A categorical action of the shifted q=0 affine algebras

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NCTS

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Remark

In this talk, we work over the field \mathbb{C} of complex numbers.

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The above data of the representation can be characterized in the following picture

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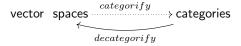
We can consider a more general case, which is the representation of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$. The third condition is replaced by $(ef-fe)|_{V_\lambda}=[\lambda]_qId_{V_\lambda}$, where $[\lambda]_q:=q^{\lambda-1}+q^{\lambda-3}+...+q^{-\lambda+1}$ is the quantum integer.

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- Geometry is a good resource for producing categories.
- It can be decategorified to recover the original vector space.



We would apply the philosophy of categorification to representations of $\mathfrak{sl}_2(\mathbb{C})$ (or $\mathcal{U}_q(\mathfrak{sl}_2)$) and we call it the categorical \mathfrak{sl}_2 (or $\mathcal{U}_q(\mathfrak{sl}_2)$) action.

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Similarly, there is also a lift of the commutator relation $(ef-fe)|_{V_{\lambda}}=[\lambda]_q Id_{V_{\lambda}}$ for $\mathcal{U}_q(\mathfrak{sl}_2)$ to categorical level.

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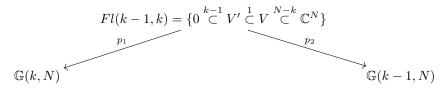
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Let $\mathcal{D}^bCon(\mathbb{G}(k,N))$ to be the bounded derived categories of constructible sheaves on $\mathbb{G}(k,N)$. These will be our weight categories $\mathcal{K}(\lambda) = \mathcal{D}^bCon(\mathbb{G}(k,N))$, where $\lambda = N-2k$.

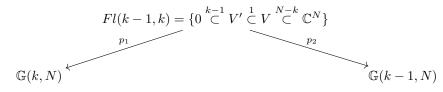
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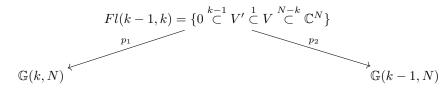


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$$\mathsf{E} := p_{2*}p_1^* : \mathcal{D}^bCon(\mathbb{G}(k,N)) = \mathcal{K}(\lambda) \to \mathcal{D}^bCon(\mathbb{G}(k-1,N)) = \mathcal{K}(\lambda+2)$$

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Remark

In this talk, all functors between derived categories are assumed to be derived. For example, we will use f^* , f_* instead of Lf^* , Rf_* respectively.

With the categories $\mathcal{K}(\lambda)$ and functors E, F defined above, we have the following theorem.

Construct categorification from geometries

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Theorem 1 (Beilinson-Lusztig-MacPherson, Chuang-Rouquier)

The categories and functors defined above gives a categorical \mathfrak{sl}_2 (or $\mathcal{U}_q(\mathfrak{sl}_2)$) action. This means that the functors defined above satisfy

$$\mathsf{EF}|_{\mathcal{K}(\lambda)} \cong \mathsf{FE}|_{\mathcal{K}(\lambda)} \bigoplus \mathsf{Id}_{\mathcal{K}(\lambda)}^{\oplus \lambda} \ \textit{if} \ \lambda \geq 0$$

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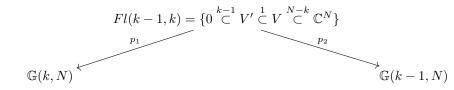
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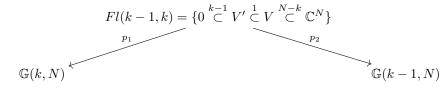
Motivated by the above result, we replace constructible sheaves with coherent sheaves.

That means that our weight categories $\mathcal{K}(\lambda)$ are bounded derived categories of coherent sheaves on $\mathbb{G}(k,N)$, which is denoted by $\mathcal{D}^bCoh(\mathbb{G}(k,N))$, where $\lambda=N-2k$.

Our functors

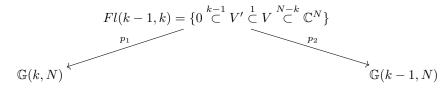


Our functors



Denoting \mathcal{V},\mathcal{V}' to be the tautological bundles on Fl(k-1,k) of rank $k,\,k-1$ respectively, then there is a natural line bundle \mathcal{V}/\mathcal{V}' on Fl(k-1,k).

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Denoting \mathcal{V},\mathcal{V}' to be the tautological bundles on Fl(k-1,k) of rank $k,\,k-1$ respectively, then there is a natural line bundle \mathcal{V}/\mathcal{V}' on Fl(k-1,k). Instead of just pullback and pushforward, we have more functors

$$\mathsf{E}_r := p_{2*}(p_1^* \otimes (\mathcal{V}/\mathcal{V}')^r) : \mathcal{D}^bCoh(\mathbb{G}(k,N)) \to \mathcal{D}^bCoh(\mathbb{G}(k-1,N))$$
$$\mathsf{F}_r := p_{1*}(p_2^* \otimes (\mathcal{V}/\mathcal{V}')^r) : \mathcal{D}^bCoh(\mathbb{G}(k-1,N)) \to \mathcal{D}^bCoh(\mathbb{G}(k,N))$$

where $r \in \mathbb{Z}$.

Problem.

We want to understand how this $L\mathfrak{sl}_2:=\mathfrak{sl}_2\otimes \mathbb{C}[t,t^{-1}]$ -like algebra acting on $\bigoplus_k \mathcal{D}^bCoh(\mathbb{G}(k,N))$, where $e\otimes t^r$ and $f\otimes t^s$ acting via the functors E_r and F_s respectively for $r,\ s\in \mathbb{Z}$.

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We can ask several natural questions, for example,

- **1** What are the categorical commutator relations between E_rF_s and F_sE_r ?
- What is the algebra that we obtain after decategorifying?
- **3** If we define the algebra, can we give a definition of its categorical action like \mathfrak{sl}_2 in the introduction?

