Galois Conjugation of TQFTs

Math-String Seminar, Kavli IPMU

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Based on work with Dr Matthew Buican

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THE ROYAL SOCIETY

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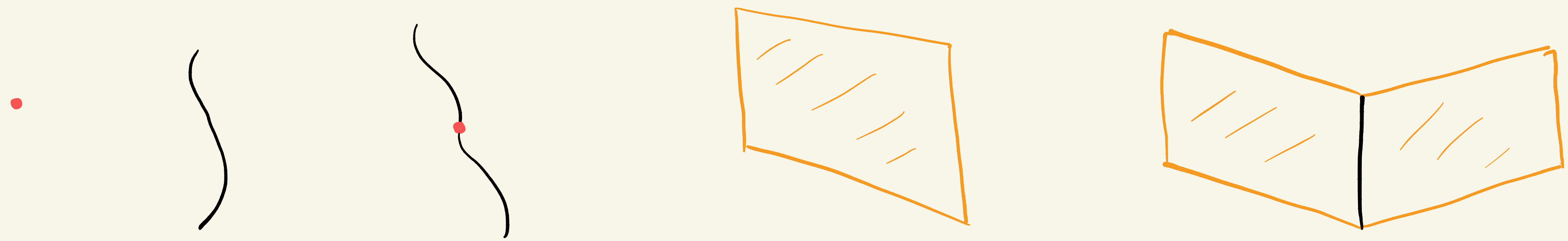
Plan of the Talk

1. Motivation
2. 2+1D Topological Quantum Field Theory
3. Galois Conjugation
4. Galois Conjugation of TQFTs
5. Galois action v/s Unitarity
6. Symmetries and Gapped Boundaries
7. Conclusion

Motivation

1. Understand the mathematical structure of QFTs.

Observables and their correlation functions are basic constituents of a quantum field theory.



Point Line Junction of Line Surface Junction of surfaces etc...

Aim: Encode the zoo of operators in a suitable mathematical structure.

What is that structure? Hard problem in general.

Life becomes easier with topological symmetry.

2. Symmetry = Topological operators [Gaiotto, Kapustin, Seiberg, Willett (2014)]

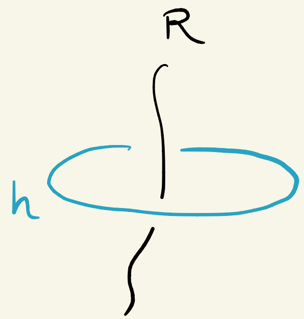
The modern view of symmetry of a QFT is in terms of the subset of allowed topological operators in the QFT.

Example: In 2+1D QFTs

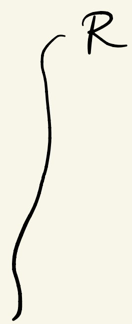
0-form symmetry

$= R(g)\, O(x)$

1-form symmetry



$= R(h)$



TQFTs are theories of topological operators.

2+1D Topological Quantum Field Theories

TQFTs are QFTs where all observables are topological.

Chern-Simons Theory

Given a 3-manifold M and Lie group G, we can define

$$S = \frac{k}{4\pi} \int_M \mathrm{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

A: Connection 1-form

Classical EOM: $F = 0$ $\Rightarrow$ No gauge-invariant local operators.

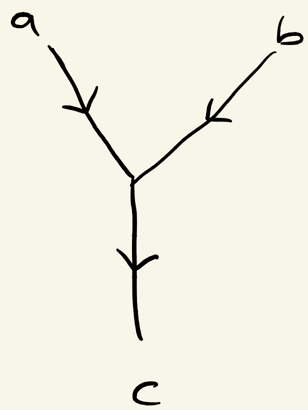
Observables: Wilson loops $\quad W_R(N) = \mathrm{Tr}_R\, P \exp\left(i \oint_N A \right)$

2+1D TQFTs are described by algebraic objects called modular tensor categories (MTC).

Basic constituents of an MTC

Objects: $a, b, c, \ldots$ labels line operators / anyons.

Fusion Rules

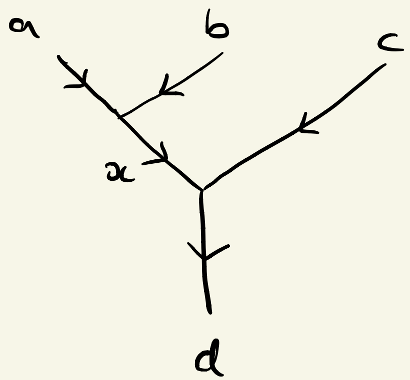


$$a \otimes b = \sum_c N_{ab}^{c}\, c \qquad \text{where } N_{ab}^{c} \in \mathbb{Z}_{\geq 0}$$

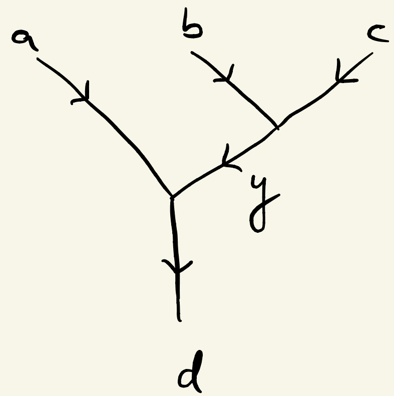
N_{ab}^{c} are dimensions of fusion spaces $V_{ab}^{c} = \mathrm{Hom}(a \otimes b, c)$.

