AN ANALYTIC VIEW OF RENORMALONS FROM INTEGRABILITY AND RESURGENCE

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Transseries in QFT

Asymptotic series in QFT

Many if not most series in QFT are asymptotic, i.e. divergent (Dyson 1953).
Typically they are of the form:

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If φ is *Borel summable* and we recover the "true" function $\varphi(z)$ from the Borel sum

$$s(\varphi)(z) = \int_0^\infty e^{-\zeta}\widehat{\varphi}(z\zeta)d\zeta. \tag{1.3}$$

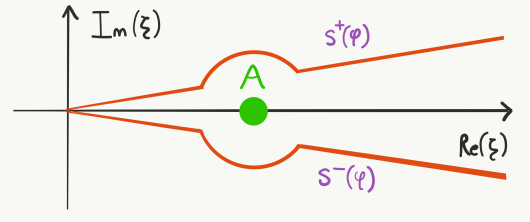
Ambiguity strikes back

If we Borel transform the example from before with $A > 0$

$$F_p(g) \sim \sum_{k\geq 0}^{\infty} (A^{-k} k!) g^k \Rightarrow \widehat{F}(\zeta) = \frac{1}{1-\zeta/A} \tag{1.4}$$

There's a pole on $\mathbb{R}^+$! We can deform the contour to go slightly above or below the real axis. But an ambiguity remains

$$s_+(F)(g) - s_-(F)(g) = 2\pi \mathrm{i} A g^{-1} \mathrm{e}^{-A/g} \tag{1.5}$$



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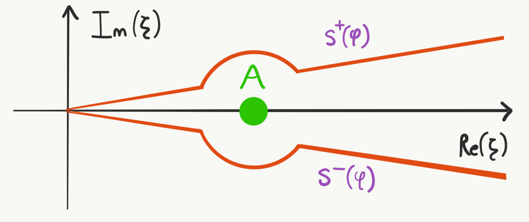

The transseries

Ambiguities can be cancelled by non-perturbative sectors. The “true” function is then given by a **trans-series**

$$\Phi(z) = \sum_{k \geq 0} c_k z^k + \sum_i \mathcal{C}_i^{\pm} \mathrm{e}^{-A_i/z} z^{b_i} \sum_{k \geq 0} c_k^{(i)} z^k + \cdots \tag{1.6}$$

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We saw an example where

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To cancel we must have

$$\mathcal{C}^+ - \mathcal{C}^- = -2\pi\mathrm{i}A, \quad b = -1, c_0 = 1. \tag{1.8}$$

Resurgence itself

Generally the closest singularity in the trans-series (top) implies a contribution to the asymptotic behaviour of the perturbative series (bottom)

$$\begin{gathered}\mathcal{C}_1^{\pm} g^{-b_1} \mathrm{e}^{-A_1/g} \left(\psi_{1,0} + \mathcal{O}(g)\right), \quad \mathcal{C}_1^{+} = \mathcal{C}_1^{-} - \mathrm{i}\mathsf{S}_1 \\ \Leftrightarrow \\ c_k \sim \frac{\mathsf{S}_1}{2\pi} A_1^{-k-b_1} \Gamma(k+b_1) \left(\psi_{1,0} + \mathcal{O}(k^{-1})\right), \qquad k \gg 1,\end{gathered} \tag{1.9}$$

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Work to do

Today I will show how, in some integrable field theories, we tackle

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Renormalons in a nutshell

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But a series can also diverge because individual Feynman diagrams through their momenta integration become too big. We call this a **renormalon** effect (discovered in renormalizable theories, and baptised in analogy with instantons.).

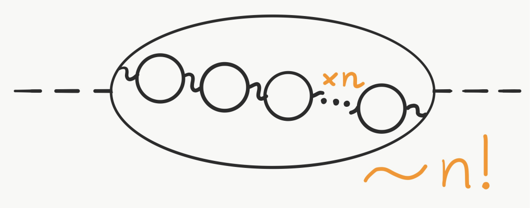

Figure 2: A typical renormalon diagram in particle physics.