Fourier-Mukai (FM) transforms

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Definition 2

Let X, Y be two smooth projective varieties. A Fourier-Mukai (FM) kernel is any object $\mathcal{P}\in\mathcal{D}^bCoh(X\times Y)$. For such \mathcal{P} we define the associated Fourier-Mukai (FM) transform, which is the functor

$$\Phi_{\mathcal{P}}: \mathcal{D}^bCoh(X) \to \mathcal{D}^bCoh(Y)$$

$$\mathcal{F} \mapsto \pi_{2*}(\pi_1^*(\mathcal{F}) \otimes \mathcal{P})$$

where π_1 , π_2 are natural projections.

FM kernels for E_r and F_s

Then the functor $\mathsf{E}_r: \mathcal{D}^bCoh(\mathbb{G}(k,N)) \to \mathcal{D}^bCoh(\mathbb{G}(k-1,N))$ isomorphic to a FM transform with the kernel

$$\mathcal{E}_r \mathbf{1}_{(k,N-k)} := \iota_* (\mathcal{V}/\mathcal{V}')^r \in \mathcal{D}^b Coh(\mathbb{G}(k,N) \times \mathbb{G}(k-1,N))$$

where $\iota:Fl(k-1,k)\to\mathbb{G}(k,N)\times\mathbb{G}(k-1,N)$ is the natural inclusion, i.e., $\mathsf{E}_r\cong\Phi_{\mathcal{E}_r\mathbf{1}_{(k,N-k)}}.$

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$$\mathcal{F}_s \mathbf{1}_{(k,N-k)} \in \mathcal{D}^b Coh(\mathbb{G}(k,N) \times \mathbb{G}(k+1,N))$$

to be the FM kernel for F_s , i.e., $F_s \cong \Phi_{\mathcal{F}_r \mathbf{1}_{(k,N-k)}}$.

First, we study the relation between the two functors

$$\mathsf{E}_r \circ \mathsf{F}_s, \ \mathsf{F}_s \circ \mathsf{E}_r : \mathcal{D}^bCoh(\mathbb{G}(k,N)) \to \mathcal{D}^bCoh(\mathbb{G}(k,N)).$$

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Then $(\mathcal{E}_r * \mathcal{F}_s)\mathbf{1}_{(k,N-k)}, \ (\mathcal{F}_s * \mathcal{E}_r)\mathbf{1}_{(k,N-k)} \in \mathcal{D}^bCoh(\mathbb{G}(k,N) \times \mathbb{G}(k,N))$ are FM kernels for the functors $\mathsf{E}_r \circ \mathsf{F}_s, \ \mathsf{F}_s \circ \mathsf{E}_r$, respectively.

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$$\begin{split} \text{Comparing E}_r \circ \mathsf{F}_s, \ \mathsf{F}_s \circ \mathsf{E}_r : \mathcal{D}^b Coh(\mathbb{G}(k,N)) \to \mathcal{D}^b Coh(\mathbb{G}(k,N)) \\ & \qquad \qquad \\ & \qquad \qquad \\ \\ \mathsf{Comparing} \ (\mathcal{E}_r * \mathcal{F}_s) \mathbf{1}_{(k,N-k)}, \ (\mathcal{F}_s * \mathcal{E}_r) \mathbf{1}_{(k,N-k)} \in \mathcal{D}^b Coh(\mathbb{G}(k,N) \times \mathbb{G}(k,N)) \end{split}$$

Theorem 3 (Hsu)

We have the following exact triangles in $\mathcal{D}^bCoh(\mathbb{G}(k,N)\times\mathbb{G}(k,N))$.

$$(\mathcal{F}_s * \mathcal{E}_r) \mathbf{1}_{(k,N-k)} \to (\mathcal{E}_r * \mathcal{F}_s) \mathbf{1}_{(k,N-k)} \to (\Psi^+ * \mathcal{H}_1) \mathbf{1}_{(k,N-k)}, \text{ if } r+s=N-k+1$$

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Remark

The exact triangles are non-split in general.

To compare $(\mathcal{E}_r * \mathcal{F}_s)\mathbf{1}_{(k,N-k)}$ and $(\mathcal{F}_s * \mathcal{E}_r)\mathbf{1}_{(k,N-k)}$, the geometries is exactly the same to the setting of constructible derived category for categorical \mathfrak{sl}_2 action.

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Geometry for FE or F_sE_r

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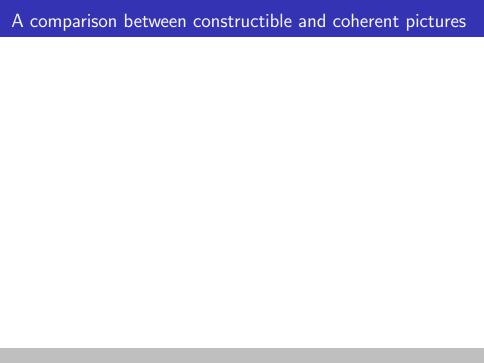
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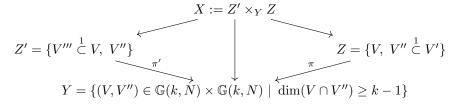
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$$Id_{\mathcal{K}(\lambda)}^{\oplus \lambda} \sim \sim Id_{\mathcal{K}(\lambda)} \otimes H_{sing}^*(\mathbb{P}^{\lambda-1}, \mathbb{C})$$

However, in the coherent setting $\mathcal{K}(\lambda) = \mathcal{D}^b Coh(\mathbb{G}(k,N))$, we do not have powerful tools like the decomposition theorem we can use for constructible sheaves.

