Gravity and Integrability

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Introduction

It is a remarkable fact that general relativity admits many exact solutions.

The equations of motion are 10 nonlinear partial differential equations (PDEs).

Nonlinear PDEs do not usually admit any exact solutions, but general relativity admits many exact solutions.

Why?

Introduction

It turns out the dimensional reduction of general relativity to two spacetime dimensions is an **integrable system**.

Solutions of general relativity with two commuting Killing vectors (two ignorable coordinates) can be viewed as solutions of the 2d integrable system.

The 2d integrable system has an infinite dimensional symmetry called the **Geroch group** that explains why it is integrable.

The existence of the Geroch group thus explains (at least in many cases) why general relativity admits many exact solutions.

The Schwarzschild metric

General relativity was published in 1915.

The first nontrivial exact solution, the **Schwarzschild metric**, was published in 1916.

Einstein was surprised to see an exact solution and wrote to Schwarzschild:

"I have read your paper with the utmost interest. I had not expected that one could formulate the exact solution of the problem in such a simple way."

The Schwarzschild metric

The **Schwarzschild metric** is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \, .$$

It describes a static black hole with mass M.

The Schwarzschild metric is a diagonal metric.

In the special case of diagonal metrics with two commuting Killing vectors, the Einstein equations reduce to a linear PDE.

So, in retrospect, it is not too surprising that there are exact solutions for diagonal metrics. (Although, it is still somewhat surprising that the Einstein equations become linear on diagonal metrics.)

The Kerr metric

The **Kerr metric** was published in 1963.

It describes a spinning black hole with mass M and spin parameter a.

It is *not* a diagonal metric, so the fact that this exact solution exists is much more surprising.

The Kerr metric is:

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar\sin^2\theta}{\Sigma} dt d\varphi \\ &+ \left(r^2 + a^2 + \frac{2Ma^2 r \sin^2\theta}{\Sigma}\right) \sin^2\theta d\varphi^2 \\ &+ \frac{\Sigma}{\Delta} dr^2 + \Sigma \, d\theta^2 \, . \end{aligned}$$

where $\Delta = r^2 - 2Mr + a^2$ and $\Sigma = r^2 + a^2 \cos^2\theta$.

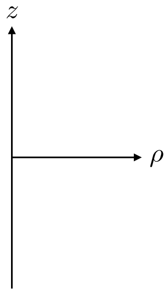
The Kerr metric

The coordinates are (t, φ, r, θ).

The metric has two commuting Killing vectors, ∂_t and ∂_φ.

Introduce (roughly) $\rho \sim r \cos\theta$ and $z \sim r \sin\theta$.

The functions appearing in the metric are functions on a two dimensional Euclidean half-plane:



The Kerr metric

The Einstein equations reduce to an integrable sigma model on the $\rho - z$ half-plane.

The Kerr metric is a solution of the 2d integrable sigma model.

The action of the **Geroch group** maps the Minkowski metric to the Kerr metric. The Kerr metric can be derived using pure algebra (no PDEs).

The double Kerr metric

The **double Kerr metric** was discovered by Kramer and Neugebauer (1980).

It describes a pair of Kerr black holes that are held apart by a massless strut:

This metric is also a solution of the 2d integrable sigma model on the $\rho - z$ half-plane.

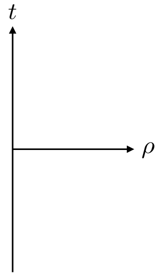
If Kerr is the single soliton solution, then double Kerr is the 2-soliton solution of this integrable system.

Cylindrical gravitational waves

Cylindrical gravitational waves are another interesting class of exact solutions of the vacuum Einstein equations.

These solutions have two *spacelike* commuting Killing vectors, ∂_z and ∂_φ.

They live on a two dimensional *Lorentzian* half-plane:



Cylindrical pulse wave

A **cylindrical gravitational pulse wave** is a cylindrical gravitational wave with a pulse profile in the radial direction.

The pulse arrives from infinity, scatters off $\rho = 0$, and returns to infinity.

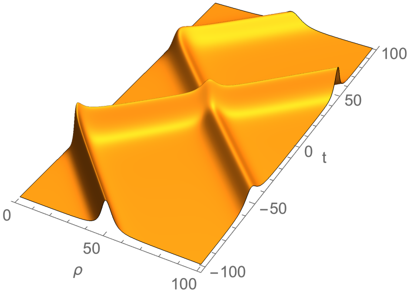
Piran, Safier, and Katz (1986) discovered an exact solution for a cylindrical pulse wave by making a double analytic continuation of the Kerr metric:

$$t = iz \,, \quad z = it \,.$$

To get a real metric, the spin parameter, a, needs to be imaginary.

