Quantum modularity of higher rank homological blocks

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Based on work with M. Cheng, D. Passaro and G. Sgroi [to appear]

Kavli IPMU - Mathematics and String Theory Seminar

02 December 2021

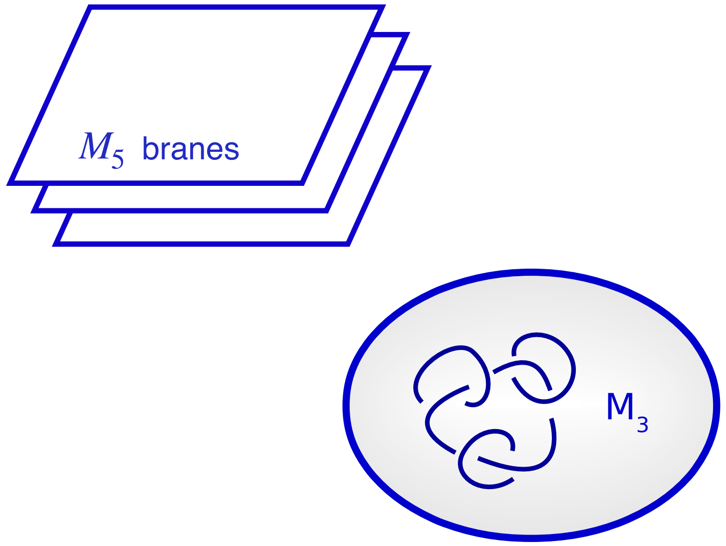
Roadmap

<u>3-manifold invariants</u> $\hat{Z}_a(M_3)$ [Gukov, Putrov, Vafa '16], [Gukov, Pei, Putrov, Vafa '17]

Context and motivation

- Physics
  - 3d SQFT
  - M-theory realisation
- counting of BPS states

3d-3d correspondence → $\hat{Z}_a(M_3)$ are q-series invariants for 3-manifolds M_3



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- **Mathematics**
  - low-dimensional topology
  - log VOA characters
  - quantum modular forms

[Cheng, Chun, Ferrari, Gukov, Harrison '18]
[Cheng, Ferrari, Sgroi '19] + in progress

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Generalisation

higher rank

✓ higher rank Log-VOA
✓ higher depth QMF
✓ relations to rank-1

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Generalisation

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Why investigate this?

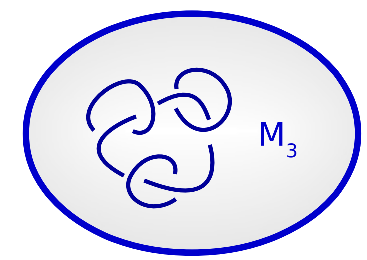
✓ higher rank Log-VOA
✓ higher depth QMF
✓ relations to rank-1

3-Manifold Invariants $\hat{Z}_a(M_3)$

Physical definition of new 3-manifold invariants $\hat{Z}_a(M_3)$ [Gukov, Pei, Putrov, Vafa '17]

... in the context of the 3d-3d correspondence

space-time	S^1	$\times$	TN	$\times$	T^*M_3	M-theory background
			$\cup$		$\cup$	
M5-branes on	S^1	$\times$	D^2	$\times$	M_3	



6d $\mathcal{N}$=(2,0) $\mathfrak{g}$-SCFT on $D^2 \times_\tau S^1 \times M_3$

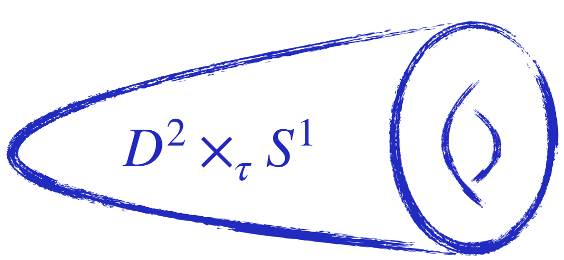
3d $\mathcal{N}$=2 gauge theory T[M_3 , $\mathfrak{g}$]

gauge group G, Lie algebra $\mathfrak{g}$,
choice of boundary condition

Topological quantum field theory

3d $G_\mathbb{C}$ Chern-Simons on M_3

abelian flat connections



2d $\mathcal{N}$=(0,2) boundary condition $\mathscr{B}_a$ index label

3d $\mathcal{N}$=2 theory

3-Manifold Invariants $\hat{Z}_a(M_3)$ and WRT invariants

$\hat{Z}_a(M_3)$ is related to the Witten-Reshetikhin-Turaev invariant of M_3 [Witten'88; Reshetikhin, Turaev '90]

$$Z_{\rm CS}(M_3;k) = \int_{\mathscr{A}} \mathscr{D}A e^{\frac{i(k-h^\vee)}{4\pi}\int_{M_3} \mathrm{Tr}(A\wedge dA + \frac{3}{2}A\wedge A\wedge A)}$$ 3d Chern-Simons partition function

↑ radial limit $|q| \to 1 \leftrightarrow \tau \to 1/k$ and sum over "a" $k \in \mathbb{Z}$ shifted CS level