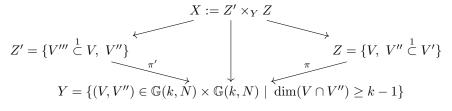
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In this setting, the last isomorphism in Theorem 7 is reflected by the vanishing of coherent cohomology.

$$(\mathcal{F}_s * \mathcal{E}_r) \mathbf{1}_{(k,N-k)} \cong (\mathcal{E}_r * \mathcal{F}_s) \mathbf{1}_{(k,N-k)}, -k+1 \leq r+s \leq N-k-1$$

$$\downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$H^*(\mathbb{P}^{N-1}, \mathcal{O}_{\mathbb{P}^{N-1}}(-r-s-k)) = 0, -N+1 \leq -r-s-k \leq -1.$$



There is an exact triangle in $\mathcal{D}^bCoh(\mathbb{G}(k,N)\times\mathbb{G}(k,N))$

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This tells us that $\mathcal{H}_1\mathbf{1}_{(k,N-k)}$ is neither isomorphic to $\Delta_*(\mathcal{V}\oplus\mathbb{C}^N/\mathcal{V})$ nor to $\Delta_*\mathbb{C}^N$.

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Remark

We prove that $(\mathcal{H}_1\mathbf{1}_{(k,N-k)})_R \cong \mathcal{H}_{-1}\mathbf{1}_{(k,N-k)} \cong (\mathcal{H}_1\mathbf{1}_{(k,N-k)})_L$.

We define the functors $H_{\pm 1}: \mathcal{D}^bCoh(\mathbb{G}(k,N)) \to \mathcal{D}^bCoh(\mathbb{G}(k,N))$ to be the FM transforms with the kernels given by $\mathcal{H}_{\pm 1}\mathbf{1}_{(k,N-k)}$.

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Comparing $\mathsf{E}_r \circ \mathsf{H}_1, \ \mathsf{H}_1 \circ \mathsf{E}_r \leadsto \mathsf{Comparing}\ (\mathcal{E}_r * \mathcal{H}_1) \mathbf{1}_{(k,N-k)}, \ (\mathcal{H}_1 * \mathcal{E}_r) \mathbf{1}_{(k,N-k)}$

Theorem 4 (Hsu)

We have the following exact triangles

$$(\mathcal{E}_{r} * \mathcal{H}_{-1}) \mathbf{1}_{(k,N-k)} \to (\mathcal{H}_{-1} * \mathcal{E}_{r}) \mathbf{1}_{(k,N-k)} \to (\mathcal{E}_{r-1} \bigoplus \mathcal{E}_{r-1}[1]) \mathbf{1}_{(k,N-k)}$$

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$$\downarrow \pi_{13}^* \qquad \qquad \downarrow$$

$$\iota_* \mathcal{V} \to (\mathcal{E} * \mathcal{H}_1) \mathbf{1}_{(k,N-k)} \to \iota_* \mathbb{C}^N / \mathcal{V} \qquad x \in \operatorname{Ext}^1$$

Here we roughly explain the technical details behind the proof.

We prove the first exact triangle and the argument is similar for the rest. Also, it suffices to prove the case r=0. To understand the object $(\mathcal{E}*\mathcal{H}_1)\mathbf{1}_{(k,N-k)}$, we have the following analysis.

$$\Delta_* \mathcal{V} \to \mathcal{H}_1 \mathbf{1}_{(k,N-k)} \to \Delta_* \mathbb{C}^N / \mathcal{V} \qquad (0, id) \in \operatorname{Ext}^1$$

$$\downarrow \pi_{12}^* \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$\pi_{12}^* \Delta_* \mathcal{V} \to \pi_{12}^* \mathcal{H}_1 \mathbf{1}_{(k,N-k)} \to \pi_{12}^* \Delta_* \mathbb{C}^N / \mathcal{V} \qquad \dots \in \operatorname{Ext}^1$$

$$\downarrow \otimes \pi_{23}^* \mathcal{E} \qquad \qquad \downarrow$$

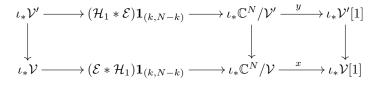
$$\pi_{12}^* \Delta_* \mathcal{V} \otimes \pi_{23}^* \mathcal{E} \to \pi_{12}^* \mathcal{H}_1 \mathbf{1}_{(k,N-k)} \otimes \pi_{23}^* \mathcal{E} \to \pi_{12}^* \Delta_* \mathbb{C}^N / \mathcal{V} \otimes \pi_{23}^* \mathcal{E} \qquad \dots \in \operatorname{Ext}^1$$

$$\downarrow \pi_{13}^* \qquad \qquad \downarrow$$

$$\iota_* \mathcal{V} \to (\mathcal{E} * \mathcal{H}_1) \mathbf{1}_{(k,N-k)} \to \iota_* \mathbb{C}^N / \mathcal{V} \qquad x \in \operatorname{Ext}^1$$

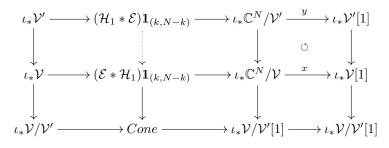
Similarly, we obtain the exact triangle

$$\iota_*\mathcal{V}' \to (\mathcal{H}_1 * \mathcal{E})\mathbf{1}_{(k,N-k)} \to \iota_*\mathbb{C}^N/\mathcal{V}'$$
 determined by $y \in \operatorname{Ext}^1$

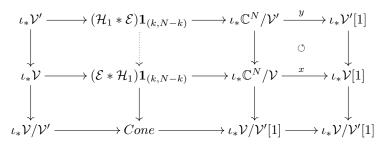


Check the commutativity of the right square induces a morphism $(\mathcal{H}_1 * \mathcal{E}) \mathbf{1}_{(k,N-k)} \to (\mathcal{E} * \mathcal{H}_1) \mathbf{1}_{(k,N-k)}$.

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Finally, we show that $Cone \cong \iota_* \mathcal{V}/\mathcal{V}' \oplus \iota_* \mathcal{V}/\mathcal{V}'[1] \cong \mathcal{E}_1 \oplus \mathcal{E}_1[1]$ to get the desire exact triangle.

The main result

Together with the study of other categorical relations, we obtain the following main result.