Double pulse wave

In an unpublished but soon-to-appear work, I have obtained an exact solution for a **double pulse wave** by making a double analytic continuation of the double Kerr metric:



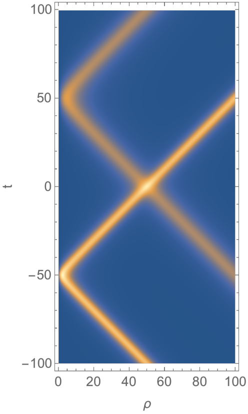
The z and φ directions are not shown.

The function plotted is related to the energy.

Double pulse wave

The pulses maintain their shapes after scattering, much like **solitons** in other integrable systems.

Unlike other integrable systems, they suffer **no time delay** after scattering (RFP arXiv:21xxx.xxxx, to appear):



Outline

- Introduction
- Double pulse wave solution
- Solution generating and the Einstein-Rosen wave
- Twistor Chern-Simons action
- Hyperbolic Kac-Moody symmetry in $d = 1$
- Open questions

Solution generating

The process of generating new solutions using the Geroch group looks like this:

$$g_{\mu\nu} \to M(\tau) \xrightarrow{\text{Geroch}} M'(\tau) \xrightarrow{\text{Riemann-Hilbert}} g'_{\mu\nu}\,.$$

$g_{\mu\nu}$ is the initial metric.

$M(\tau)$ is an invariant called the **monodromy matrix**. It is valued in $\mathrm{SL}(2,\mathbb{C})$. It is a function of a complex parameter, τ, called the **spectral parameter**.

The action of the Geroch group gives a new monodromy matrix, $M'(\tau)$.

We recover a new vacuum metric, $g'_{\mu\nu}$, by solving a Riemann-Hilbert problem using the inverse scattering technique.

Solution generating: example

I would like to discuss the previous diagram using an example: the Einstein-Rosen wave.

This example is from RFP, arXiv:2106.13252.

The **Einstein-Rosen metric** describes a cylindrical gravitational wave:

$$ds^2 = e^{2\gamma - 2\psi}(-dt^2 + d\rho^2) + e^{-2\psi}\rho^2 d\varphi^2 + e^{2\psi} dz^2 \,,$$

where

$$\begin{aligned} \psi &= J_0(\rho)\cos t \,, \\ \gamma &= \frac{1}{2}\rho^2 J_0(\rho)^2 + \frac{1}{2}\rho^2 J_1(\rho)^2 - \rho J_0(\rho) J_1(\rho)\cos^2 t \,. \end{aligned}$$

The metric is a function of t and ρ only.

The monodromy function

The Einstein-Rosen metric is diagonal. Diagonal metrics have diagonal monodromy matrices. A diagonal $\mathrm{SL}(2, \mathbb{C})$ matrix is characterized by a single number. So a diagonal metric with two commuting Killing vectors can be assigned a **monodromy function**, $m(\tau) \in \mathbb{C}$.

The monodromy function of the Einstein-Rosen wave is

$$m(\tau) = \cos \tau \,.$$

It is remarkable that the complicated metric on the previous slide can be reconstructed from this simple monodromy function using only algebra.

$g_{\mu\nu} \to m(\tau)$

First, let me explain how to get the monodromy function from the metric.

Evaluate the metric function $\psi(\rho, t) = J_0(\rho)\cos t$ at the $\rho = 0$ boundary and complexify $t = \tau$:

$$\psi(\rho = 0, t = \tau) = \cos\tau = m(\tau)\,.$$

Thus the monodromy function is a kind of complexified boundary data.

The monodromy matrices of nondiagonal metrics arise in a similar way.

$m(\tau) \to g_{\mu\nu}$

Now let me describe how to recover the Einstein-Rosen metric from its monodromy function. There are basically two steps.

Step one is a change of variables. Let

$$\tau = \frac{\rho}{2}(\zeta + \zeta^{-1}) - t\,.$$

ζ is a new spectral parameter. This equation has a natural interpretation on twistor space, which I will unfortunately not have time to explain.

After this change of variables, the monodromy function becomes:

$$\cos\tau = \frac{1}{2}e^{-it}e^{i\frac{\rho}{2}(\zeta+\zeta^{-1})} + \frac{1}{2}e^{it}e^{-i\frac{\rho}{2}(\zeta+\zeta^{-1})}\,.$$

Note the return of the spacetime coordinates, ρ and t.

