{S}_{\mathrm{d}} = \prod_{i \in Q_0} \mathfrak{S}_{\mathrm{d}_i}$ acts by permuting variables.

The 1d CoHA

Given a quiver Q, Kontsevich and Soibelman define the CoHA

$$\mathcal{A}_Q = \bigoplus_{\mathrm{d}\in\mathbb{N}^{Q_0}} \mathrm{H}(\mathfrak{M}_{\mathrm{d}}(Q), \mathbb{Q})[-\chi_Q(\mathrm{d},\mathrm{d})]$$

via pullback and pushforward of cohomology in the correspondence diagram

$$\mathfrak{M}(Q) \times \mathfrak{M}(Q) \xleftarrow{\pi_1 \times \pi_3} \mathfrak{Exact}(Q) \xrightarrow{\pi_2} \mathfrak{M}(Q).$$

where

- $\chi_Q(\mathrm{d},\mathrm{d}) = \sum_{i\in Q_0} \mathrm{d}_i\mathrm{d}_i - \sum_{a\in Q_1} \mathrm{d}_{s(a)}\mathrm{d}_{t(a)}$,
- $\mathfrak{Exact}(Q)$ is the stack of short exact sequences $0 \to \rho' \to \rho \to \rho'' \to 0$, and
- π_1, π_2, π_3 map such a sequence to the modules ρ', ρ, ρ'' respectively.

Shuffle algebra

Recall that $\mathcal{A}_{Q,\mathrm{d}}$ is the ring of $\mathfrak{S}_{\mathrm{d}}$-symmetric functions in the variables $x_{1,1},\ldots x_{1,\mathrm{d}_1}, x_{2,1},\ldots x_{n,\mathrm{d}_n}$ where $n=|Q_0|$. Multiplication in $\mathcal{A}_Q$ is given by the explicit formula, for $\mathrm{d}'+\mathrm{d}''=\mathrm{d}$:

$$f(x_{1,1},\ldots x_{1,\mathrm{d}'_1}, x_{2,1},\ldots, x_{n,\mathrm{d}'_n}) \star g(x_{1,1},\ldots x_{1,\mathrm{d}''_1}, x_{2,1},\ldots, x_{n,\mathrm{d}''_n}) =$$

$$\sum_{\sigma\in\mathrm{Sh}_{\mathrm{d}',\mathrm{d}''}} \sigma\Big(f(x_{1,1},\ldots x_{1,\mathrm{d}'_1},\ldots, x_{n,\mathrm{d}'_n}) \star g(x_{1,\mathrm{d}'_1+1},\ldots x_{1,\mathrm{d}_1}, x_{2,\mathrm{d}'_2+1},\ldots, x_{n,\mathrm{d}_n})$$

$$\prod_{\substack{a\in Q_1\\ 1\leq r\leq \mathrm{d}'_{s(a)}\\ \mathrm{d}'_{t(a)}+1\leq t\leq \mathrm{d}_{t(a)}}} (x_{s(a),r}-x_{t(a),t}) \prod_{\substack{i\in Q_0\\ 1\leq r\leq \mathrm{d}'_i\\ 1+\mathrm{d}'_i\leq t\leq \mathrm{d}_i}} (x_{i,r}-x_{i,t})^{-1}\Big)$$

- Here $\mathrm{Sh}_{\mathrm{d}',\mathrm{d}''}\subset\mathfrak{S}_{\mathrm{d}}$ is the set of all permutations that preserve the ordering of the variables $x_{i,1},\ldots,x_{i,\mathrm{d}'_i}$ and $x_{i,\mathrm{d}'_i+1},\ldots,x_{i,\mathrm{d}''_i}$.
- I.e. we recover the Feigin–Odesskii shuffle algebra this way.

A boson-fermion correspondence

Using the explicit formula, we can do some basic calculations.