Theorem 5 (Hsu)

- (1) The resulting algebra acting on $\bigoplus_k \mathcal{D}^b Coh(\mathbb{G}(k,N))$ is a new algebra, which we call it the shifted q=0 affine algebra. Denoted by $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_2)$.
- (2)We give a definition of the categorical $\mathcal{U}_{0,N}(L\mathfrak{sl}_2)$ action.
- (3)We verify that there is a categorical $\mathcal{U}_{0,N}(L\mathfrak{sl}_2)$ action on
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Our main result answers these natural questions that arising from the study of the $L\mathfrak{sl}_2$ -like algebra action.

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Our main result answers these natural questions that arising from the study of the $L\mathfrak{sl}_2$ -like algebra action.

Remark

More generally, we constructed a categorical $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_n)$ action on the derived categories of coherent sheaves on n-step partial flag varieties.

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Fixing a triangulated category $\mathcal{D}\text{,}$ which we may assume it is $\mathbb{C}\text{-linear.}$

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Definition 6

An object $E \in \mathsf{Ob}(\mathcal{D})$ is called exceptional if

$$\operatorname{Hom}_{\mathcal{D}}(E, E[l]) = \begin{cases} \mathbb{C} & \text{if } l = 0\\ 0 & \text{if } l \neq 0. \end{cases}$$

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Then we define the notion of exceptional collections.

Definition 7

An ordered collection $\{E_1,...,E_n\}$, where $E_i\in \mathsf{Ob}(\mathcal{D})$ for all $1\leq i\leq n$, is called an exceptional collection if each E_i is exceptional and moreover $\mathrm{Hom}_{\mathcal{D}}(E_i,E_i[l])=0$ for all i>j and all $l\in\mathbb{Z}$.

Semiorthogonal decompositions

Then we define the notion of semiorthogonal decompositions, which can be thought of as a generalization of exceptional collections.

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A semiorthogonal decomposition (SOD for short) of \mathcal{D} is a sequence of full triangulated subcategories $\mathcal{A}_1,...,\mathcal{A}_n$ such that

- there is no non-zero Homs from right to left, i.e. $\operatorname{Hom}_{\mathcal{D}}(A_i, A_j) = 0$ for all $A_i \in \operatorname{Ob}(\mathcal{A}_i)$, $A_j \in \operatorname{Ob}(\mathcal{A}_j)$ where $1 \leq j < i \leq n$.
- ② \mathcal{D} is generated by $\mathcal{A}_1,...,\mathcal{A}_n$, i.e. the smallest full triangulated subcategory containing $\mathcal{A}_1,...,\mathcal{A}_n$ equal to \mathcal{D} .

We will use the notation $\mathcal{D}=\langle \mathcal{A}_1,...,\mathcal{A}_n \rangle$ for a semiorthogonal decomposition of \mathcal{D} with components $\mathcal{A}_1,...,\mathcal{A}_n$.

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Remark

Constructing SOD when $\mathcal{D}=\mathcal{D}^bCoh(X)$ is an active research area in algebraic geometry. There has been many developments due to Bondal-Orlov, Kuznetsov...etc.

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For a full triangulated subcategory $\mathcal{C} \subset \mathcal{D}$, we define $\mathcal{C}^{\perp} = \{X \in \mathsf{Ob}(\mathcal{D}) \mid \mathrm{Hom}_{\mathcal{D}}(C,X) = 0 \ \forall \ C \in \mathsf{Ob}(\mathcal{C})\}$ to be the right orthogonal to \mathcal{C} in \mathcal{D} . It is a triangulated subcategories of \mathcal{D} .

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Remark

An exceptional collection is called full if the subcategory $\mathcal A$ is zero.

The simplest example is given by Beilinson for projective space $\mathbb{P}^{N-1}=\mathbb{G}(1,N)$.

Theorem 9 (Beilinson)

There is a full exceptional collection (thus a SOD)

$$\mathcal{D}^bCoh(\mathbb{P}^{N-1}) = \langle \mathcal{O}_{\mathbb{P}^{N-1}}(-N+1), \mathcal{O}_{\mathbb{P}^{N-1}}(-N+2), ..., \mathcal{O}_{\mathbb{P}^{N-1}} \rangle.$$

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Theorem 10 (M. Kapranov)

There is a full exceptional collection (thus a SOD)

$$\mathcal{D}^bCoh(\mathbb{G}(k,N)) = \langle \, \mathbb{S}_{\lambda} \mathcal{V} \, \rangle_{\lambda \in P(N-k,k)}.$$

Since we construct an action of $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_2)$ on $\bigoplus_k \mathcal{D}^bCoh(\mathbb{G}(k,N))$ via using FM kernels, we try to relate the Kapranov exceptional collection to this action.

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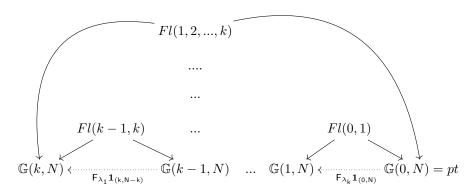
$$Fl(1,2,...,k)$$

$$Fl(k-1,k)$$

$$Fl(0,1)$$

$$\mathbb{G}(k,N) \longleftarrow \mathbb{G}(k-1,N)$$
 ...
$$\mathbb{G}(1,N) \longleftarrow \mathbb{G}(0,N) = pt$$

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More precisely, by using the Borel-Weil-Bott theorem we get

$$\mathbb{S}_{\lambda}\mathcal{V}\cong\mathcal{F}_{\lambda_{1}}st...st\mathcal{F}_{\lambda_{k}}\mathbf{1}_{(0,N)}$$

where $\lambda=(\lambda_1,...,\lambda_k)\in P(N-k,k)$. Note that $\mathcal{F}_{\lambda_1}*...*\mathcal{F}_{\lambda_k}\mathbf{1}_{(0,N)}$ is the FM kernel for the functor $\mathsf{F}_\lambda\mathbf{1}_{(0,N)}:=\mathsf{F}_{\lambda_1}...\mathsf{F}_{\lambda_k}\mathbf{1}_{(0,N)}$.