$m(\tau) \rightarrow g_{\mu\nu}$

Step two is to expand the exponentials using

$$e^{\pm i \frac{\rho}{2}(\zeta+\zeta^{-1})} = \sum_{n=-\infty}^{\infty} (\pm 1)^n i^n J_n(\rho) \zeta^n \, .$$

The right hand side is a sum over Bessel functions.

Now the monodromy function is an infinite sum with three parts:

$$\cos\tau = m_+ + m_0 + m_-$$

m_+ are the positive powers of ζ, m_- are the negative powers of ζ, and m_0 are the terms independent of ζ.

$m(\tau) \rightarrow g_{\mu\nu}$

One of the functions in the Einstein-Rosen metric is

$$\psi(t,\rho) = m_0 = J_0(\rho)\cos t\,.$$

In other words, one of the functions in the metric is the "zero mode" of the monodromy function.

Where is the other function, $\gamma(t,\rho)$, from the Einstein-Rosen metric?

$m(\tau) \to g_{\mu\nu}$

It turns out:

$$\gamma(t, \rho) = 2 \, \mathrm{Res}_{\zeta=0} \left(m_- \frac{d}{d\zeta} m_+ \right).$$

The other piece of the metric comes from a 2-cocycle on the $\mathrm{SL}(2, \mathbb{C})$ loop group!

The above equation is a special case of a general formula due to Breitenlohner and Maison (1987).

It is remarkable that this 2-cocycle is hidden in the geometry of astrophysical black holes!

For details on the Einstein-Rosen example, see RFP, arXiv:2106.13252.

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The Lax operator

A basis object in the theory of integrable systems is the **Lax operator**, L.

It is a matrix valued function of the spacetime coordinates, x^μ, and the spectral parameter, τ.

It satisfies the **flatness condition**

$$dL + L \wedge L = 0\,.$$

The flatness condition encodes the equations of motion of an integrable system.

The Lax operator

This story can be reformulated on **twistor space**, Z.

Twistor space is six real dimensional. As a real manifold

$$Z = \mathbb{R}^4 \times S^2\,.$$

Now x^μ and τ are all just coordinates on a twistor space.

The Lax operator can be realized as a $(0,1)$ gauge field, A, on twistor space.

The flatness condition becomes the **partial flatness condition**

$$\bar{\partial} A + A \wedge A = 0\,.$$

The Penrose-Ward correspondence

This reformulation of integrable systems on twistor space is called the **Penrose-Ward correspondence**.

A shortcoming is that it is formulated at the level of the equations of motion.

Twistor Chern-Simons theory

A recent development is the observation (Costello, Bittleston-Skinner, RFP) that the partial flatness condition

$$\bar{\partial} A + A \wedge A = 0\,,$$

is the equation of motion of the **twistor Chern-Simons action**

$$S = \frac{1}{2\pi i} \int \Omega \wedge \mathrm{Tr} \left(A \wedge \bar{\partial} A + \frac{2}{3} A \wedge A \wedge A \right).$$

Ω is a meromorphic $(3,0)$ form on twistor space.

This result was inspired by the recent 4d Chern-Simons theory of Costello and Yamazaki.

Twistor Chern-Simons theory

S is almost the action of holomorphic Chern-Simons theory.

But in that case, Ω is a holomorphic $(3,0)$ form.

Twistor space is not Calabi-Yau, so it admits no holomorphic $(3,0)$ form (absent $N = 4$ supersymmetry).

We thus need to choose Ω to be a meromorphic $(3,0)$ form.

Twistor Chern-Simons theory

To get a well-defined theory, we will need boundary conditions on A at the poles of Ω.

The problem of classifying integrable systems thus becomes related to the problem of classifying boundary conditions for twistor Chern-Simons theory.

As we will now show, the 2d sigma model that governs dimensionally reduced general relativity fits well into this framework.

Holomorphic coordinates

As a complex manifold, twistor space is

$$\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{CP}^1$$

Let λ and μ be holomorphic coordinates on the $\mathcal{O}(1)$ fibers.

Let ζ be the holomorphic coordinate on $\mathbb{CP}^1$.

Choose

$$\Omega = \frac{d\lambda \wedge d\mu \wedge d\zeta}{\zeta^2}\,.$$

We will need boundary conditions on A at the poles at $\zeta = 0$ and $\zeta = \infty$.

Twistor lines and spacetime

Before stating the boundary conditions, we need to recall how spacetime emerges from this picture.

Global sections of

$$O(1) \oplus O(1) \to \mathbb{CP}^1$$

are called **complex twistor lines**.