- Given a $\mathbb{N}^{Q_0}$-graded vector space V, which we assume also has a cohomological grading, we denote by $\mathrm{Sym}(V)$ the $\mathbb{N}^{Q_0}$-graded free supercommutative algebra generated by V. Note that by the Koszul sign rule, we can write this as

$$\mathrm{Sym}(V) \cong \mathrm{Comm}(V^{\mathrm{even}}) \otimes \bigwedge(V^{\mathrm{odd}}).$$

  where $\mathrm{Comm}(V^{\mathrm{even}})$ is the free commutative algebra generated by the part of V lying in even cohomological degree.
- Let $Q^{(n)}$ denote the quiver with one vertex and n loops. $\mathcal{A}_{Q^{(1)}} \cong \mathrm{Sym}(\mathbb{Q}[u])$ where u^i lives in cohomological degree $2i$ and $\mathbb{N}^{Q_0}$-degree 1. I.e. this is a free **commutative** algebra on countably many generators.
- $\mathcal{A}_{Q^{(0)}} \cong \mathrm{Sym}\left((\mathbb{Q}[u])[-1]\right)$ with same gradings, but where $[-1]$ denotes cohomological shift. I.e. this is a free **exterior** algebra on the same countable set of generators.

CoHAs in dimensions one, two and three

- The CoHAs that we have met so far live in the 1d world: they are built out of the stack of $\mathbb{C}Q$-modules, which is a homologically one dimensional category (analogous to vector bundles on a curve). A result of Efimov states that they are *always* free supercommutative algebras.
- If we want to study Yangians, philosophically we should pass to the 2d setting. It is known that Yangians can be built out of the geometric representation theory of Nakajima quiver varieties. These quiver varieties parameterise representations of preprojective algebras (to be introduced later), which live in a homologically **two**-dimensional category, which will have a more complicated "preprojective" CoHA $\mathcal{A}_{\Pi_Q}$ attached to it.
- In order to prove results about $\mathcal{A}_{\Pi_Q}$,and fermionize it, we will have to pass to CoHAs associated to *three*-dimensional categories.

Kac polynomials and Kac-Moody Lie algebras

Given a quiver Q we build the Kac–Moody Lie algebra, with generators e_i, h_i, f_i for $i \in Q_0$, and relations including the Serre relations

$$[e_i, \bullet]^{a_{ij}+1}(e_j) = 0$$

for a_{ij} the number of edges between i and j in the underlying graph of Q. The positive half $\mathfrak{n}_Q^+$, generated by the e_i, is a free Lie algebra modulo these Serre relations.

Theorem (Kac)

For each dimension vector $\mathrm{d} \in \mathbb{N}^{Q_0}$ there is a polynomial $\mathtt{a}_{Q,\mathrm{d}}(t) \in \mathbb{Z}[t]$ such that for q a prime power, $\mathtt{a}_{Q,\mathrm{d}}(q)$ is the number of isomorphism classes of absolutely indecomposable d-dimensional $\mathbb{F}_q Q$-modules.

1. (Hausel): If Q has no loops, there is an equality $\mathtt{a}_{Q,\mathrm{d}}(0) = \dim \mathfrak{n}_{Q,\mathrm{d}}^+$
2. (Hausel, Letellier, Villegas): For general Q we have $\mathtt{a}_{Q,\mathrm{d}}(t) \in \mathbb{N}[t]$.

   (For indivisible d, both results were proved earlier by Crawley-Boevey and Van den Bergh, using work of Nakajima on cohomology of moduli spaces of representations of deformed preprojective algebras.)

Preprojective CoHAs: definitions

- Given a quiver Q we define the double $\overline{Q}$ by adding an arrow a^* for every $a \in Q_1$, with $s(a^*) = t(a)$ and $t(a^*) = s(a)$.
- Then we define the **preprojective algebra** $\Pi_Q = \mathbb{C}\overline{Q}/\langle \sum_{a\in Q_1}[a, a^*]\rangle$.
- We write $\mathfrak{M}(\Pi_Q)$ for the moduli stack of finite-dimensional Π_Q-modules.
- E.g. $\Pi_{Q^{(1)}} \cong \mathbb{C}[x, x^*]$, and $\mathfrak{M}(\Pi_Q)$ is the stack of coherent sheaves on $\mathbb{C}^2$ with zero-dimensional support.
- Via pullback and pushforward of Borel–Moore homology in the diagram