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Question: Given an (abstract) categorical $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_2)$ (or $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_n)$) action \mathcal{K} . Do the functors

$$\mathsf{F}_{\lambda}\mathbf{1}_{(0,N)} := \mathsf{F}_{\lambda_1}...\mathsf{F}_{\lambda_k}\mathbf{1}_{(0,N)} : \mathcal{K}(0,N) \to \mathcal{K}(k,N-k), \ \lambda \in P(N-k,k)$$

behave like an exceptional collection?

Proposition 11 (Hsu)

The functors $\{\mathsf{F}_{\lambda}\mathbf{1}_{(0,N)}\mid \lambda\in P(N-k,k)\}$ satisfy the following properties

- (1) $\operatorname{Hom}(\mathsf{F}_{\lambda}\mathbf{1}_{(0,N)},\mathsf{F}_{\lambda}\mathbf{1}_{(0,N)}) \cong \operatorname{Hom}(\mathbf{1}_{(0,N)},\mathbf{1}_{(0,N)})$ (exceptional-like property)
- $(2) \ \operatorname{Hom}(\mathsf{F}_{\lambda}\mathbf{1}_{(0,N)},\mathsf{F}_{\lambda'}\mathbf{1}_{(0,N)}) \cong 0, \ \textit{if} \ \lambda <_{l} \lambda' \ (\textit{semiorthogonal property})$

where $<_l$ is the lexicographical order.

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Remark

When the weight categories are $\mathcal{K}(k,N-k)=\mathcal{D}^bCoh(\mathbb{G}(k,N))$, we have $\mathrm{Hom}(\mathbf{1}_{(0,N)},\mathbf{1}_{(0,N)})\cong\mathbb{C}$. This recovers that $\mathbb{S}_\lambda\mathcal{V}$ is exceptional.

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Remark

The first property (1) also implies that the functors $F_{\lambda}\mathbf{1}_{(0,N)}: \mathcal{K}(0,N) \to \mathcal{K}(k,N-k)$ are fully faithful for $\lambda \in P(N-k,k)$.

Then we have the following result.

Theorem 12 (Hsu)

Given a categorical $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_2)$ action \mathcal{K} . There is a SOD

$$\mathcal{K}(k, N-k) = \langle \mathcal{A}, \mathcal{K}_{\lambda}(k, N-k) \rangle_{\lambda \in P(N-k,k)}$$

where $\mathcal{A} := \langle \mathcal{K}_{\lambda}(k, N-k) \rangle_{\lambda \in P(N-k,k)}^{\perp}$ and $\mathcal{K}_{\lambda}(k, N-k) := \langle \mathsf{F}_{\lambda} \mathbf{1}_{(0,N)}(\mathcal{K}(0,N)) \rangle$ is the minimal full triangulated subcategories of $\mathcal{K}(k, N-k)$ generated by the class of objects which are the essential images of $\mathsf{F}_{\lambda} \mathbf{1}_{(0,N)}$.

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Remark

In fact, we prove the above theorem for general \mathfrak{sl}_n action.

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Let $G=\mathrm{SL}_N(\mathbb{C})$ and $B\subset G$ be the Borel of upper triangular matrices. Consider the type A full flag variety

$$G/B = \{0 = V_0 \stackrel{1}{\subset} V_1 \stackrel{1}{\subset} \dots \stackrel{1}{\subset} V_N = \mathbb{C}^N\}$$

and similarly the partial flag variety

$$G/P_i = \{0 \stackrel{1}{\subset} V_1 \stackrel{1}{\subset} V_2 \stackrel{1}{\subset} ... V_{i-1} \stackrel{2}{\subset} V_{i+1} \stackrel{1}{\subset} ... V_N = \mathbb{C}^N \}$$

where P_i is a minimal parabolic subgroup and $1 \le i \le N-1$.

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 π_i induces pullback $\pi_i^*: K(G/P_i) \to K(G/B)$ and pushforward $\pi_{i*}: K(G/B) \to K(G/P_i)$ on the K-theory.

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For each $1 \le i \le N-1$, let $\pi_i: G/B \to G/P_i$ be the natural projection, which is a \mathbb{P}^1 -fibration for all i.

 π_i induces pullback $\pi_i^*:K(G/P_i)\to K(G/B)$ and pushforward $\pi_{i*}:K(G/B)\to K(G/P_i)$ on the K-theory.

We denote \mathcal{V}_i to be the tautological bundle of rank i on G/B, and $\mathcal{L}_i = \mathcal{V}_i/\mathcal{V}_{i-1}$ the natural line bundles. Let $x_i = [\mathcal{L}_i] \in K(G/B)$ be the class in the Grothendieck group.

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$$\begin{split} \delta_i^2 &= \delta_i \; (\text{idempotent}) \\ \delta_i \delta_{i+1} \delta_i &= \delta_{i+1} \delta_i \delta_{i+1} \; (\text{Braid relations}) \end{split}$$

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and thus we have an action $H_N(0)$ on K(G/B). By abusing of notations, we still denote x_i for the linear operators on K(G/B) that defined by multiplication with $x_i = [\mathcal{L}_i]$.

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Remark

Lusztig introduced an q-analogue version of δ_i , which is called the Demazure-Lusztig operator. Together with x_j , he proved that there is an action of the affine Hecke algebra on $K^{G \times \mathbb{C}^*}(G/B)$.

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Theorem 13 (Hsu)

There is a categorical action of the q=0 affine Hecke algebra $\mathcal{H}_N(0)$ on $\mathcal{D}^bCoh(G/B)$, where the generators δ_i,x_j act by lifting to the FM transformation $\Phi_{\mathcal{T}_i}$, $\Phi_{\mathcal{X}_j}$ respectively.