$\hat{Z}_a(M_3;q)$ → has a q-series expansion with integer powers and integer coefficients $q = e^{2\pi i\tau}$

convergent for $|q| < 1 \leftrightarrow \tau \in \mathbb{H}$

$$Z_{CS}(M_3;k) = (i\sqrt{2k})^{N(b_1(M_3)-1)/2} \sum_{a,b\in\pi_0\mathscr{M}^{ab}_{flat}(M_3,G)} e^{2\pi i k\,\mathrm{CS}(a)} \left[S_{ab}\hat{Z}_b(M_3;q)\right]_{\tau\to 1/k}$$ [Gukov, Putrov, Vafa '16]

$Z_a(1/k)$

3-Manifold Invariants $\hat{Z}_a(M_3)$

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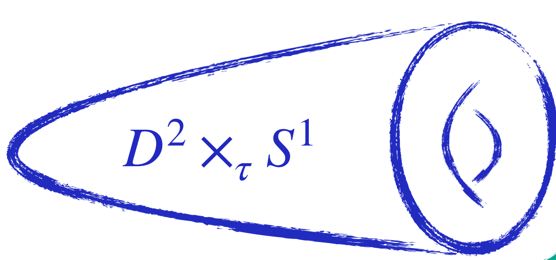

2d $\mathcal{N}$=(0,2) boundary condition $\mathscr{B}_a$

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3-Manifold Invariants $\hat{Z}_a(M_3)$

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… in the context of the 3d-3d correspondence

- $\hat{Z}_a(M_3)$ as the supersymmetric index of T[M_3, $\mathfrak{g}$] "half-index"

$$\hat{Z}_a(M_3) = Z_{T[M_3,\mathfrak{g}]}(D^2 \times_\tau S^1; \mathscr{B}_a) = \sum_{i,j} (-1)^i q^j \dim \mathscr{H}_a^{i,j}$$

BPS Hilbert spaces $\tau \in \mathbb{H}$

when a Lagrangian description of T[M_3] is known, compute by localisation [Yoshida, Sugiyama '14]

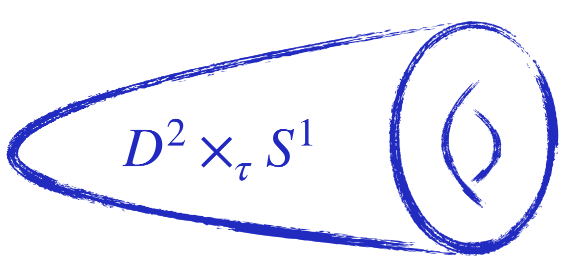
$$\hat{Z}_a = \int \frac{dx}{2\pi i x} F_{3d}(x)\, \Theta^{(a)}_{2d}(x)$$

[Gukov, Putrov, Vafa '16]

conjectured relation to 3d $\mathcal{N}$=2 superconformal index

$$Z(S^2 \times_\tau S^1) = \sum_a |\mathscr{W}_a| \hat{Z}_a(M_3; q)\, \hat{Z}_a(M_3; q^{-1}) \quad \in \mathbb{Z}[[q]]$$

[Gukov, Pei, Putrov, Vafa '17]



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- $\hat{Z}_a(M_3)$ from resurgence in 3d theory T$[M_3\,,\mathfrak{g}]$

[Gukov, Marino, Putrov '16]
[Gukov, Pei, Putrov, Vafa '17], [Chun '17] …
[Cheng, Chun, Ferrari, Gukov, Harrison '18]

↪ is a Borel resummation of a perturbative series $\hat{Z}_a^{\mathrm{pert}}\left(1/k\right) = \sum_{m\geq 1} N_m^b \left(2\pi \mathrm{i}/k\right)^m \in \mathbb{Q}[[2\pi \mathrm{i}/k]]$

$\hat{Z}_a\left(1/k\right) = \hat{Z}_a\left(-k\right) + \mathrm{pert.\ series\ in}\ k^{-1}$ (at rank-1, Seifert M_3)

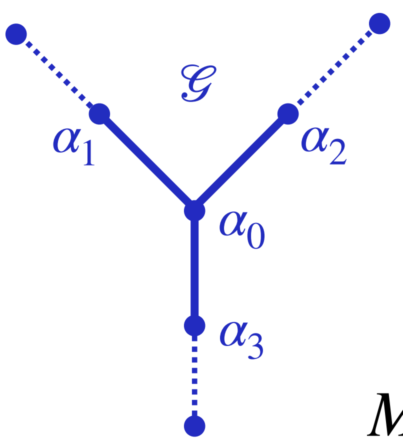
3-Manifold Invariants $\hat{Z}_a(M_3)$

from the WRT inv. $Z_{CS}(M_3;k)$

Mathematical definition of the 3-manifold invariants $\hat{Z}_a(M_3)$ [Gukov, Pei, Putrov, Vafa '17]

$\mathfrak{g} = A_1$

... in the case where $M_3(\mathscr{G})$ is a plumbed 3-manifold, with plumbing graph $\mathscr{G}$