More generally, $V^{b}_{a_1 a_2 \cdots a_n} = \mathrm{Hom}(a_1 \otimes a_2 \otimes \cdots a_n, b)$.

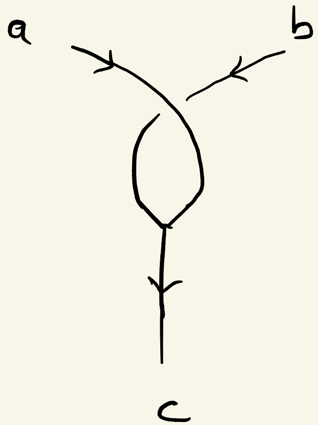
Fusion is associative.



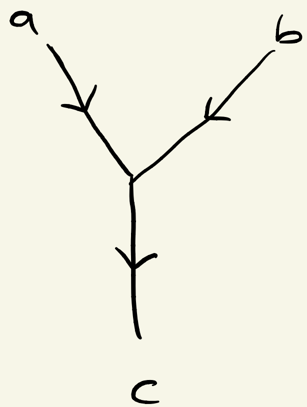
$$= \sum_y \left(F^{d}_{abc}\right)^{x}_{y}$$



Fusion is commutative.

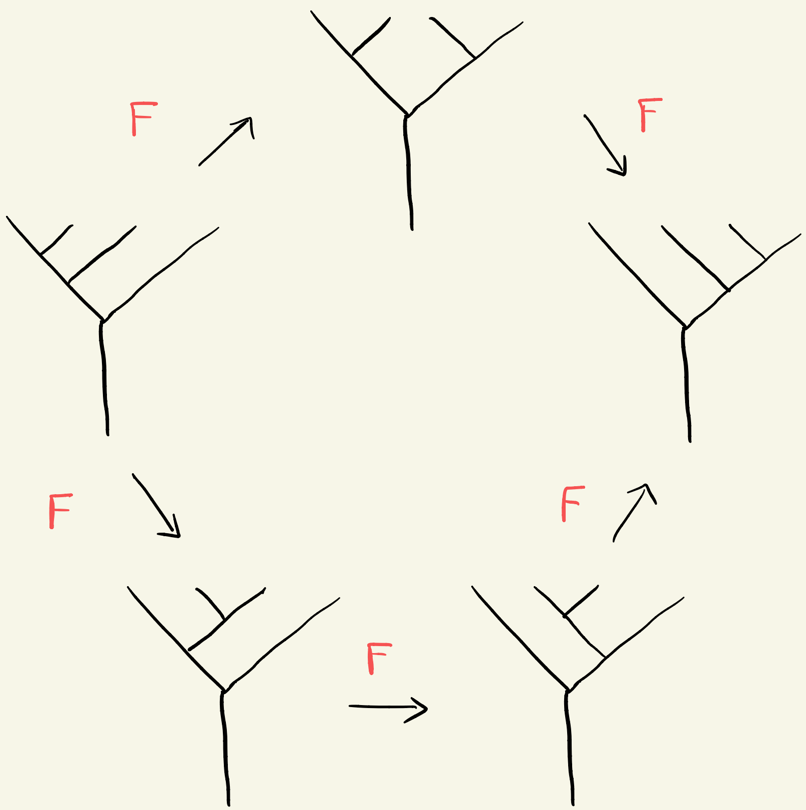


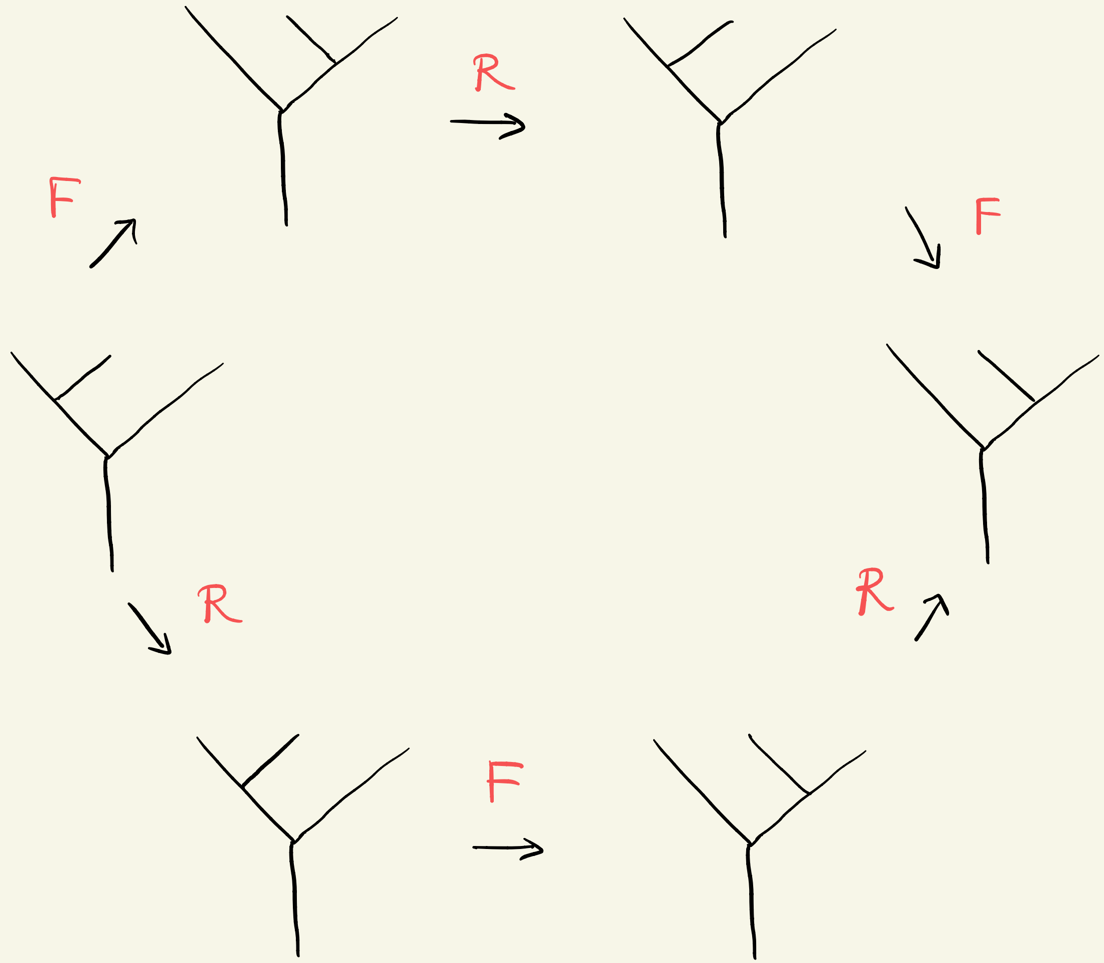
$$= R^{c}_{ab}$$



Pentagon and Hexagon constraints. [Moore, Seiberg (1989)]

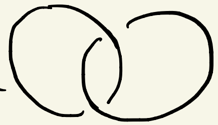
[Etingof, Gelaki, Nikshych, Ostrik (2015)]



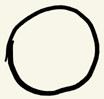


Explicit solutions to Pentagon and Hexagon equations depend on a choice of basis of V^c_{ab}. $\Rightarrow$ F and R are "gauge dependent".

Modular Data

$$S_{ab} = \frac{1}{D}$$ a  b $$= \frac{1}{D} \sum_c d_c R^c_{\bar b a} R^c_{a \bar b}$$

$$\theta_a =$$ a  $$= \frac{1}{d_a} \sum_c d_c R^c_{aa}$$

where $d_a =$  a are invariants of the unknot labelled by a and