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Remark

One way to prove this theorem is to verify the categorical relations directly by calculating many convolutions of the FM kernels $\mathcal{T}_i, \mathcal{X}_j$. However, instead of proving this theorem by direct calculation, we prove this theorem by relating this action to the categorical action of shifted q=0 affine algebra.

Relation to shifted q=0 affine algebra

For any $\underline{k}=(k_1,...,k_n)\in\mathbb{N}^n$ such that $\sum_i k_i=N$, we define

$$Fl_{\underline{k}}(\mathbb{C}^N) = \{ 0 \stackrel{k_1}{\subset} V_1 \stackrel{k_2}{\subset} \dots \stackrel{k_n}{\subset} V_n = \mathbb{C}^N \}$$

to be the n-step partial flag variety. We will simply use the notation $Fl_{\underline{k}}$ if there is no ambiguity.

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In particular, when n=N, there is a categorical action of $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_{\color{red}N})$ on $\bigoplus_{\underline{k}}\mathcal{D}^bCoh(Fl_{\underline{k}})$ and thus descend to action on $\bigoplus_{\underline{k}}K(Fl_{\underline{k}})$.

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In particular, when n=N, there is a categorical action of $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_N)$ on $\bigoplus_{\underline{k}} \mathcal{D}^b Coh(Fl_{\underline{k}})$ and thus descend to action on $\bigoplus_{\underline{k}} K(Fl_{\underline{k}})$. Note that in this notation, we have $G/B=Fl_{(1,1,\ldots,1)}$, and K(G/B) is one of the direct summand. The main idea is to interpret the Demazure operators δ_i in terms of elements in $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_N)$.

A simple observation

Recall that to construct δ_i , we need

$$G/P_i = \{0 \stackrel{1}{\subset} V_1 \stackrel{1}{\subset} V_2 \stackrel{1}{\subset} ... V_{i-1} \stackrel{2}{\subset} V_{i+1} \stackrel{1}{\subset} ... V_N = \mathbb{C}^N\}$$

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Observe that

$$V_{i-1} \stackrel{?}{\subset} V_{i+1} = V_{i-1} \stackrel{0}{\subset} V_{i-1} \stackrel{?}{\subset} V_{i+1}$$
$$= V_{i-1} \stackrel{2}{\subset} V_{i+1} \stackrel{0}{\subset} V_{i+1}$$

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Observe that

$$\begin{split} V_{i-1} \overset{2}{\subset} V_{i+1} &= \textcolor{red}{V_{i-1}} \overset{0}{\subset} \textcolor{blue}{V_{i-1}} \overset{2}{\subset} V_{i+1} \\ &= V_{i-1} \overset{2}{\subset} V_{i+1} \overset{0}{\subset} V_{i+1} \end{split}$$

So

$$G/P_i = Fl_{(1,1,\dots,1)+\alpha_i} = Fl_{(1,1,\dots,1)-\alpha_i}$$

where $\alpha_i = (0,...,-1,1,...,0)$ is the simple root with -1 at the ith position.

We have the following picture

$$K(G/P_i = Fl_{(1,1,\dots,1)-\alpha_i}) \xrightarrow{f_{i,s}} K(G/B = Fl_{(1,1,\dots,1)}) \xrightarrow{e_{i,r}} K(G/P_i = Fl_{(1,1,\dots,1)+\alpha_i})$$

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As a result, the Demazure operators can be written as

$$\delta_i = e_i f_{i,1}(\psi_i^+)^{-1} 1_{(1,1,\dots,1)} = f_i e_{i,-1}(\psi_i^-)^{-1} 1_{(1,1,\dots,1)}.$$

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Remark

Here we simply denote e_i for $e_{i,0}$, f_i for $f_{i,0}$. Similarly for \mathcal{E}_i and \mathcal{F}_i .

Lifting

Lifting to the categorical level, as a result, we have the isomorphisms of FM kernels

$$\mathcal{T}_{i} \cong \mathcal{E}_{i} * \mathcal{F}_{i,1} * (\Psi_{i}^{+})^{-1} \mathbf{1}_{(1,1,\dots,1)} \cong \mathcal{F}_{i} * \mathcal{E}_{i,-1} * (\Psi_{i}^{-})^{-1} \mathbf{1}_{(1,1,\dots,1)}$$
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Thus the categorical relations that we need to verify for $\mathcal{H}_N(0)$ can be deduced from the categorical relations in $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_N)$.

In particular, the categorical commutator relations

$$(\mathcal{F}_{i,1} * \mathcal{E}_i) \mathbf{1}_{(1,1,\dots,1)} \to (\mathcal{E}_i * \mathcal{F}_{i,1}) \mathbf{1}_{(1,1,\dots,1)} \to \Psi_i^+ \mathbf{1}_{(1,1,\dots,1)}, (\mathcal{E}_{i,-1} * \mathcal{F}_i) \mathbf{1}_{(1,1,\dots,1)} \to (\mathcal{F}_i * \mathcal{E}_{i,-1}) \mathbf{1}_{(1,1,\dots,1)} \to \Psi_i^- \mathbf{1}_{(1,1,\dots,1)},$$

are precisely the (categorical) affine Hecke relations

$$\mathcal{X}_i * \mathcal{T}_i \to \mathcal{T}_i * \mathcal{X}_{i+1} \to \mathcal{X}_{i+1},$$

 $\mathcal{T}_i * \mathcal{X}_i \to \mathcal{X}_{i+1} * \mathcal{T}_i \to \mathcal{X}_{i+1}.$

For $\underline{k} = (k_1, ..., k_n)$ with n < N, the pullback induced by the natural projection $\pi : G/B = Fl_{(1,1,...,1)} \to Fl_k$ makes $K(Fl_k)$ as a submodule of K(G/B).

Generalization to Fl_k

For $\underline{k}=(k_1,...,k_n)$ with n< N, the pullback induced by the natural projection $\pi:G/B=Fl_{(1,1,...,1)}\to Fl_{\underline{k}}$ makes $K(Fl_{\underline{k}})$ as a submodule of K(G/B). One natural question is to generalize the construction of the Demazure operators δ_i from G/B to Fl_k .