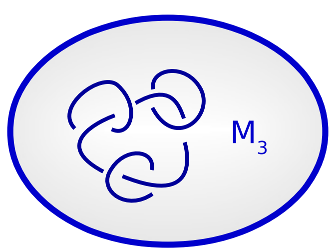


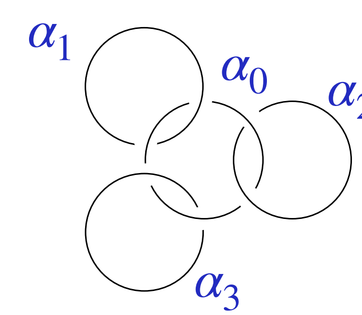
$\mathscr{G} := (V, E, \alpha)$ is a weighted graph $\alpha : V \to \mathbb{Z}$

... this data is encoded in the adjacency matrix M

$$M_{vv'} = \begin{cases} \alpha(v) & \text{if } v = v' \\ 1 & \text{if } (v, v') \in E \\ 0 & \text{otherwise} \end{cases}$$

$M_3(\mathscr{G})$ from Dehn surgery along the corresponding framed link





$M_3(\mathscr{G})$ weakly negative if M^{-1} negative definite when restricted to subspace of high-valency vertices

... for 3-star graphs, this means $(M^{-1})_{00} < 0$

Example: Brieskorn sphere $\Sigma(p_1, p_2, p_3)$ $\quad p_i \in \mathbb{Z}$ coprime

$$M_3(\mathscr{G}) = \Sigma(p_1, p_2, p_3) = \{(x, y, z) \in \mathbb{C}^3 \,|\, x^{p_1} + y^{p_2} + z^{p_3} = 0\} \cap S^5$$

3-Manifold Invariants $\hat{Z}_a(M_3)$

Mathematical definition of the 3-manifold invariants $\hat{Z}_a(M_3)$ [Gukov, Pei, Putrov, Vafa '17]

$\mathfrak{g} = A_1$

$$\hat{Z}_a(M_3;q) = q^{\Delta_a} \oint \prod_{v\in V} \frac{dz_v}{2\pi i z_v} \left(z_v - z_v^{-1}\right)^{2-\deg(v)} \Theta_a^M(q;\mathbf{z})$$

... the contour integral picks the $[z^0]$ term

$\Delta_a \in \mathbb{Q}$

$$\hat{Z}_a = \int \frac{dx}{2\pi i x} F_{3d}(x)\, \Theta_{2d}^{(a)}(x)$$

$$\Theta_a^M(q;\mathbf{z}) = \sum_{\ell \in 2M\mathbb{Z}^{|V|} \pm a} q^{-\ell^T M^{-1} \ell} \mathbf{z}^{\ell}$$

well defined only if $M_3(\mathcal{G})$ is weakly negative

the sum is over a positive definite lattice and $\Theta_a^M(\tau), \hat{Z}_a(\tau)$ converge for $|q| < 1$

Example: Brieskorn sphere $\Sigma(4,5,7)$ [Cheng, Chun, Ferrari, Gukov, Harrison '18]

$$\hat{Z}_0(\Sigma(4,5,7);q) = -\,q^{\Delta}(\widetilde{\theta}^1_{140,57} - \widetilde{\theta}^1_{140,97} - \widetilde{\theta}^1_{140,113} + \widetilde{\theta}^1_{140,153})$$

false θ-function

$$\widetilde{\theta}^1_{p,r} = \sum_{\substack{l \in \mathbb{Z} \\ l = r \bmod 2p}} \operatorname{sgn}(l)\, q^{\frac{l^2}{4p}}$$

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<u>Mathematical definition of the 3-manifold invariants</u> $\hat{Z}_a(M_3)$ [Gukov, Pei, Putrov, Vafa '17]

$\mathfrak{g} = A_1$

for the general case of a Brieskorn sphere $\Sigma(p_1, p_2, p_3)$

$$q^{-\Delta}\hat{Z}_0(\Sigma(p_1,p_2,p_3);q) = \widetilde{\theta}^1_{p,r_1} - \widetilde{\theta}^1_{p,r_2} - \widetilde{\theta}^1_{p,r_3} + \widetilde{\theta}^1_{p,r_4} := \widetilde{\theta}^{1,p+K}_{r_1}$$

$p = p_1 p_2 p_3$

$K = \{1,\ p_1p_2,\ p_2p_3,\ p_1p_3\}$

$r_4 = p - p_1p_2 - p_1p_3 - p_2p_3$

$r_i = r_4 + 2p_jp_k\ ,\ i \neq j \neq k$

orbit in Weil representation of $SL_2(\mathbb{Z})$

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✓ Definition

- Physics
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counting of BPS states

3d-3d correspondence → $\hat{Z}_a(M_3)$ are q-series invariants for 3-manifolds M_3

- Mathematics
  - low-dimensional topology
  - log VOA characters
  - quantum modular forms

[Cheng, Chun, Ferrari, Gukov, Harrison '18]

$\hat{Z}_a(M_3)$ as characters of Log VOAs

Log VOA's underlie 2d Log CFT's, whose correlation functions have log-singularities

free boson OPE $\varphi(z)\varphi(w) \sim \log(z-w)$

[Gurarie '93]
[Pearce, Rasmussen, Zuber '06]
[Feigin, Tipunin '10]