$$D = \sqrt{\sum_c d_c^2}.$$

Unitary TQFT :

$\exists$ a basis such that F and R are unitary and $d_a > 0 \ \forall a$.

$$d_a = \underbrace{\text{Frobenius - Perron dimension}}_{\text{unique largest real eigenvalue}} \text{ of } (N_a)^c_b.$$

Abelian TQFTs

In this case the fusion rules form an abelian group A.

$$a \otimes b = c$$

Equivalently, $d_a = 1 \quad \forall\, a$.

In this case F and R are determined by θ.

Let $A = \mathbb{Z}_{k_1} \otimes \cdots \otimes \mathbb{Z}_{k_n}$ with basis e_i. $\vec{p} = \sum_i p_i e_i$

$$F(\vec{p}, \vec{q}, \vec{r}) = \prod_i \begin{cases} 1 & \text{if } q_i + r_i < k_i \\ \theta(e_i)^{p_i k_i} & \phantom{\text{if }} q_i + r_i \geq k_i \end{cases}$$

$$R(\vec{p}, \vec{q}) = \prod_i \theta(e_i)^{p_i q_i} \prod_{i<j} S(e_i, e_j)^{p_i q_j}$$

[Quinn (1999)]

Examples

Toric Code or Spin(16)$_1$ CS Theory

Anyons : $1, e, m, \epsilon$.