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Motivating from the above interpretation of δ_i for G/B, and since there is a categorical action of $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_n)$ on $\bigoplus_{\underline{k}} \mathcal{D}^b Coh(Fl_{\underline{k}})$, we can try to generalize δ_i to get operators acting on $\mathcal{D}^b Coh(Fl_k)$ and $K(Fl_k)$.

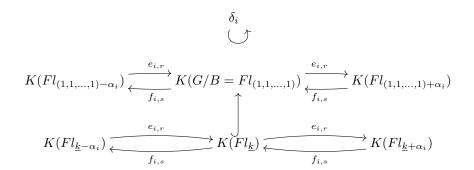
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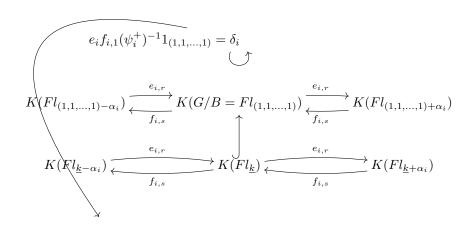
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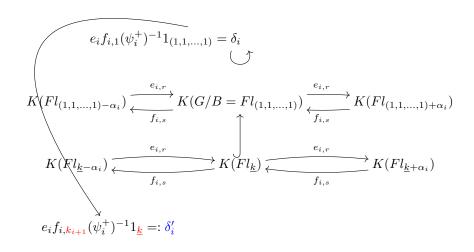
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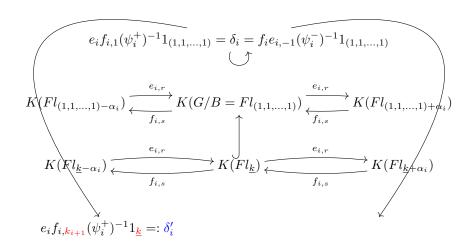
Remark

Note that the submodule $K(Fl_{\underline{k}})$ is not invariant under the action of δ_i , so $\delta_i|_{K(Fl_k)}$ does not work.









$$K(Fl_{(1,1,\dots,1)-\alpha_{i}}) \xrightarrow{e_{i,r}} K(G/B = Fl_{(1,1,\dots,1)}) \xrightarrow{e_{i,r}} K(Fl_{(1,1,\dots,1)+\alpha_{i}}) \xrightarrow{e_{i,r}} K(Fl_{(1,1,\dots,1$$

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Now we have the operators

$$\delta_i'=e_if_{i,k_{i+1}}(\psi_i^+)^{-1}1_{\underline{k}},\ \delta_i''=f_ie_{i,-k_i}(\psi_i^-)^{-1}1_{\underline{k}}:K(Fl_{\underline{k}})\to K(Fl_{\underline{k}}).$$

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Lifting to categorical level by using FM transforms, we denote

$$\begin{split} \mathcal{T}_i' &:= \mathcal{E}_i * \mathcal{F}_{i,k_{i+1}} * (\Psi_i^+)^{-1} \mathbf{1}_{\underline{k}} \\ \mathcal{T}_i'' &:= \mathcal{F}_i * \mathcal{E}_{i,-k_i} * (\Psi_i^-)^{-1} \mathbf{1}_{\underline{k}} \end{split}$$

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Remark

In general, we have $\delta_i' \neq \delta_i''$ and thus $\mathcal{T}_i' \ncong \mathcal{T}_i''$. They are equal/isomorphic only when $\underline{k} = (1,1,...,1)$.

Demazure operators for Fl_k

From the categorical commutator relations,

$$(\mathcal{F}_{i,k_{i+1}} * \mathcal{E}_i) \mathbf{1}_{\underline{k}} \to (\mathcal{E}_i * \mathcal{F}_{i,k_{i+1}}) \mathbf{1}_{\underline{k}} \to \Psi_i^+ \mathbf{1}_{\underline{k}},$$

$$(\mathcal{E}_{i,-k_i} * \mathcal{F}_i) \mathbf{1}_{\underline{k}} \to (\mathcal{F}_i * \mathcal{E}_{i,-k_i}) \mathbf{1}_{\underline{k}} \to \Psi_i^- \mathbf{1}_{\underline{k}},$$

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since $\Psi_i^+ \mathbf{1}_{\underline{k}}$, $\Psi_i^- \mathbf{1}_{\underline{k}}$ are certain line bundles, we can convolution their inverse to get the following exact triangle

$$\mathcal{F}_{i,k_{i+1}} * \mathcal{E}_{i} * (\Psi_{i}^{+})^{-1} \mathbf{1}_{\underline{k}} \to \mathcal{T}_{i}' = \mathcal{E}_{i} * \mathcal{F}_{i,k_{i+1}} * (\Psi_{i}^{+})^{-1} \mathbf{1}_{\underline{k}} \to \mathcal{O}_{\Delta},$$

$$\mathcal{E}_{i,-k_{i}} * \mathcal{F}_{i} * (\Psi_{i}^{-})^{-1} \mathbf{1}_{\underline{k}} \to \mathcal{T}_{i}'' = \mathcal{F}_{i} * \mathcal{E}_{i,-k_{i}} * (\Psi_{i}^{-})^{-1} \mathbf{1}_{\underline{k}} \to \mathcal{O}_{\Delta}.$$

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$$(\mathcal{F}_{i,k_{i+1}} * \mathcal{E}_i) \mathbf{1}_{\underline{k}} \to (\mathcal{E}_i * \mathcal{F}_{i,k_{i+1}}) \mathbf{1}_{\underline{k}} \to \Psi_i^+ \mathbf{1}_{\underline{k}},$$