- Two cases: triplet (1,p) & singlet (1,p) algebras $p \in \mathbb{Z}_+$ [Kausch '91] [Feigin, Tipunin '10]

screening charge $Q = \oint dz\, e^{\alpha\varphi(z)}$ were $\alpha = -\sqrt{2/p}$

triplet (1,p) algebra: $\mathscr{W}(p) = \ker_{\mathscr{V}_L} Q \quad \subset \quad \mathscr{V}_L$ lattice VOA   lattice $L = \sqrt{2p}\mathbb{Z}$

$\cup \qquad\qquad \cup$ [Feigin, Gainutdinov, Semikhatov, Tipunin '06]

singlet (1,p) algebra: $\mathscr{M}(p) = \ker_{\mathscr{H}} Q \quad \subset \quad \mathscr{H}$ Heisenberg algebra

- Characters of irreducible modules [Adamovich, Milas '07]

$$F^{(s)}(q) = [z^0]\left(F_s^{\mathscr{W}(p)}(q,z)\right) = \eta(q)^{-1}\sum_{n\geq 0}\left(q^{\frac{1}{4p}(2pn+p-s)^2} - q^{\frac{1}{4p}(2pn+p+s)^2}\right) \qquad s = 1,\ldots,p$$

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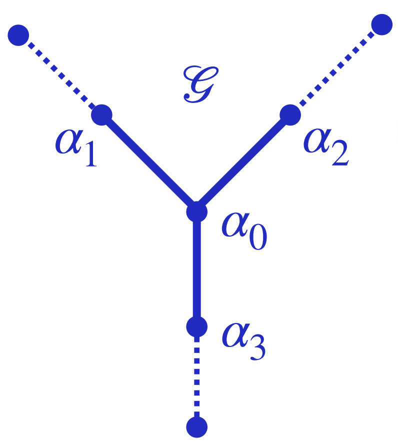

$$q^c \hat{Z}_a(M_3(\mathscr{G}); \tau) \in \text{span}_{\mathbb{Z}}\{\widetilde{\theta}^1_{m,r}, r \in \mathbb{Z}/2m\} \;,\quad m \in \mathbb{Z}_+$$

[Cheng, Chun, Ferrari, Gukov, Harrison '18]

recall the example $\hat{Z}_0(\Sigma(4,5,7); q) \sim \widetilde{\theta}^1_{140,57} - \widetilde{\theta}^1_{140,97} - \widetilde{\theta}^1_{140,113} + \widetilde{\theta}^1_{140,153}$

♣ Characters of irreducible modules [Adamovich, Milas '07]

$$F^{(s)}(q) \sim \widetilde{\theta}^1_{p,p-s}$$

false θ-function

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$\hat{Z}_a(M_3)$ as characters of Log VOAs

higher rank Lie algebra $\mathfrak{g}$

Log VOA's underlie 2d Log CFT's, whose correlation functions have log-singularities

r=rank($\mathfrak{g}$) free bosons OPE $\varphi_\alpha(z)\varphi_\beta(w) \sim (\alpha,\beta)\log(z-w)$

$\alpha, \beta \in \Lambda_{\text{root}}$

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screening charge $Q = \oint dz\, e^{\alpha\varphi(z)}$ were $\alpha = -\sqrt{2/p}$

triplet (1,p) algebra: $\mathscr{W}(p) = \ker_{\mathscr{V}_L} Q$

lattice $L = \sqrt{p}\Lambda_{\text{root}}$

singlet (1,p) algebra: $\mathscr{M}(p) = \ker_{\mathscr{H}} Q$

- Characters of irreducible modules at higher rank $F^{(\mathbf{s})}_{\mathfrak{g}}(q) = [z^0]\left(F^{\mathscr{W}(p)_{\mathfrak{g}}}_{\mathbf{s}}(q,z)\right)$ [Bringmann, Milas '16]

$$F^{(1,1)}_{A_2}(q) = \eta(q)^{-2} \sum_{\substack{m_1,m_2\geq 1 \\ m_1\equiv m_2 \,(\text{mod } 3)}} \min(m_1,m_2)\, q^{\frac{p}{3}(m_1^2+m_2^2+m_1m_2)-m_1-m_2+\frac{1}{p}} \left(1-q^{m_1}\right)\left(1-q^{m_2}\right)\left(1-q^{m_1+m_2}\right)$$

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Quantum modular forms

A modular form f(τ) of weight w, multiplier system χ with respect to $\Gamma \subseteq SL_2(\mathbb{Z})$