Fusion Rules : $e \otimes e = m \otimes m = \epsilon \otimes \epsilon = 1, \quad e \otimes m = \epsilon, \quad m \otimes \epsilon = e,$

$e \otimes \epsilon = m$

$d_a = 1 \ \forall a$

TQFT determined by twists. $\theta_1 = \theta_e = \theta_m = 1, \quad \theta_\epsilon = -1$

Semion model or SU(2)$_1$ CS Theory

Anyons : $1, s$. Fusion Rules : $s \otimes s = 1$

$\theta_1 = 1, \quad \theta_s = i. \quad d_a = 1 \ \forall a$

Fibonacci Model or $(G_2)_1$ CS Theory

Anyons : $1, \tau$ Fusion Rules : $\tau \otimes \tau = 1 + \tau$

Twists : $\theta_1 = 1$, $\theta_\tau = e^{\frac{4\pi i}{5}}$

Quantum dimensions : $d_1 = 1$, $d_\tau = \dfrac{1+\sqrt{5}}{2} = \varphi$

$$F^{\tau}_{\tau\tau\tau} = \begin{pmatrix} \varphi^{-1} & \varphi^{-1/2} \\ \varphi^{-1/2} & -\varphi^{-1} \end{pmatrix} ; \quad R^{1}_{\tau\tau} = e^{\frac{2\pi i}{5}} ; \quad R^{\tau}_{\tau\tau} = e^{-\frac{2\pi i 3}{10}}$$

$$S = \frac{1}{\sqrt{2+\varphi}} \begin{pmatrix} 1 & \varphi \\ \varphi & -1 \end{pmatrix} ; \quad T = \mathrm{diag}\left(1, \; e^{\frac{4\pi i}{5}}\right)$$

Fusion Categories and Symmetries of 1+1D QFTs.

[Bhardwaj, Tachikawa (2017)]

Suppose a 1+1D QFT has a symmetry G. This is related to a fusion category Vec_G^{α}.

Symmetry	$\longleftrightarrow$	Fusion category
Elements of G		Objects
Group multiplication		Fusion rules
Anomaly of G $\alpha \in H^3(G, U(1))$		F symbols

Galois Conjugation

Consider the polynomial

$$x^4 = 1 \quad \Longrightarrow \quad x = 1,\ -1,\ i,\ -i$$

Polynomial over $\mathbb{Q}$ but all solutions not in $\mathbb{Q}$.

Define $\mathbb{Q}(i) = \{ a + ib \mid a, b \in \mathbb{Q} \}$. $\mathbb{Q}(i)$ is a field extension of $\mathbb{Q}$.

Galois group of $\mathbb{Q}(i)$

$\mathrm{Gal}(\mathbb{Q}(i)) \ni$ automorphisms of $\mathbb{Q}(i)$ which fix $\mathbb{Q}$.

||

preserves algebraic operations

$$g(a + ib) = a + g(i)b = a - ib \quad , \quad g \in \mathrm{Gal}(\mathbb{Q}(i)) \cong \mathbb{Z}_2$$

Now consider the polynomial $x^2 = i$ which is defined over $\mathbb{Q}(i)$.

We get, $x = \pm\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2} \notin \mathbb{Q}(i)$.

We can extend the field further to get

$$\mathbb{Q}(i, \sqrt{2}) = \{a + ib + \sqrt{2}c + i\sqrt{2}d \mid a, b, c, d \in \mathbb{Q}\}.$$

$\mathrm{Gal}(\mathbb{Q}(i, \sqrt{2}))$ generated by $g(i) = -i$ and $h(\sqrt{2}) = -\sqrt{2}$.

$g \in \mathrm{Gal}(\mathbb{Q}(i))$ acts on $x^2 = i$ to give $x^2 = -i$.

$$\Rightarrow \quad x = \mp\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2} = gh\left(\pm\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}\right).$$

For every $\ell \in \mathrm{Gal}(\mathbb{Q}(i)) \ \exists \ \tilde{\ell} \in \mathrm{Gal}(\mathbb{Q}(i, \sqrt{2}))$ such that

$$\tilde{\ell}\big|_{\mathbb{Q}(i)} = \ell$$

Galois Conjugation of TQFTs [Davidovich, Hagge, Wang (2013)]

The Pentagon and Hexagon equations are multivariable polynomials over $\mathbb{Q}$.