$$\mathfrak{M}(\Pi_Q) \times \mathfrak{M}(\Pi_Q) \xleftarrow{\pi_1\times\pi_3} \mathfrak{Exact}(\Pi_Q) \xrightarrow{\pi_2} \mathfrak{M}(\Pi_Q),$$

where $\mathfrak{Exact}(\Pi_Q)$ is the stack of short exact sequences of Π_Q-modules, Schiffmann and Vasserot defined the cohomological Hall algebra structure on

$$\mathcal{A}_{\Pi_Q} = \bigoplus_{\mathrm{d}\in\mathbb{N}^{Q_0}} \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}_{\mathrm{d}}(\Pi_Q), \mathbb{Q})[-2\chi_Q(\mathrm{d}, \mathrm{d})]$$

Preprojective CoHAs: results and main question

1. There is a cohomologically graded, $\mathbb{N}^{Q_0}$-graded Lie sub-algebra $\mathfrak{g}_{\Pi_Q} \subset (\mathcal{A}_{\Pi_Q}, [\cdot,\cdot])$ called the BPS Lie algebra, and a PBW isomorphism
$$\mathrm{Sym}(\mathfrak{g}_{\Pi_Q} \otimes \mathrm{H}(\mathrm{pt}/\mathbb{C}^*)) \xrightarrow{\cong} \mathcal{A}_{\Pi_Q}.$$
2. $$\chi_{q^{1/2}}(\mathfrak{g}_{\Pi_Q,\mathrm{d}}) := \sum_i \dim(\mathfrak{g}^i_{\Pi_Q,\mathrm{d}}) q^{i/2} = \mathtt{a}_{Q,\mathrm{d}}(q^{-1})$$
3. There is an isomorphism of Lie algebras $\mathfrak{g}^0_{\Pi_Q} \cong \mathfrak{n}^+_{Q'}$, where Q' is obtained from Q by removing all vertices supporting loops.

Main question

(1) and (3) suggest that $\mathcal{A}_{\Pi_Q}$ is a kind of generalised (half) Yangian. (2) guarantees that everything is in even cohomological degree, so Sym yields usual free **commutative** algebra. Is there a "fermionised" Hall algebra/Yangian, in which the BPS Lie algebra $\mathfrak{g}_{\Pi_Q}$ is moved to odd cohomological degree?

Jacobi algebras

- Given $a \in Q_1$, if $W = a_1 \dots a_n \in \mathbb{C}Q$ is a single cyclic word we define

$$\partial W/\partial a = \sum_{a_m = a} a_{m+1}a_{m+2}\dots a_n a_1 \dots a_{m-1}$$

  and define $\partial W/\partial a$ for general W by extending linearly.
- We define $\mathrm{Jac}(Q, W) = \mathbb{C}Q/\langle \partial W/\partial a \mid a \in Q_1\rangle$.

Example

Let Q be any quiver, define the tripled quiver $\tilde{Q}$ by adding a loop ω_i at each vertex of the doubled quiver $\overline{Q}$. Set $\tilde{W} = (\sum_{i\in Q_0}\omega_i)(\sum_{a\in Q_1}[a, a^*])$. Then there is an isomorphism identifying $\sum_{i\in Q_0}\omega_i$ with ω

$$\Psi\colon\ \mathrm{Jac}(\tilde{Q}, \tilde{W}) \cong \Pi_Q[\omega].$$

Example

If we start with $Q^{(1)}$ in the above example, then $\mathrm{Jac}(\tilde{Q}, \tilde{W}) \cong \mathbb{C}[x, x^*, \omega]$.