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What do we know about renormalons?

What's in a name?

From renormalization and diagrammatic arguments, Parisi (1978) and 't Hooft (1979) argued that in asymptotically free theories, the Borel transform of an observable $F(g)$ should have singularities at

$$\zeta = \frac{\ell}{2|\beta_0|} \tag{2.10}$$

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But is this true in general? To study the perturbative series we need a lot of coefficients, this can be very hard!

Bethe(r) idea

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Integrable models like

- $O(N)$ Gross-Neveu model (GN), a fermion vector model
- $O(N)$ non-linear sigma model (NLSM), a vector valued sigma-model
- $\mathcal{N} = 1$ supersymmetric $O(N)$ non-linear sigma model
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are both very rich asymptotically free theories and admit a Bethe ansatz solution. For example, their mass gaps were computed exactly in the 90's (Forgacs et al. Hasenfratz et al., Evans et al., Balog et al., ...)

In order to use integrability to our advantage, we add a chemical potential $h > m$ coupled to a conserved charge Q such that it excites a single species of particles of the lowest mass m in the ground state

$$\mathrm{H} \to \mathrm{H} - h\mathrm{Q}. \tag{2.11}$$

In this case the ground state, populated by particles, can be described by the **Bethe ansatz integral equation**

$$\epsilon(\theta) - \int_{-B}^{B} K(\theta - \theta')\epsilon(\theta')\mathrm{d}\theta' = h - m\cosh\theta, \quad \epsilon(\pm B) = 0, \tag{2.12}$$

where ϵ is like a Fermi density over rapidities θ. B is a function of h specified by the “Fermi level”, and the kernel K is specified by the S-matrix of the excited particles, which is known exactly thanks to integrability.

In a previous episode...

Thanks to a method by Volin, at weak coupling ($B \gg 1$) one can can turn the integral equation into a series of recursive algebraic solution that give the perturbative expansion of the solution and some observables. This can be done exactly (we get 40-50 coefficients) or numerically (Abbott et al. got $\sim$2000 coeffiecients for the $O(4)$ NLSM).

An interesting observable is the free energy

$$\mathcal{F}(h) = -\frac{m}{2\pi} \int_{-B}^{B} \epsilon(\theta) \cosh \theta \mathrm{d}\theta \,. \tag{2.13}$$

Using Volin's method we tested that the leading large order behaviour of the perturbative series of $\mathcal{F}(h)$ matched Parisi's prediction with $\ell = 2$.

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Using Volin's method we tested that the leading large order behaviour of the perturbative series of $\mathcal{F}(h)$ matched Parisi's prediction with $\ell = 2$. But the integral equation is an **exact** solution, it should know the full trans-series!

New renormalons in Gross-Neveu

The Gross Neveu model

Let us focus on the example of the Gross-Neveu model. We have N Majorana fermion χ with a 4 point interaction

$$\mathcal{L} = \frac{\mathrm{i}}{2}\bar{\chi} \cdot \partial\!\!\!/\chi + \frac{g^2}{8}(\bar{\chi} \cdot \chi)^2 \tag{3.14}$$

The theory is asymptotically free, so we the running coupling is evaluated at scale $\mu = h$. The following expansions are equivalent

$$\bar{g}(h) \ll 1 \quad \Leftrightarrow \quad h \gg \Lambda \quad \Leftrightarrow \quad B \gg 1 \tag{3.15}$$

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but they all have different complications. The most simple choice is to write our results as a function of

$$\frac{1}{\alpha} + \Delta \log \alpha = \log \frac{h}{\Lambda}, \quad \alpha \sim 2|\beta_0|\bar{g}(h)^2 \sim 1/B \tag{3.16}$$

An adventure in (Fourier) space

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The previous integral equation can be taken into Fourier space to apply Wiener-Hopf type methods. Our bestiary becomes

- We split the Fourier transform of the kernel into

$$1 - K(\omega) = \frac{1}{G_+(\omega)\, G_+(-\omega)} \tag{3.17}$$

  such that $G_+(\omega)$ is analytic in the upper half plane. This function is our key ingredient.
- $u(\omega)$ is an unknown function. Finding $u(\omega)$ is equivalent to solving $\epsilon(\theta)$.