We can construct the field extension $K_c := \mathbb{Q}(F, R)$. The Galois group of this field extension acts on F and R.

defining number field of the TQFT

$$q \in \mathrm{Gal}(\mathbb{Q}(F, R)) \; ; \quad F \to q(F) \qquad R \to q(R)$$

$q(F)$ and $q(R)$ is another solution to the Pentagon and Hexagon equations.

$$\mathrm{TQFT}_{F,R} \xleftrightarrow{\text{Galois Conjugation}} \mathrm{TQFT}_{q(F),\, q(R)}$$

Examples

Toric Code

Anyons : $1, e, m, \epsilon$

TQFT determined by twists. $\theta_1 = \theta_e = \theta_m = 1, \quad \theta_\epsilon = -1$

$\mathbb{Q}(F_{\text{Toric}}, R_{\text{Toric}}) = \mathbb{Q}$. Galois action is trivial.

Semion model

Anyons : $1, s$ and $\theta_1 = 1, \quad \theta_s = i$

$\mathbb{Q}(F_{\text{Semion}}, R_{\text{Semion}}) = \mathbb{Q}(i)$. Galois action is complex conjugation.

$$q(\theta_s) = -i$$

Semion $\longleftrightarrow$ Anti-Semion

Double - Semion = Semion $\boxtimes$ Anti - Semion

$\theta_{(1,1)} = 1$, $\theta_{(1,\bar{s})} = -i$, $\theta_{(s,1)} = i$, $\theta_{(s,\bar{s})} = 1$

We know that under Galois action

$$\text{Semion} \longleftrightarrow \text{Anti-Semion}$$

Therefore,

$$
\begin{aligned}
q(\text{Double-Semion}) &= q(\text{Semion}) \boxtimes q(\text{Anti-Semion}) \\
&= \text{Anti-Semion} \boxtimes \text{Semion} \\
&= \text{Double-Semion}
\end{aligned}
$$

Because $(1, \bar{s}) \longleftrightarrow (s, 1)$ under Galois action.

Galois action can be a symmetry of the theory!

Fibonacci Model

Anyons : $1, \tau$ Fusion Rules : $\tau \otimes \tau = 1 + \tau$

$$\tau \times \tau \times \tau = \tau(1+\tau) = \tau + 1 + \tau = 2\tau + 1$$

Twists : $\theta_1 = 1$, $\theta_\tau = e^{\frac{4\pi i}{5}}$

Quantum dimensions : $d_1 = 1$, $d_\tau = \frac{1+\sqrt{5}}{2} = \varphi$

$$F^{\tau}_{\tau\tau\tau} = \begin{pmatrix} \varphi^{-1} & \varphi^{-1/2} \\ \varphi^{-1/2} & -\varphi^{-1} \end{pmatrix} ; \quad R^{1}_{\tau\tau} = e^{\frac{2\pi i}{5}} ; \quad R^{\tau}_{\tau\tau} = e^{-\frac{2\pi i 3}{10}}$$

$$S = \frac{1}{\sqrt{2+\varphi}} \begin{pmatrix} 1 & \varphi \\ \varphi & -1 \end{pmatrix} ; \quad T = \mathrm{diag}\left(1, e^{\frac{4\pi i}{5}}\right)$$

Defining number field $= \mathbb{Q}\left(\sqrt{\varphi}, e^{\frac{2\pi i}{10}}, \sqrt{5}\right) = \mathbb{Q}\left(e^{\frac{2\pi i}{20}}\right)$

$\exists$ a Galois action which $\sqrt{5} \to -\sqrt{5}$. So $d_\tau \to \frac{1-\sqrt{5}}{2} < 0$

Galois action need not preserve unitarity!

Galois action and unitarity

Recall: Unitary TQFT $\longrightarrow$ $d_a > 0$, F and R unitary.

Galois action can violate both!

Suppose F is unitary, then

$$F^\dagger F = \mathbb{1}$$

After Galois action,

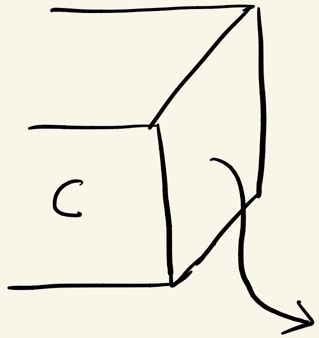
$$\sigma(F^\dagger)\,\sigma(F) \neq (\sigma(F))^\dagger \sigma(F), \qquad \sigma \in \mathrm{Gal}(K_C).$$

unless σ commutes with complex conjugation.

If F and R are unitary in a CM field, then

$$\sigma(F) \text{ and } \sigma(R) \text{ unitary } \forall\ \sigma \in \mathrm{Gal}(K_C).$$

Gapped Boundaries and Galois action



Trivial TQFT

gapped boundary A

A is determined by a Lagrangian algebra in C.

$\downarrow$

"a bunch of bosons which can be condensed"

$$\underset{\substack{\cap \\ C}}{A} = \sum_{a \in C} n_a\, a \quad \text{such that} \quad \begin{array}{ll} \text{(i)} & \theta_a = 1 \ \forall a \\ \text{(ii)} & \dim(C) = \dim(A)^2 \\ \text{(iii)} & n_a n_b \le N_{ab}^c n_c \end{array}$$

(for unitary TQFT)

$$\dim(C) := \sum_{a \in C} d_a^2$$

[Cong, Cheng, Wang (2016)]

All of these conditions are preserved under Galois action!