Cohomological DT theory

- Given a quiver Q with potential W, we consider the function $\mathrm{Tr}(W)$ on $\mathfrak{M}(Q)$.
- As subspaces of $\mathfrak{M}(Q)$ there are equalities $\mathfrak{M}(\mathrm{Jac}(Q,W)) = \mathrm{crit}(\mathrm{Tr}(W)) = \mathrm{supp}({}^{\mathfrak{p}}\phi_{\mathrm{Tr}(W)}\mathbb{Q})$
- Via usual correspondence diagram Kontsevich and Soibelman define a Hall algebra structure on

$$\mathcal{A}_{Q,W} = \bigoplus_{\mathrm{d}\in\mathbb{N}^{Q_0}} \mathrm{H}(\mathfrak{M}(Q), {}^{\mathfrak{p}}\phi_{\mathrm{Tr}(W)}\mathbb{Q})[-\chi_Q(\mathrm{d},\mathrm{d})].$$

Theorem

Assume Q is symmetric. There is a $\mathbb{N}^{Q_0}$-graded+cohomologically graded Lie sub-algebra $\mathfrak{g}_{Q,W} \subset \mathcal{A}_{Q,W}$ and a PBW isomorphism

$$\mathrm{Sym}\left(\mathfrak{g}_{Q,W} \otimes \mathrm{H}(\mathrm{pt}\,/\mathbb{C}^*)\right) \xrightarrow{\cong} \mathcal{A}_{Q,W}.$$

We call $\mathfrak{g}_{Q,W}$ the *BPS Lie algebra* associated to $\mathrm{Jac}(Q,W)$.

Dimensional reduction

Theorem

Let $X \times \mathbb{A}^n$ be a smooth scheme, let f be regular function with weight one with respect to the scaling action of $\mathbb{A}^n$, let $Z = Z(\partial f/\partial x_1, \dots, \partial f/\partial x_n)$.

$$\mathrm{H}(X \times \mathbb{A}^n, {}^{\mathfrak{p}}\phi_f \mathbb{Q}) \cong \mathrm{H}^{\mathrm{BM}}(Z, \mathbb{Q}).$$

- Considering the function $\mathrm{Tr}(\tilde{W}) = \mathrm{Tr}\left((\sum_{i \in Q_0} \omega_i)(\sum_{a \in Q_1} [a, a^*])\right)$, this has weight one with respect to the scaling action on the loops ω_i.
- Thus, leaving out a cohomological shift, there is an isomorphism

$$\mathrm{H}(\mathfrak{M}_\gamma(\widetilde{Q}), {}^{\mathfrak{p}}\phi_{\mathrm{Tr}(\tilde{W})}\mathbb{Q}) \cong \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}_\gamma(\Pi_Q), \mathbb{Q})$$

- This gives an isomorphism of algebras $\mathcal{A}_{\tilde{Q}, \tilde{W}} \cong \mathcal{A}_{\Pi_Q}$.

An aside: Relation to 1d CoHAs

There is a 2-torus T which acts by scaling the a, a^*, ω_i in such a way that $\mathrm{Tr}(\tilde{W})$ is T-invariant. There is an embedding $\mathcal{A}^T_{\Pi_Q} \hookrightarrow \mathcal{A}^T_{\tilde{Q}}$ of Hall algebras defined by extending the gauge group by T.

Recasting the question

Let us recap the construction so far:

- Given a quiver Q (not necessarily symmetric), we form the *tripled quiver* $\tilde{Q}$ by adding loops ω_i to each vertex i of the doubled quiver.
- Fix the canonical cubic potential $\tilde{W} = (\sum_{i \in Q_0} \omega_i)(\sum_{a \in Q_1} [a, a^*])$.
- For this data, we consider the KS Hall algebra

$$\mathcal{A}_{\tilde{Q},\tilde{W}} = \bigoplus_{\mathrm{d} \in \mathbb{N}^{Q_0}} \mathrm{H}(\mathfrak{M}(Q), {}^{\mathfrak{p}}\phi_{\mathrm{Tr}(\tilde{W})}\mathbb{Q})[-\chi_Q(\mathrm{d},\mathrm{d})].$$

- This is isomorphic to $\mathcal{A}_{\Pi_Q}$ via dimensional reduction, and we have the PBW isomorphism

$$\mathrm{Sym}(\mathfrak{g}_{\Pi_Q} \otimes \mathrm{H}(\mathrm{pt}/\mathbb{C}^*)) \xrightarrow{\cong} \mathcal{A}_{\tilde{Q},\tilde{W}}$$

  where $\mathfrak{g}_{\Pi_Q} := \mathfrak{g}_{\tilde{Q},\tilde{W}}$ is a BPS Lie algebra extending $\mathfrak{n}_Q^+$, living in even cohomological degrees.