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$$\begin{split} &\mathcal{F}_{i,k_{i+1}} * \mathcal{E}_i * (\Psi_i^+)^{-1} \mathbf{1}_{\underline{k}} \to \underline{\mathcal{T}_i'} = \mathcal{E}_i * \mathcal{F}_{i,k_{i+1}} * (\Psi_i^+)^{-1} \mathbf{1}_{\underline{k}} \to \mathcal{O}_{\Delta}, \\ &\mathcal{E}_{i,-k_i} * \mathcal{F}_i * (\Psi_i^-)^{-1} \mathbf{1}_{\underline{k}} \to \underline{\mathcal{T}_i''} = \mathcal{F}_i * \mathcal{E}_{i,-k_i} * (\Psi_i^-)^{-1} \mathbf{1}_{\underline{k}} \to \mathcal{O}_{\Delta}. \end{split}$$

We denote $\mathcal{S}_i':=\mathcal{F}_{i,k_{i+1}}*\mathcal{E}_i*(\Psi_i^+)^{-1}\mathbf{1}_{\underline{k}}$ and $\mathcal{S}_i'':=\mathcal{E}_{i,-k_i}*\mathcal{F}_i*(\Psi_i^-)^{-1}\mathbf{1}_{\underline{k}}.$

Theorem 14 (Hsu)

(1) Idempotent

$$\mathcal{T}_{i}^{'}*\mathcal{T}_{i}^{'}\cong\mathcal{T}_{i}^{'},\ \mathcal{T}_{i}^{''}*\mathcal{T}_{i}^{''}\cong\mathcal{T}_{i}^{''}.$$

(2) There exist exact triangles in $\mathcal{D}^bCoh(Fl_{\underline{k}} \times Fl_{\underline{k}})$

$$\begin{split} \mathcal{T}_{i+1}' * \mathcal{T}_{i}^{'} * \mathcal{T}_{i+1}' * \mathcal{S}_{i}^{'} &\to \mathcal{T}_{i}^{'} * \mathcal{T}_{i+1}' * \mathcal{T}_{i}^{'} &\to \mathcal{T}_{i+1}' * \mathcal{T}_{i}^{'} * \mathcal{T}_{i+1}', \\ \mathcal{T}_{i}^{''} * \mathcal{T}_{i+1}^{''} * \mathcal{T}_{i}^{''} * \mathcal{S}_{i+1}^{''} &\to \mathcal{T}_{i+1}^{''} * \mathcal{T}_{i}^{''} * \mathcal{T}_{i+1}^{''} &\to \mathcal{T}_{i}^{''} * \mathcal{T}_{i+1}^{''} * \mathcal{T}_{i}^{''}. \end{split}$$

(3) Vanishing

$$\mathcal{T}_{i}^{'} * \mathcal{T}_{i+1}^{'} * \mathcal{T}_{i}^{'} * \mathcal{S}_{i+1}^{'} \cong \mathcal{T}_{i+1}^{''} * \mathcal{T}_{i}^{''} * \mathcal{T}_{i+1}^{''} * \mathcal{S}_{i}^{''} \cong 0$$

Theorem 14 (Hsu)

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(3) Vanishing

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Remark

In particular, when $\underline{k}=(1,1,...,1)$ (which is G/B), we have $\mathcal{T}_i'\cong\mathcal{T}_i''$ and $\mathcal{S}_i'\cong\mathcal{S}_i''$. So (2) and (3) imply the categorical braid relations. Thus the categorical action of $H_N(0)$ on $\mathcal{D}^bCoh(G/B)$ is a direct consequence of this theorem.

When n=2, we obtain the action of $H_2(0)$ on $K(\mathbb{G}(k,N))$, where the generator acts by $\delta'=ef_{N-k}(\psi^+)^{-1}1_{(k,N-k)}$.

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Theorem 15 (Hsu)

$$\delta^{'}([\mathbb{S}_{\lambda}\mathcal{V}]) = \begin{cases} 0 & \text{if } \lambda_{1} = N - k \\ [\mathbb{S}_{\lambda}\mathcal{V}] & \text{if } 0 \leq \lambda_{1} \leq N - k - 1, \end{cases}$$

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Remark

In fact, we calculate the action of δ_i' on the basis given by the Kapranov exceptional collection for the n-step partial flag variety Fl_k .

- Introduction
 - Categorical \mathfrak{sl}_2 action
 - An example from geometry
- The motivation and the main result
- Applications
 - Semiorthogonal decomposition
 - Demazure operators
- Some related works

Relate to the action of (quantum) loop algebra

Consider the K-theory of cotangent bundle of n-step partial flag varieties (\mathfrak{sl}_n Nakajima quiver varieties) $K(T^*Fl_k)$.

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$$\bigoplus_{\underline{k}} K(T^*Fl_{\underline{k}}) \xrightarrow{Thom} \bigoplus_{\underline{k}} K(Fl_{\underline{k}})$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad$$

Consider the affine Grassmannain $Gr_{GL_N}:=GL_N(\mathcal{K})/GL_N(\mathcal{O})$ where $\mathcal{O}=\mathbb{C}[[t]]$ is the formal power series ring and $\mathcal{K}=\mathbb{C}((t))$ its fraction field.

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$$\mathfrak{U}_{q,-2Nlpha^ee}$$
 \longrightarrow $K_{loc}^{\widetilde{GL_N}(\mathcal{O})\rtimes\widetilde{\mathbb{C}^*}}(Gr_{GL_N})$ (quantized K-theoretic Coulomb branch)

where $\mathfrak{U}_{q,-2Nlpha^ee}$ are certain truncated shifted quantum affine algebra.

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$$\mathcal{U}_{q,\mu} \nearrow \qquad \bigoplus_{\underline{k}} K_{loc}^{\widetilde{T} \times \widetilde{\mathbb{C}^*}}(\mathfrak{Q}_{\underline{k}}) \qquad \qquad \text{(K-theory of Laumon based parabolic quasiflag spaces)}$$

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It would also be interesting to see the relations to these works, e.g. lifting the above results from K-theory to derived categories by using our categorical action.

Thank you for your attention.