An adventure in (Fourier) space - part II

The integral equation is now

$$u(\omega) = \frac{\mathrm{i}}{\omega} + \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{R}} \frac{\mathrm{e}^{2\mathrm{i}B\omega'}}{\omega + \omega' + \mathrm{i}0} \rho(\omega') u(\omega') \mathrm{d}\omega', \quad \rho(\omega) = -\frac{\omega + \mathrm{i}}{\omega - \mathrm{i}} \frac{G_+(-\omega)}{G_+(\omega)}. \tag{3.18}$$

The free energy can be found through

$$\mathcal{F}(h) = -\frac{h^2}{2\pi} u(\mathrm{i}) G_+(0)^2 \left\{ 1 - \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{R}} \frac{\mathrm{e}^{2\mathrm{i}B\omega'}}{\omega - \mathrm{i}} \rho(\omega') u(\omega') \mathrm{d}\omega' \right\}. \tag{3.19}$$

What you need to know from these equations

- We need to solve for $u(\omega)$ to find the free energy.
- The key ingredient is $\rho(\omega)$ which is constructed from the kernel of the original equation and ultimately derives from the S-matrix.

To the complex plane and beyond

Let us take the integral

$$\frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{\mathrm{e}^{2\mathrm{i}B\omega'}}{\omega - \mathrm{i}} \rho(\omega') u(\omega') \mathrm{d}\omega', \quad \rho(\omega) = -\frac{\omega + \mathrm{i}}{\omega - \mathrm{i}} \frac{G_+(-\omega)}{G_+(\omega)}, \tag{3.20}$$

because of $\mathrm{e}^{2\mathrm{i}B\omega'}$ we must deform upwards in the complex plane, and we can do so around the positive imaginary axis but...

- $G_+(-\omega)$ is discontinuous along the positive imaginary axis
- $G_+(-\omega)$ has poles along the imaginary axis, whose residues have different values depending on the branch. The poles are at

$$\omega = \mathrm{i}\xi_k, \quad \xi_k = (2k+1)\frac{N-2}{N-4}, \quad k \in \mathbb{N} \tag{3.21}$$

So we must careful about how we proceed.

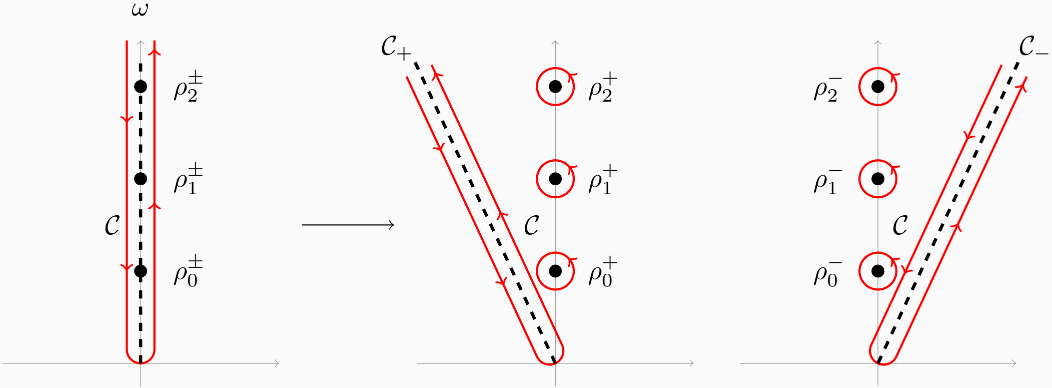

Figure 3: Deforming the contour in the complex plane.