If $\mathcal{C}$ admits a gapped boundary, so does $\sigma(\mathcal{C})$ $\forall$ $\sigma \in \mathrm{Gal}(K_{\mathcal{C}})$.

Example : Consider the discrete gauge theory based on group $\mathbb{Z}_N$ and twist $\omega \in H^3(\mathbb{Z}_N, U(1))$. $D(\mathbb{Z}_N)^\omega$

$$\omega(a,b,c) = e^{\frac{2\pi i p}{N} a\,(b+c \,-\, b+c \bmod N)} \quad , \; p \in \{0, 1, \ldots, N-1\}.$$

Anyons : (a,m) $\qquad a, m \in \{0, 1, \ldots, N-1\}$.

$$(a,m) \otimes (b,n) = \left(a+b \bmod N,\ \left[m+n - \frac{2p}{N}\,(a+b - a+b \bmod N)\right] \bmod N\right)$$

$$\theta_{(a,m)} = e^{\frac{2\pi i}{N} am} \; e^{-\frac{2\pi i}{N^2} pa^2}$$

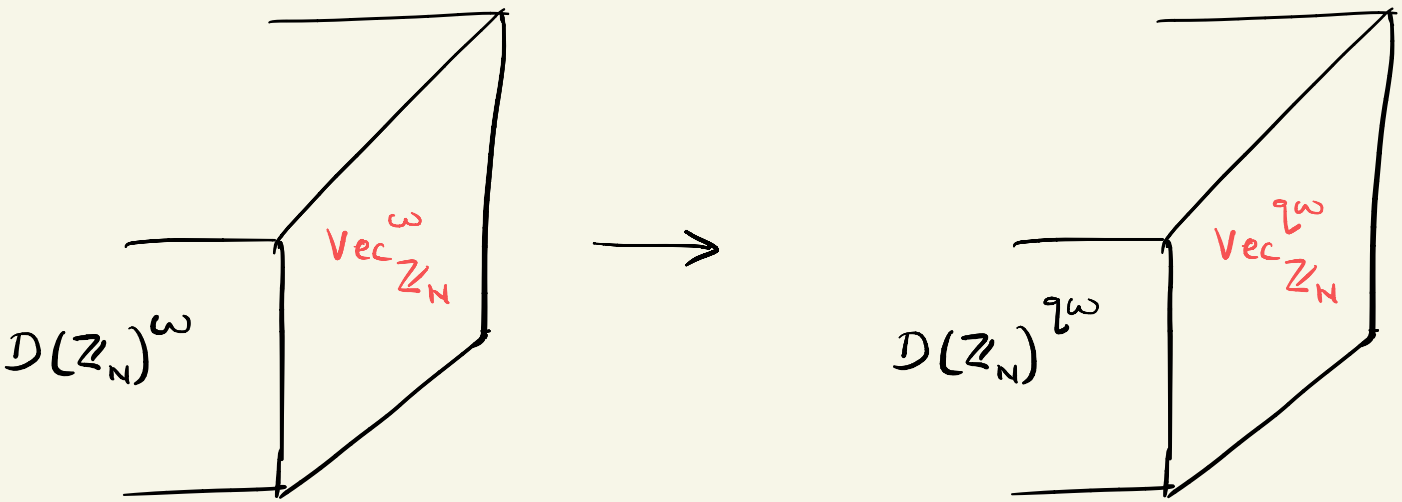
[Coste, Gannon, Ruelle (2000)]

The anyons $(0, m)$ form the $\mathrm{Rep}(\mathbb{Z}_N)$ which can be condensed to get the gapped boundary $\mathrm{Vec}^{\omega}_{\mathbb{Z}_N}$.

$\exists$ Galois action

$$D(G)^{\omega} \to D(G)^{q\omega} \qquad \gcd(q, N) = 1$$

Under this Galois action,



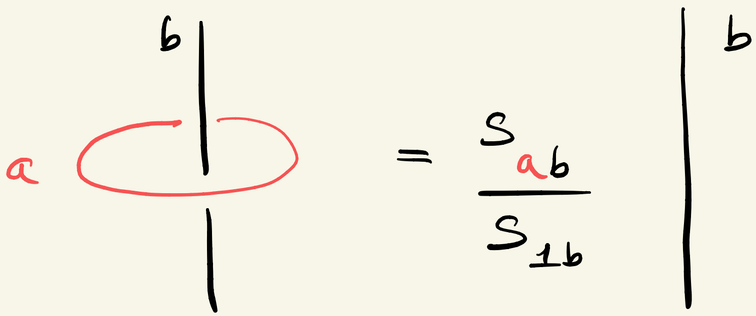
Symmetries and Galois Conjugation

1-form symmetry

[Barkeshli, Bonderson, Cheng, Wang (2014)]

1-form symmetry of a TQFT: subset of abelian anyons in the MTC.