The goal

Find a 3d "fermionisation" of the Yangian $\mathcal{A}_{\Pi_Q}$, given by a quiver Q' and potential W' such that $\mathfrak{g}_{Q',W'} = \mathfrak{g}_{\tilde{Q},\tilde{W}}[-1](= \mathfrak{g}_{\Pi_Q}[-1])$.

Flopping curves and fermionisation I

- Let $G \subset \mathrm{SL}_2(\mathbb{C})$ be a finite group, with associated (isolated) Kleinian singularity $X_0 = \mathbb{C}^2/G$, and let $Y_0 \to X_0$ be a minimal resolution of X_0.
- Explicitly, the exceptional fibre in Y_0 is a chain of $\mathbb{P}^1$s with incidence graph an ADE type Dynkin diagram Γ.
- The space Y_0 has a universal deformation Y parametrised by $\mathfrak{h}$ the Cartan algebra for the simple Lie algebra corresponding to Γ, so that we have a Cartesian diagram

$$\begin{array}{ccc} Y_0 & \longrightarrow & Y \\ \downarrow & & \downarrow \pi \\ 0 & \longrightarrow & \mathfrak{h}. \end{array}$$

Flopping curves and fermionisation II

- For $\alpha \in \mathfrak{h}$ define the (smooth) threefold Y^α via Cartesian diagram

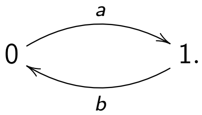
$$\begin{array}{ccc} Y^\alpha & \longrightarrow & Y \\ \downarrow & & \downarrow \pi \\ \mathbb{A}^1 & \xrightarrow{t \mapsto t\cdot\alpha} & \mathfrak{h}. \end{array}$$

- If α belongs to $i^\perp$ for i a vertex of Γ, the curve C_i corresponding to i deforms along $\mathbb{A}^1$, otherwise it is rigid.
- Fix $i \in \Gamma_0$, and $n \in \mathbb{N}$, and let $\mathcal{M}_{i,n}$ be the moduli space of semistable coherent sheaves with the same Chern classes as $\mathcal{O}_{C_i}(n)$, then either $\mathcal{M}_{i,n} \cong \mathbb{A}^1$ if $\alpha \in i^\perp$ or $\mathcal{M}_{i,n} \cong \mathrm{pt}$ if not (since C_i is rigid).
- The degree $[\mathcal{O}_{C_i}(n)]$ piece of the BPS Lie algebra $\mathfrak{g}_{Y^\alpha}$ is given by $\mathrm{H}(\mathcal{M}_{i,n}, \mathbb{Q})[\dim(\mathcal{M}_{i,n}) - 1]$, so it is one-dimensional, and even or odd depending on which root hyperplanes α avoids.
- So for example the degree $[\mathcal{O}_{C_i}(n)]$ piece of $\mathfrak{g}_{Y^\alpha}$ is even if $\alpha = 0$, and odd if α is generic.

The McKay correspondence

- Set $G = \mathbb{Z}/2\mathbb{Z}$ in the previous slides. Then Y_0 is the minimal resolution of $V(xy + z^2) \subset \mathbb{A}^3$, and $\mathfrak{h} \cong \mathbb{A}^1$.
- Let Q be the affine type A_1 quiver

$Q =$



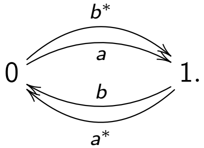
- By a result of Kapranov and Vasserot, there is a derived equivalence between Π_Q and Y_0, so a derived equivalence between $\mathrm{Jac}(\tilde{Q}, \tilde{W}) \cong \Pi_Q[x]$ and $Y^0 = Y_0 \times \mathbb{A}^1$. We've already met the calculation of the BPS cohomology for this Jacobi algebra: via explicitly calculating Kac polynomials we find

$$\mathfrak{g}_{\tilde{Q},\tilde{W},(m,n)} \cong \begin{cases} \mathbb{Q}[2] \oplus \mathbb{Q} & \text{if } m = n \\ \mathbb{Q} & \text{if } m = n \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