Non-perturbative effects appear

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\frac{\mathrm{e}^{2\mathrm{i}B\omega'}}{\omega-\mathrm{i}}\rho(\omega')u(\omega')\mathrm{d}\omega' = \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{C}^\pm}\frac{\mathrm{e}^{-2B\xi}}{\xi-1}\operatorname{disc}\rho(\mathrm{i}\xi)u(\xi)\mathrm{d}\xi + \mathrm{e}^{-2B}\rho(\mathrm{i}\pm 0)u(\mathrm{i}) + \sum_{n\geq 1}\mathrm{e}^{-2B\xi_n}\rho_n^\pm\frac{u(\mathrm{i}\xi_n)}{\xi_n-1} \tag{3.22}$$

where $\rho_n^\pm$ are the residues of $\rho(\omega)$ in the two branches.

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$$\begin{aligned}\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\frac{\mathrm{e}^{2\mathrm{i}B\omega'}}{\omega-\mathrm{i}}\rho(\omega')u(\omega')\mathrm{d}\omega' =& \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{C}^\pm}\frac{\mathrm{e}^{-2B\xi}}{\xi-1}\operatorname{disc}\rho(\mathrm{i}\xi)u(\xi)\mathrm{d}\xi \\ &+\mathrm{e}^{-2B}\rho(\mathrm{i}\pm 0)u(\mathrm{i})+\sum_{n\geq 1}\mathrm{e}^{-2B\xi_n}\rho_n^\pm\frac{u(\mathrm{i}\xi_n)}{\xi_n-1}\end{aligned} \tag{3.22}$$

where $\rho_n^\pm$ are the residues of $\rho(\omega)$ in the two branches.

Because we want the perturbative expansion, we must expand in $1/B$. Furthermore, we also need to do this in the equation for u, where we find the same structure, and to apply boundary conditions.

After filling in the details...

The sketch of what happens in the end is

$$\begin{aligned} \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{C}^\pm}\frac{\mathrm{e}^{-2B\xi}}{\xi-1}\,\mathrm{disc}\,\rho(\mathrm{i}\xi)u(\xi)\mathrm{d}\xi &\to\cdots\to \quad \frac{c_0}{B}\left\{1+\mathcal{O}\left(\frac{1}{B}\right)\right\} \\ \mathrm{e}^{-2B}\rho(\mathrm{i}\pm 0)u(\mathrm{i}) &\to\cdots\to \quad \mathrm{e}^{-2B}\mathcal{C}_0^\pm \\ \mathrm{e}^{-2B\xi_n}\rho_n^\pm\frac{u(\mathrm{i}\xi_n)}{\xi_n-1} &\to\cdots\to \quad \mathrm{e}^{-2B\xi_n}\mathcal{C}_n^\pm\left\{1+\mathcal{O}\left(\frac{1}{B}\right)\right\} \end{aligned} \tag{3.23}$$

where the $\mathcal{C}_{0,n}^\pm$ depend on the branch choice through the $\rho_n^\pm$.

We find an formal series with exponential suppressed terms and ambiguous coefficients: the trans-series!

$$\mathcal{F}(h) = -\frac{h^2}{2\pi}\left\{(1+\mathcal{O}(\alpha)) - \mathrm{e}^{-\frac{2}{\alpha}}\alpha^{\frac{2}{N-2}}\mathcal{C}_0^{\pm} + \sum_{k\geq 1}\mathrm{e}^{-\frac{2k}{\alpha}\frac{N-2}{N-4}}\alpha^{\frac{2k}{N-4}}\mathcal{C}_k^{\pm}\,(1+\mathcal{O}(\alpha))\right\} \tag{3.24}$$

(remember $\alpha \sim 1/B \sim 2|\beta_0|\bar{g}(h)^2$).