$a \in \mathcal{C}$ such that $d_a = 1$ forms an abelian group A.



Since Galois action preserves the fusion rules, it preserves the 1-form symmetry group A.

0-form symmetry

0-form symmetry of a TQFT: permutations of the anyons such that all gauge invariant data is left invariant.

$g \in G$

$$F^{g(d)}_{g(a)\,g(b)\,g(c)} \cong F^{d}_{abc}$$

$$R^{g(c)}_{g(a)g(b)} \cong R^{c}_{ab}$$

$\cong$: up to gauge transformation

Therefore, gauge invariant data remain invariant.

$$S_{g(a)g(b)} = S_{ab}\,, \quad N^{g(c)}_{g(a)g(b)} = N^{c}_{ab}$$

$$\theta_{g(a)} = \theta_{a}$$

[Barkeshli, Bonderson, Cheng, Wang (2014)]

Under a Galois action,

$$S_{g(a)g(b)} = S_{ab} \implies \tau\left(S_{g(a)g(b)}\right) = \tau\left(S_{ab}\right), \quad \tau \in \mathrm{Gal}(K_C).$$

$$\theta_{g(a)} = \theta_a \qquad\qquad \tau\left(\theta_{g(a)}\right) = \tau\left(\theta_a\right)$$

Therefore, permutations which preserve gauge invariant data are preserved under Galois action.

Example: 0-form symmetries of abelian TQFTs are automorphims of the fusion rules which preserve the twists.

$$g \in G : \quad N_{g(a)g(b)}^{g(c)} = N_{ab}^{c} \quad \text{and} \quad \theta_{g(a)} = \theta_a .$$

Both conditions are preserved under Galois action.

$\Rightarrow$ 0-form symmetry is preserved under Galois action.

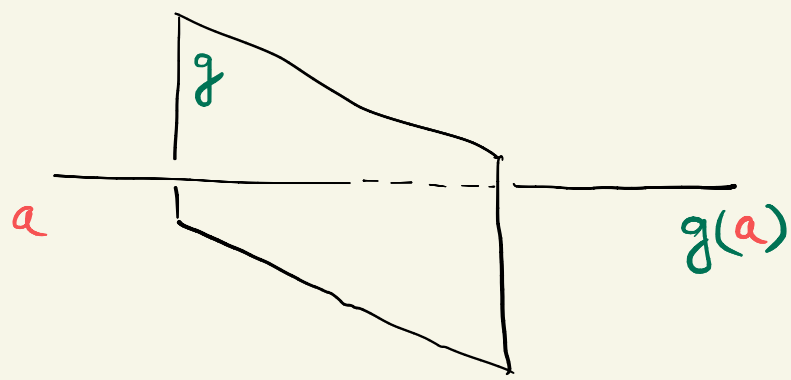
2-group Symmetry

[Barkeshli, Bonderson, Cheng, Wang (2014)]
[Benini, Cordova, Hsin (2018)]

The 0-form symmetry G and 1-form symmetry A can interact non-trivially.

G acts on anyons $\Rightarrow$ G acts on abelian anyons $\Rightarrow$ G acts on A.

Therefore, we have some $\rho : G \to \mathrm{Aut}(A)$.

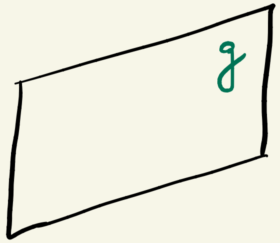


G and A together forms an algebraic object called a 2-group.

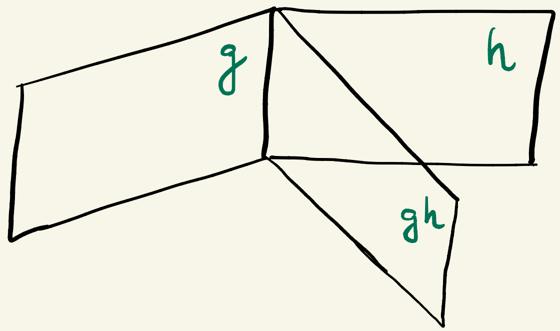
2-group $(G, A, \rho, [\beta])$ is a monoidal category with invertible morphisms.