$\gamma = \begin{pmatrix} a, b \\ c, d \end{pmatrix} \in \Gamma$ acts on $\mathbb{H}$ by a fractional linear transformation $\gamma\tau = \dfrac{a\tau + b}{c\tau + d}$

is a holomorphic function of $\tau \in \mathbb{H}$ with $f|_{w,\chi}\gamma(\tau) := (c\tau + d)^{-w}\chi(\gamma)^{-1}f(\gamma\tau)$

if $f|_{w,\chi}\gamma(\tau) = f(\tau)\ ,\quad \forall \gamma \in \Gamma$

- Examples of modular forms: θ-functions

$$\theta(\tau) = \sum_{k\in\mathbb{Z}} q^{\frac{k^2}{2}}$$ where $q = e^{2\pi i \tau}$

$$\theta^0_{p,r}(\tau) = \sum_{\substack{k\in\mathbb{Z} \\ k=r \text{ mod } 2p}} q^{\frac{k^2}{4p}}$$ weight 1/2

$$\theta^1_{p,r}(\tau) = \sum_{\substack{k\in\mathbb{Z} \\ k=r \text{ mod } 2p}} k q^{\frac{k^2}{4p}}$$ weight 3/2

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The radial limit $|q| \to 1 \Leftrightarrow \tau \to \alpha \in \mathbb{Q}$ defines a function on the boundary of $\mathbb{H}$, on $\mathbb{Q}$

$$f(\alpha) := \lim_{t\to 0^+} f(\alpha + it)$$

Quantum modular forms (QMF's) are defined at the boundary of $\mathbb{H}$, on $\mathbb{Q} \cup \{i\infty\}$

Quantum modular forms

A quantum modular form of weight w, multiplier system χ with respect to $\Gamma \subseteq SL_2(\mathbb{Z})$

is a function $f : \mathbb{Q} \to \mathbb{C}$ such that, $\forall \gamma \in \Gamma$, the function $p_\gamma(x) : \mathbb{Q} \backslash \{\gamma^{-1}(\infty)\} \to \mathbb{C}$

defined by $p_\gamma(x) := f(x) - f|_{w,\chi}\gamma(x)$ has a better analytic behaviour than $f(x)$. [Zagier '10]

The function $\gamma \to p_\gamma$ is a cocycle in Γ, which means $p_{\gamma_1\gamma_2} = p_{\gamma_1}|_{w,\chi}\gamma_2 + p_{\gamma_2}$

A strong quantum modular form is a QMF f which associates to each element $x \in \mathbb{Q}$

a formal power series over $\mathbb{C}$, so that $p_\gamma(x + \mathrm{i}t) := f(x + \mathrm{i}t) - f|_{w,\chi}\gamma(x + \mathrm{i}t)$ $t \to 0^+, \gamma \in \Gamma$

holds as an identity between countable collections of formal power series.

characters of Log-VOA's, $\hat{Z}_a^{SU(2)}(M_3; \tau)$

[Cheng, Chun, Ferrari, Gukov, Harrison '18] [Bringmann, Milas '15]

Quantum modular forms & $\hat{Z}_a(M_3)$

Given a modular form g of weight w, its Eichler integrals

holomorphic $\tilde{g}(\tau) = c_{(w)} \int_{\tau}^{i\infty} g(\tau')(\tau'-\tau)^{w-2} d\tau'$ non-holomorphic $g^*(\tau,\bar{\tau}) = c_{(w)} \int_{-\bar{\tau}}^{i\infty} g(\tau')(\tau'+\tau)^{w-2} d\tau'$

MF g with $w \in \frac{1}{2}\mathbb{Z}$ and Fourier expansion $g = \sum_{n>0} a_g(n) q^n \longleftrightarrow \tilde{g} = \sum_{n\geq 1} a_g(n) n^{1-w} q^n$ [Lawrence, Zagier '99] $c_{(w)} = \frac{(2\pi i)^{w-1}}{\Gamma(w-1)}$

are QMF's, since $\tilde{g} - \tilde{g}|_{2-w}\gamma$ and $g^* - g^*|_{2-w}\gamma$ are period integrals.

$$\left(\tilde{g} - \tilde{g}|_{2-w}\gamma\right)(\tau) = \int_{\gamma^{-1}(\infty)}^{\infty} g(\tau')(\tau'-\tau)^{w-2} d\tau'$$

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Given a modular form g of weight w, its Eichler integrals

holomorphic $\tilde{g}(\tau) = c_{(w)} \int_{\tau}^{i\infty} g(\tau')(\tau' - \tau)^{w-2} d\tau'$ non-holomorphic $g^*(\tau, \bar{\tau}) = c_{(w)} \int_{-\bar{\tau}}^{i\infty} g(\tau')(\tau' + \tau)^{w-2} d\tau'$

MF g with $w \in \frac{1}{2}\mathbb{Z}$ and Fourier expansion $g = \sum_{n>0} a_g(n) q^n \longleftrightarrow \tilde{g} = \sum_{n \geq 1} a_g(n) n^{1-w} q^n$ [Lawrence, Zagier '99] $c_{(w)} = \frac{(2\pi i)^{w-1}}{\Gamma(w-1)}$

are QMF's, since $\tilde{g} - \tilde{g}|_{2-w}\gamma$ and $g^* - g^*|_{2-w}\gamma$ are period integrals.

$$\left(\tilde{g} - \tilde{g}|_{2-w}\gamma\right)(\tau) = \int_{\gamma^{-1}(\infty)}^{\infty} g(\tau')(\tau' - \tau)^{w-2} d\tau'$$