New renormalons

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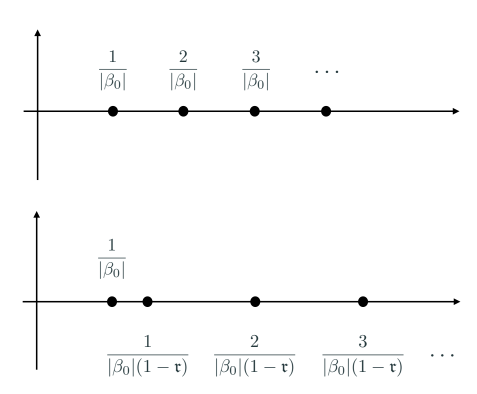


Instead of Borel singularities

$$\zeta = \frac{k}{|\beta_0|}, \quad k \in \mathbb{N}$$

we have

$$\zeta = \frac{1}{|\beta_0|},$$

and then

$$\zeta = \frac{k}{|\beta_0|(1-\mathfrak{r})}, \quad \mathfrak{r} = \frac{2}{N-2}$$

So our trans-series is very different from what the standard lore predicted!

Extraordinary claims require extraordinary evidence

If this is the trans-series for the exact function of the observable $\mathcal{F}(h)$ we should be able to test it

- By comparing the discontinuity of the $\mathcal{C}_n^{\pm}$ with the large order behaviour of the perturbative series found with Volin's method.
- By comparing the resummation of the perturbative series with the numeric solution of the exact integral equation and see what exponentially suppressed terms are missing.

We have done many such tests with success, I will present two of the most important.

Large order behaviour

One important test is to take long perturbative series from Volin's method and compare with the Stokes constants $\mathcal{C}_0^+ - \mathcal{C}_0^- = -\mathrm{i}\mathsf{S}_0$.

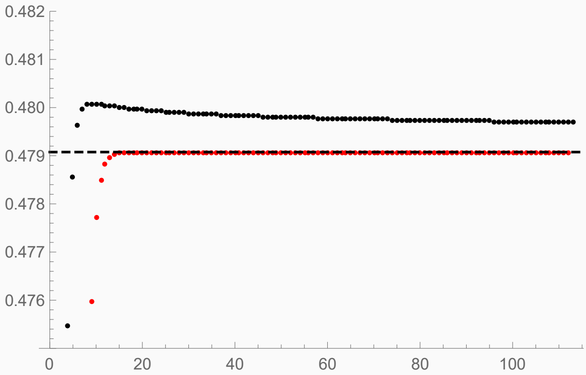
With a series that grows

$$c_k \sim \frac{\mathsf{S}_0}{2\pi} A_1^{-k-b_1} \Gamma(k + b_1)$$

we can construct an auxiliary series s_k such that

$$s_k \sim \mathsf{S}_0, \quad k \gg 1$$

We plot the series s_k for $N = 7$.



Because the $\mathrm{e}^{-\frac{2}{\alpha}}$ term is very simple in

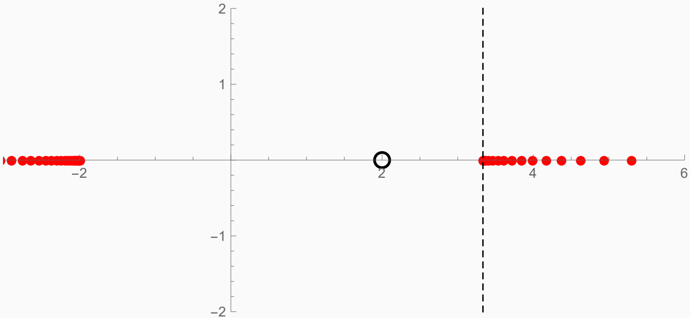
$$\mathcal{F}(h) = -\frac{h^2}{2\pi}\left\{ (1+\mathcal{O}(\alpha)) - \mathrm{e}^{-\frac{2}{\alpha}}\alpha^{\frac{2}{N-2}}\mathcal{C}_0^{\pm} + \mathrm{e}^{-\frac{2}{\alpha}\frac{N-2}{N-4}}\alpha^{\frac{2}{N-4}}\mathcal{C}_1^{\pm}\left(1+\mathcal{O}(\alpha)+\cdots\right)\right\}$$