The noncommutative conifold

- Keeping Y_0 the minimal resolution of $V(xy+z^2)$ as before, there is a unique nontrivial deformation given by picking $\alpha \neq 0$ in $\mathfrak{h} \cong \mathbb{A}^1$. The resulting variety $Y^\alpha = Y$ is the Grothendieck-Springer resolution for $\mathrm{SL}_2(\mathbb{C})$ (a.k.a. the resolved conifold).
- This also has a noncommutative model: fix $W_{\mathtt{KW}} = aa^*b^*b - aba^*b^*$ as a potential for

$\overline{Q} =$



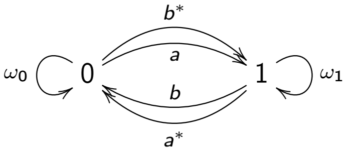
then $\mathrm{Jac}(\overline{Q}, W_{\mathtt{KW}})$ is derived equivalent to Y^α.
- Via purity and work of Morrison, Mozgovoy, Nagao and Szendrői:

$$\mathfrak{g}_{\overline{Q},W_{\mathtt{KW}},m,n} \cong \begin{cases} \mathbb{Q}[2] \oplus \mathbb{Q} & \text{if } m = n \\ \mathbb{Q}[-1] & \text{if } m = n \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

Towards the recipe for fermionisation

- Keep same Q as previous slides, so

$\tilde{Q} =$



- Set $\tilde{W}^{(-1,1)} = \tilde{W} + \frac{1}{2}\omega_0^2 - \frac{1}{2}\omega_1^2$.

An elementary observation

There is an isomorphism $\mathrm{Jac}(\tilde{Q}, \tilde{W}^{(-1,1)}) \cong \mathrm{Jac}(\overline{Q}, W_{\mathtt{KW}})$! Since after the noncommutative change of variables

$$\omega_0 \mapsto \omega_0 - a^* a + bb^*$$
$$\omega_1 \mapsto \omega_1 - aa^* + b^* b$$

the potential transforms to $W = \frac{1}{2}(\omega_0^2 - \omega_1^2) + b^* baa^* - bb^* a^* a$.

Central extensions of Π_Q

- Let $\mu \in \mathbb{C}^{Q_0}$. We define the central extension (following Etingof+Rains)

$$\Pi_Q^\mu = \mathbb{C}[\omega]\overline{Q}/\langle \sum_{a \in Q_1} [a, a^*] - \sum_{i \in Q_0} \mu_i \omega_i \rangle.$$

  E.g. there is an isomorphism $\Pi_Q^0 \cong \Pi_Q[\omega]$.

- Set $\tilde{W}^\mu = \tilde{W} - \frac{1}{2}\sum_{i \in Q_0} \mu_i \omega_i^2$. Then there is an isomorphism

$$\Pi_Q^\mu \cong \mathrm{Jac}(\tilde{Q}, \tilde{W}^\mu).$$

Recapping the geometric example

In our running example, $\mathrm{Jac}(\tilde{Q}, \tilde{W}^{(0,0)})$ was derived equivalent to $Y_0 \times \mathbb{A}^1$, in which the $\mathbb{P}^1$ could deform in an $\mathbb{A}^1$-family, while $\mathrm{Jac}(\tilde{Q}, \tilde{W}^{(-1,1)})$ was derived equivalent to the resolved conifold, in which $\mathbb{P}^1$ was rigid; this flipped the parity of the BPS invariants coming from curve classes. But the contributions from dimension vectors (n, n) retained their parity...