Example: rank-1 invariants for Brieskorn spheres $\hat{Z}_0(\Sigma(p_1, p_2, p_3); q)$

$$\theta^1_{p,r}(\tau) = \sum_{\substack{k \in \mathbb{Z} \\ k = r \text{ mod } 2p}} k q^{\frac{k^2}{2p}}$$ θ-functions $\rightarrow$ $$\widetilde{\theta}^1_{p,r}(\tau) = \sum_{\substack{k \in \mathbb{Z} \\ k = r \text{ mod } 2p}} \operatorname{sgn}(k)\, q^{\frac{l^2}{4p}}$$ false θ-functions

$$\hat{Z}_0(\Sigma(p_1, p_2, p_3); q) \sim \widetilde{\theta}^1_{p,r_1} - \widetilde{\theta}^1_{p,r_2} - \widetilde{\theta}^1_{p,r_3} + \widetilde{\theta}^1_{p,r_4}$$

[Cheng, Chun, Ferrari, Gukov, Harrison '18]

Roadmap

3-manifold invariants $\hat{Z}_a(M_3)$

Context and motivation

what is there to gain from knowing this?

- Mathematics
  - low-dimensional topology
  - log VOA characters
  - quantum modular forms

[Cheng, Chun, Ferrari, Gukov, Harrison '18]

♣ $\hat{Z}_a$ invariants have been calculated for $\tau \in \mathbb{H}$, but what happens for $\tau \in \mathbb{H}_-$?

$$Z(S^2 \times_\tau S^1) = \sum_a |\mathscr{W}_a| \hat{Z}_a(M_3; q)\, \hat{Z}_a(M_3; q^{-1})$$

$\hat{Z}_a(M_3)$ when M_3 not weakly negative

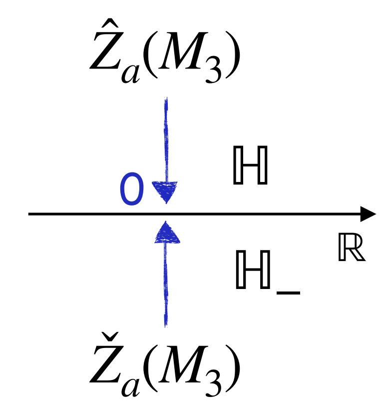
The 3-manifold invariants $\hat{Z}_a(M_3; q)$ were defined for M_3 weakly negative, but $\hat{Z}_a(M_3; q^{-1})$?

[Gukov, Pei, Putrov, Vafa '17]

From $Z_{\mathrm{CS}}(M_3; k) = \sum_{a,b} e^{2\pi i k \, \mathrm{lk}(a,a)} \left[S_{ab} \hat{Z}_b(M_3; \tau) \right]_{\tau \to 1/k}$ & $Z_{\mathrm{CS}}(-M_3; k) = Z_{\mathrm{CS}}(M_3; -k)$ $\quad k \to \infty$

expect $\hat{Z}_a(-M_3; q) = \hat{Z}_a(M_3; q^{-1})$ ♣

but this means defining $\hat{Z}_a(M_3)$ to be a convergent q-series both for $|q| \lessgtr 1$



! $\hat{Z}_a(M_3)$ does not converge for $|q| > 1$ as defined from M_3 topological data

! need asymptotic agreement in radial lim; use quantum modularity to find companion $\check{Z}_a(M_3)$

[Cheng, Chun, Ferrari, Gukov, Harrison '18], [Cheng, Ferrari, Sgroi '19] rank 1

More general quantum modularity

higher rank Lie algebra $\mathfrak{g}$

For higher rank $\mathfrak{g}$ and invariants $\hat{Z}^G_a(M_3)$ expect more general quantum modularity.

A depth-N QMF is a function $f : \mathbb{Q} \to \mathbb{C}$ such that $p_\gamma := f - f|_w \gamma$ is a sum of QMF's of depth N'<N, multiplied by some real-analytic functions, $\forall \gamma \in \Gamma$.

[Bringmann, Kaszian, Milas '17]

… relevant for [Cheng, Coman, Passaro, Sgroi, *to appear*]

Iterated non-holomorphic Eichler integral

$$I_{f_1,f_2}(\tau,\bar{\tau}) := \int_{-\bar{\tau}}^{i\infty} dz_1 \int_{z_1}^{i\infty} dz_2 \, \frac{f_1(z_1) f_2(z_2)}{(-i(z_1+\tau))^{2-w_1}(-i(z_2+\tau))^{2-w_2}}$$

is a depth-2 QMF

More general quantum modularity

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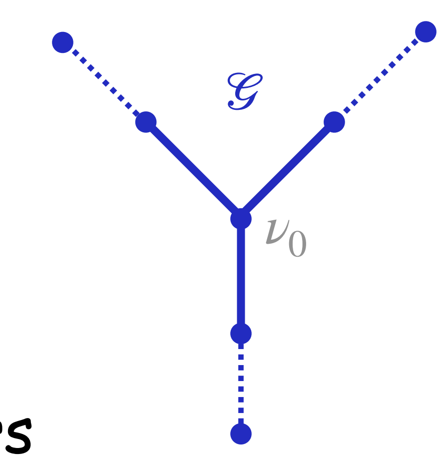
For higher rank $\mathfrak{g}$ and invariants $\hat{Z}^G_a(M_3)$ expect more general quantum modularity.