↑ Postnikov class

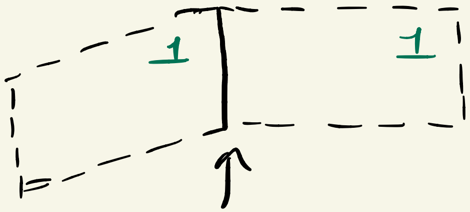
Objects $g, h, \ldots$



Morphisms : $\mathrm{Hom}(g \otimes h, gh)$

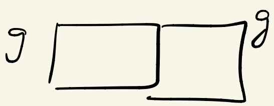


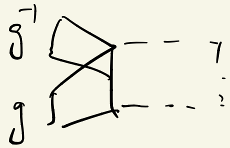
Note : $\mathrm{Hom}(1, 1) \cong A$



Given some $f \in \mathrm{Hom}(g, g)$, we have some $a \in \mathrm{Hom}(g \otimes g^{-1}, 1)$

$\cong \mathrm{Hom}(1, 1) \cong A$





Therefore, given

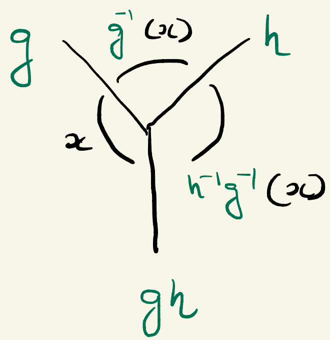
$$\alpha : g\otimes(h\otimes k) \longrightarrow (g\otimes h)\otimes k \in H^3(G, U(1))$$

we get a $\beta \in H^3_\rho(G, A)$. [Baez, Lauda (2003)]

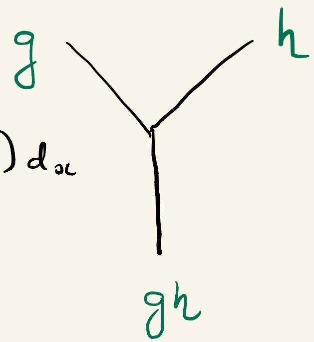
Picture: Action of G is associative only up to a 1-form symmetry action by β.

β satistifies

$$\frac{S_{\beta(g,h,k)\,a}}{S_{1a}} = \frac{\eta_a(gh,k)\,\eta_a(g,h)}{\eta_{g^{-1}(a)}(h,k)\;\eta_a(g,hk)} \; ;$$



$= \eta_x(g,h)\, d_x$



where η is found by solving algebraic constraints involving F and R.

[Barkeshli, Bonderson, Cheng, Wang (2014)]

Under a Galois action $q \in \mathrm{Gal}(K_c)$

$$F \longrightarrow q(F) \;;\quad R \longrightarrow q(R) \;;\quad S \longrightarrow q(S).$$

Galois action on F and R induces a Galois action $q'(\eta)$ on η.

Therefore, we have

$$q\left(\frac{S_{\beta(g,h,k)a}}{S_{1a}}\right) = q'\left(\frac{\eta_a(gh,k)\,\eta_a(g,h)}{\eta_{g^{-1}(a)}(h,k)\,\eta_a(g,hk)}\right)$$

So the element $\beta \in H^3_\rho(G, A)$ doesn't change under Galois action.

In particular, if β is non-trivial for C, it is non-trivial for $\sigma(C)$.

$\Rightarrow$ Galois action preserves the full 2-group symmetry.

Conclusion

Galois conjugation gives us a way to jump between theories in the space of TQFTs.

Galois conjugation may not preserve unitarity.

If a TQFT admits gapped boundaries, so do all of its Galois conjugates.

Galois conjugation preserves 0-form, 1-form and 2-group symmetry of a TQFT.

Q. What are the defects that implement Galois action?

Q. What about higher dimensional TQFTs?

Q. Can Galois action on the topological operators of a QFT be lifted to some action on the full QFT?

Thank You

Discrete Gauge Theory

Given a finite group G, we can construct a TQFT called discrete gauge theory $D(G)$.

The anyons are labelled by $([g], \pi_g)$ $\quad \pi_g \in \mathrm{Ir}(N_g)$.

Given an MTC, how do we know if it is a discrete gauge theory?

A Lagrangian subcategory L of an MTC C is a fusion subcategory satisfying

(i) $\theta_a = 1 \quad \forall \quad a \in L$

(ii) $\left(\sum_{a \in L} d_a^2 \right)^2 = D^2$

Note : Conditions (i) and (ii) are algebraic.

A Lagrangian subcategory L is equivalent to $\mathrm{Rep}(G)$ for some finite group G.

An MTC C describes the line operators in a discrete gauge theory based on group G if $L \cong \mathrm{Rep}(G) \subset C$.

$$\begin{array}{ccc} C & \longleftrightarrow & q(C) \\ \cup & & \cup \\ L & & q(L) \end{array}$$

If L is Lagrangian in C, then $q(L)$ is Lagrangian in $q(C)$.

Moreover,

$$\text{if} \quad L \cong \mathrm{Rep}(G) \Rightarrow q(L) \cong \mathrm{Rep}(G).$$

$\Rightarrow$ Discrete gauge theory $D(G)$ is invariant under Galois conjugation.