The main result

Theorem

Let $\mu \in \mathbb{C}^{Q_0}$. *Set* $\tilde{W}^{\mu} = \sum_{a \in Q_1} [a, a^*] \sum_{i \in Q_0} \omega_i - \frac{1}{2} \sum_{i \in Q_0} \mu_i \omega_i^2$. *Then the BPS Lie algebra for the Jacobi algebra* $\mathrm{Jac}(\tilde{Q}, \tilde{W}^{\mu})$ *satisfies*

$$\chi_{q^{1/2}}(\mathfrak{g}_{\tilde{Q}, \tilde{W}^{\mu}, \mathrm{d}}) = \begin{cases} q^{1/2}\, \mathtt{a}_{Q,\mathrm{d}}(q^{-1}) & \text{if } \mathrm{d} \cdot \mu \neq 0 \\ \mathtt{a}_{Q,\mathrm{d}}(q^{-1}) & \text{if } \mathrm{d} \cdot \mu = 0. \end{cases}$$

There are isomorphisms $\mathrm{Sym}\left(\mathfrak{g}_{\tilde{Q}, \tilde{W}^{\mu}} \otimes \mathrm{H}(\mathrm{pt}/\mathbb{C}^*)\right) \xrightarrow{\cong} \mathcal{A}_{\tilde{Q}, \tilde{W}^{\mu}}$ *of graded vector spaces.*

- Recall that the BPS Lie algebra $\mathfrak{g}_{\Pi_Q}$ is given by setting $\mu = 0$ in the above theorem. So the theorem gives a recipe for partially fermionising the CoHA/half-Yangian $\mathcal{A}_{\Pi_Q}$, depending on which hyperplanes we choose μ to avoid.
- For something on the actual algebra structure, this should be related to recent work of Costello (especially the conifold) and Galakhov, Li, and Yamazaki. Alternatively, can use shuffle algebra embeddings.

Bonus result

- Let $\mu \in \mathbb{C}^{Q_0}$, $n \in \mathbb{Z}_{\geq 1}$. Set $\tilde{W}_n^{\mu} = \sum_{a \in Q_1} [a, a^*] \sum_{i \in Q_0} \omega_i - \frac{1}{2} \sum_{i \in Q_0} \mu_i \omega_i^n$. The BPS Lie algebra for the Jacobi algebra $\mathrm{Jac}(\tilde{Q}, \tilde{W}^{\mu})$ satisfies

$$\mathfrak{g}_{\tilde{Q}, \tilde{W}_n^{\mu}, \mathrm{d}} \cong \begin{cases} \mathfrak{g}_{\tilde{Q}, \tilde{W}, \mathrm{d}} \otimes \mathrm{H}(\mathbb{A}^1, {}^{\mathfrak{p}}\phi_{x^n} \mathbb{Q}_{\mathbb{A}^1}[1]) & \text{if } \mathrm{d} \cdot \mu \neq 0 \\ \mathfrak{g}_{\tilde{Q}, \tilde{W}, \mathrm{d}} & \text{if } \mathrm{d} \cdot \mu = 0. \end{cases}$$

- The result is more general, and is proved using *deformed* dimensional reduction. Setting $n = 1$, the potential is linear in the ω_i, and *usual* dimensional reduction yields an isomorphism

$$\mathrm{H}(\mathfrak{M}_{\mathrm{d}}(\widetilde{Q}), {}^{\mathfrak{p}}\phi_{\mathrm{Tr}(\tilde{W}_n^{\mu})} \mathbb{Q}) \cong \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}_{\mathrm{d}}(\Pi_{Q,\mu}), \mathbb{Q})$$

where $\Pi_{Q,\mu} = \mathbb{C}\overline{Q} / \langle \sum_{a \in Q_1} [a, a^*] - \sum_{i \in Q_0} e_i \mu_i \rangle$ is the deformed preprojective algebra.

Theorem

$$\bigoplus_{\mathrm{d}} \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}_{\mathrm{d}}(\Pi_{Q,\mu}), \mathbb{Q}[\mathrm{vdim}]) \cong \mathrm{Sym}\left(\bigoplus_{\mathrm{d} \cdot \mu = 0} \mathfrak{g}_{\tilde{Q}, \tilde{W}, \mathrm{d}} \otimes \mathrm{H}(\mathrm{pt} / \mathbb{C}^*, \mathbb{Q}) \right).$$