→ A depth-N QMF is a function $f: \mathbb{Q} \to \mathbb{C}$ such that $p_\gamma := f - f|_w \gamma$ is a sum of QMF's of depth N'<N, multiplied by some real-analytic functions, $\forall \gamma \in \Gamma$.

Examples: characters of $\mathfrak{g}$ Log-VOA's, $\hat{Z}^G_a(M_3)$

Brieskorn sphere $M_3(\mathscr{G}) = \Sigma(p_1, p_2, p_3)$ and $G = SU(3)$



For negative definite plumbed M_3 Seifert with 3 exceptional fibers

$$\hat{Z}^{SU(3)}_a(M_3; q) \sim \sum_{\{s_1,s_2\}} (-1)^{l_s} F^{(s_1,s_2)}(q)$$ with $F^{(s_1,s_2)}(q)$ generalised A_2 false θ-functions, depth-2 QMF

[Cheng, Chun, Fegin, Ferrari, Gukov, Harrison, Passaro *upcoming*] [Cheng, Coman, Passaro, Sgroi, *to appear*]

doubly graded q-series; the topological data of M_3 determines $\{s_1, s_2\}$

$$F^{(s_1,s_2)}(q) = \sum_{w\in W} (-1)^{\ell(w)} \sum_{\substack{\vec{n}\in\Lambda\cap P^+ \\ \vec{n}\in w^{-1}(\vec{k})+D\Lambda}} \min(n_1, n_2)\, q^{\frac{1}{2mD}|-\vec{s}+mw(\vec{n})|^2}$$

" $F(q) \xrightarrow[\text{companions}]{\tau \to -\tau} \mathbb{E}(q)$ "

What do we see at higher rank?

✔ $\hat{Z}_a^{SU(3)}(M_3)$ are ~ linear combinations of generalised A_2 false θ-functions

✔ companions $\check{Z}_a^{SU(3)}(M_3)$ are ~ iterated Eichler integrals

$$\theta^{\ell}_{p,r} = \sum_{\substack{k\in\mathbb{Z} \\ k=r \bmod 2p}} k^{\ell} q^{\frac{k^2}{2p}}$$

depth-2 QMF $\quad I_{f_1,f_2}(\tau,\bar{\tau}) := \int_{-\bar{\tau}}^{i\infty} dz_1 \int_{z_1}^{i\infty} dz_2 \, \frac{f(z_1)f(z_2)}{(-i(z_1+\tau))^{2-w_1}(-i(z_2+\tau))^{2-w_2}}$

$\prod \theta^1_{p,r}\theta^1_{3p,r'}$ and $\prod \theta^1_{p,r}\theta^0_{3p,r'}$

[Cheng, Coman, Passaro, Sgroi, *to appear*]

integrands have a nice structure with respect to $SL_2(\mathbb{Z})$ and the M_3 topological data

$$\hat{Z}_a^{SU(3)}(\tau) \sim \sum_{\{s_1,s_2\}} (-1)^{l_s} F^{(s_1,s_2)}(\tau) \xrightarrow{\tau \to -\tau} \sum_{\{s_1,s_2\}} (-1)^{l_s} \mathbb{E}^{(s_1,s_2)}(\tau)$$

with $\mathbb{E}^{(s_1,s_2)}$ ~ iterated Eichler integrals

Some of the hidden detail

[Cheng, Coman, Passaro, Sgroi, *to appear*]

What do we see at higher rank?

$$\hat{Z}_a^{SU(3)}(\tau) \sim \sum_{\{s_1,s_2\}} (-1)^{l_s} F^{(s_1,s_2)}(\tau)$$

"companions"

- Lie algebra $\mathfrak{g} = sl_3$ & 3-manifolds M_3 are Brieskorn spheres $\Sigma(p_1, p_2, p_3)$

"companions" in the sense $F\left(\frac{h}{k}+\frac{it}{2\pi}\right) \sim \sum_{m\geq 0} a_{h,k}(m)\, t^m \quad \underset{t\to 0^+}{\longleftrightarrow} \quad \mathbb{E}\left(\frac{h}{k}+\frac{it}{2\pi}\right) \sim \sum_{m\geq 0} a_{-h,k}(m)\,(-t)^m \qquad \frac{h}{k}\in\mathbb{Q}$

$$\sum_{\{s_1,s_2\}} (-1)^{l_s}\, \mathbb{E}^{(s_1,s_2)}(\tau) = \sum_{\substack{i,j,k=1 \\ i\neq j,\ j<k}}^{3} \int_{-\bar{\tau}}^{i\infty} \frac{dw}{\sqrt{-i(w+\tau)}} \sum_{\zeta} \theta^{1,p+K}_{\zeta\cdot \mathbf{p}^i_{jk}}(w)\, \mathscr{B}^{p+K_{\zeta,j,k}}_{(2-\zeta)\cdot\mathbf{p}^i_{jk}}(\tau,w) + {}''\theta^{\bullet}_{p,r} \to \theta^{\bullet}_{p,p-r}{}''$$