we can subtract its contribution to the asymptotics and see the effects of the term $\mathrm{e}^{-\frac{2}{\alpha}\frac{N-2}{N-4}}$.

The singularity marks the spot

With $N = 7$, we plot the singularities of and approximation of the Borel transform (they approximate a cut) after subtracting the leading order behaviour.

The removed singularity is at 2 (i.e. $1/|\beta_0|$) and the next predicted singularity is at $10/3$ (i.e. $5/3|\beta_0|$).



Large N

In the large N limit we match known results but the new renormalons move to the "traditional" renormalon predictions

$$\frac{\ell}{|\beta_0|}\frac{N-2}{N-4} \to \frac{\ell}{|\beta_0|} \tag{3.25}$$

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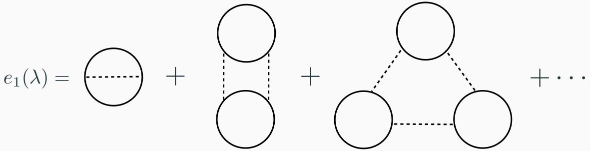
This also shows that these effects are "renormalons", at large N we see they are the result of ring diagrams, which grow factorially.



Non-linear sigma model

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- $O(N)$ **non-linear sigma model**

$$\mathcal{L} = \frac{1}{2g^2}\mathbf{S}\cdot\mathbf{S}, \quad \mathbf{S}^2 = 1 \tag{3.26}$$

and we find the Borel singularities

$$\zeta_{IR} = \frac{1}{|\beta_0|} \quad \text{and} \quad \zeta_\ell = \frac{\ell(N-2)}{|\beta_0|} \quad \ell \in \mathbb{N} \tag{3.27}$$

The first one is a traditional IR renormalons, while the other might correspond to unstable instantons.

Other examples

- $\mathcal{N} = 1$ SUSY non-linear sigma model

$$\zeta_\ell = \frac{\ell}{|\beta_0|}\frac{N-2}{N-4}, \quad \zeta'_\ell = \frac{\ell(N-2)}{|\beta_0|} \quad \text{and} \quad \zeta_{\ell_1} + \zeta'_{\ell_2} \tag{3.28}$$

There is no traditional IR renormalons, but there are new renormalons like in GN and instanton-like singularities like in the NLSM.

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- $SU(N)$ **principal chiral field**

$$\zeta_{IR} = \frac{1}{|\beta_0|}, \quad \zeta_\ell = \frac{\ell}{|\beta_0|}\frac{N}{N-1}, \quad \zeta'_\ell = \frac{\ell N}{|\beta_0|} \quad \text{and} \quad \zeta_{\ell_1}+\zeta'_{\ell_2} \tag{3.29}$$

A traditional IR renormalon, new renormalons and instanton-like singularities.

Conclusion

Take-home ideas

- Resurgence is a useful and important tool in making sense of perturbative and non-perturbative QFT. Through resurgence we can relate non-perturbative effects and large order behaviour of the perturbative series.

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- Integrable models are great toy models for resurgence and renormalons.
- Using integrability we found a way of extracting the analytic trans-series, which we tested extensively.
- Standard renormalon predictions are sometimes wrong! Borel singularities appear in weirder places than expected.
- Large N can be deceiving.
- Would be very interesting to understand renormalons from first principles to compare to our results, which give a guiding template.

Thank you!

arxiv reference for more details: 2111.11951