$K_{\zeta,j,k} \subset \{1, p_j p_k, p_i p_k\} \qquad \mathbf{p}^i_{jk} = (p_j p_k, p_i(p_j+p_k)) \qquad \zeta \in \{(1,1),(1,2),(2,1),(2,2)\}$

P linear operator

$$\theta^{1,p+K}_r(w) \sim \sum_{r' \bmod 2p} P^{p+K}_{r,r'}\, \theta^1_{p,r'}(w)$$

$$c_0\, \mathscr{B}^{p+K_r}_r(\tau,z) \in \mathrm{Span}_{\mathbb{Z}}\left[\left(\theta^1_{3p,3r}\right)^*, \left(\theta^0_{3p,3r}\right)^*\right]$$

[Cheng, Coman, Passaro, Sgroi, *to appear*]

What do we see at higher rank?

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rank-1 $\theta^{1,p+K}_{p_1p_2+p_1p_3+p_2p_3} = -\,\theta^1_{p,-p_1p_2-p_1p_3+p_2p_3} + \theta^1_{p,p_1p_2-p_1p_3+p_2p_3} + \theta^1_{p,p_1p_3-p_1p_2+p_2p_3} - \theta^1_{p,p_1p_2+p_1p_3+p_2p_3}$

$$\hat{Z}^{SU(2)} \sim \tilde{\theta}^{1,p+K}_{p_1p_2+p_1p_3+p_2p_3}$$

$$c_0\, \mathscr{B}^{p+K_r}_r(\tau,z) \in \mathrm{Span}_{\mathbb{Z}}\left[\left(\theta^1_{3p,3r}\right)^*, \left(\theta^0_{3p,3r}\right)^*\right]$$

What do we see at higher rank?

✓ $\hat{Z}_a^{SU(3)}(M_3)$ are ~ linear combinations of generalised A_2 false θ-functions

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for M_3 Brieskorn spheres

✓ integrand knows about the lower rank companion $\hat{Z}_0^{SU(2)}(\tau) \sim \widetilde{\theta}^{1,p+K}_{r^*}(\tau)$

Iterated Eichler integrals elsewhere

$b_2^+ = 1$

Topologically twisted $\mathcal{N}$=4 SU(N) / U(N) SYM theory on compact M_4 (Vafa-Witten)

partition function Z_N, for G=SU(N), is not modular under an S-transformation

[Vafa, Witten '94]

complexified gauge coupling $\tau \to -1/\tau$

for pure SU(2) SYM and $M_4 = \mathbb{P}^2$

- modular anomaly is an integral of a MF; can be traded for a holomorphic anomaly

[Minahan, Nemeschansky, Vafa, Warner '98] [Manschot '17]

by adding a non-holomorphic period integral to the partition function

- higher rank: modular transformation includes a shift by iterated integrals of θ-series

[Manschot '17]

- holomorphic anomaly of Z_N factorises into partition functions at lower rank

[Minahan, Nemeschansky, Vafa, Warner '98] [Manschot '17]

$$\partial_{\bar{\tau}} Z_N \sim \sum_k k(N-k) Z_k Z_{N-k}$$

constrains Z_N

- interpretation: the non-holomorphic contributions are generated by Q-exact terms due to boundaries of the moduli space

[Vafa, Witten '94] proposed, [Dabholkar, Putrov, Witten '20] verified by example

What conclusions can be drawn

- ✓ $\hat{Z}_a^{SU(3)}(M_3;\tau)$ are ~ linear combinations of generalised A_2 false θ-functions

$\hat{Z}_0^{SU(2)}(\tau) \sim \widetilde{\theta}_{r_1}^{1,p+K}(\tau)$

- ✓ companions $\check{Z}_a^{SU(3)}(M_3)$ are ~ iterated Eichler integrals

$$\hat{Z}_a^{SU(3)}(\tau) \sim \sum_{\{s_1,s_2\}} (-1)^{l_s} F^{(s_1,s_2)}(\tau) \xrightarrow{\tau \to -\tau} \sum_{\{s_1,s_2\}} (-1)^{l_s} \mathbb{E}^{(s_1,s_2)}(\tau) \in \mathrm{Span}_{\mathbb{Z}}\left[\left(\theta_r^{1,p+K}\, \mathscr{B}_{r'}^{p+K_{r'}}\right)^*(\tau,\bar{\tau})\,;\, r,r' \in \mathbb{Z}/2p\right]$$

$$\#\, \mathscr{B}_r^{p+K_r}(\tau,z) \in \mathrm{Span}_{\mathbb{Z}}\left[\left(\theta^1_{3p,3r}\right)^*, \left(\theta^0_{3p,3r}\right)^*\right]$$

- ✓ combinatorial structure ~ topological data

depth-2 QMF

<u>To do ...</u>

- ☐ What happens for more general families of 3-manifolds
- ☐ Extract prediction for generic building blocks
- ☐ Explore links to the Log VOAs
- ☐ What insights can be obtained about T[M$_3$] from the quantum modularity of $\hat{Z}(M_3)$

Thank you!