Twistor space inherits a **real structure** from the fiberwise action of the antipodal map $\zeta \to -1/\overline{\zeta}$.

Complex twistor lines invariant under the real structure are called **real twistor lines**. The parameter space of real twistor lines is $\mathbb{R}^4$.

Twistor lines and spacetime

Euclidean spacetime is the parameter space space of real twistor lines.

Each point in twistor space lies on a unique real twistor line.

We have a **nonholomorphic projection**

$$Z \to \mathbb{R}^4$$

from twistor space to Euclidean spacetime.

Explicitly,

$$(\lambda, \mu, \zeta) \to (u, v) \in \mathbb{C}^2$$

with

$$u = \frac{\lambda - \bar{\mu}\zeta}{1 + \zeta\bar{\zeta}}\,, \quad v = \frac{\mu + \bar{\lambda}\zeta}{1 + \zeta\bar{\zeta}}\,.$$

Coordinates

The complex coordinates (u, v) are related to ordinary (real) spacetime coordinates by

$$u = \rho e^{i\varphi}\,, \quad v = z + it\,.$$

Define $w = z + i\rho$.

We are going to state the boundary conditions on A using the (nonholomorphic) twistor coordinates

$$(w, \bar{w}, \varphi, t, \zeta, \bar{\zeta})$$

Boundary conditions

Write

$$A = A_w \overline{\partial} w + A_{\bar{w}} \overline{\partial} \bar{w} + A_{\bar{\zeta}} d\overline{\zeta} \, .$$

The boundary conditions we need are

$$\begin{aligned}
\zeta = 0: \quad & A_w = O(\zeta)\,, \qquad && A_{\bar{w}} = O(\zeta)\,, \qquad && A_{\bar{\zeta}} = O(1)\,, \\
\zeta = \infty: \quad & A_w = O(1/\zeta)\,, && A_{\bar{w}} = O(1/\zeta)\,, && A_{\bar{\zeta}} = O(1/\zeta^2)\,.
\end{aligned}$$

These boundary conditions ensure that the twistor Chern-Simons action has no poles at $\zeta = 0$ and $\zeta = \infty$.

A is trivial on real twistor lines

We further assume that A is trivial on real twistor lines. This assumption is a standard part of the Penrose-Ward correspondence. It is physically sensible because real twistor lines correspond to points in spacetime.

This assumption implies

$$A_{\bar{\zeta}} = \hat{\sigma}^{-1}\partial_{\bar{\zeta}}\hat{\sigma}\,,$$

where $\hat{\sigma}$ is a Lie group valued field.

We are free to multiply $\hat{\sigma} \to g\hat{\sigma}$, for constant g.

Use this freedom to fix $\hat{\sigma} = \mathrm{id.}$ at $\zeta = \infty$.

Let σ be the value of $\hat{\sigma}$ at $\zeta = 0$

Dimensional reduction

For dimensional reduction, assume the fields are independent of t and φ.

For the application to 2d gravity, further assume σ and $\hat{\sigma}$ are positive definite and symmetric ($\sigma = \sigma^T$) $\mathrm{SL}(2, \mathbb{R})$ matrices.

Solution

Define $z = e^{-i\varphi}\zeta$.

The boundary conditions and the $F_{\bar{z}w} = F_{\bar{z}\bar{w}} = 0$ equations of motion imply

$$A = A_0 + A_1\,,$$

with

$$\begin{aligned} A_0 &= \hat{\sigma}^{-1}\bar{\partial}\hat{\sigma}\,, \\ A_1 &= -\frac{i}{z+i}\hat{\sigma}^{-1}(\partial_w\sigma)\sigma^{-1}\hat{\sigma}\bar{\partial}w + \frac{i}{z-i}\hat{\sigma}^{-1}(\partial_{\bar{w}}\sigma)\sigma^{-1}\hat{\sigma}\bar{\partial}\bar{w}\,. \end{aligned}$$

This is closely related to the Lax operator

$$L = -\frac{i}{z+i}(\partial_w\sigma)\sigma^{-1}dw + \frac{i}{z-i}(\partial_{\bar{w}}\sigma)\sigma^{-1}d\bar{w}\,.$$

2d action

Reinserting A into the twistor Chern-Simons action and carrying through the dimensional reduction gives the action (RFP)

$$4 \int du\, dv\, d\bar{u}\, d\bar{v}\ \mathrm{Tr}\left[(\partial_w \sigma)\sigma^{-1}(\partial_{\bar{w}}\sigma)\sigma^{-1}\right] .$$

Integrating over t and φ gives a 2d action.

This 2d action describes the dimensional reduction of general relativity to two Euclidean spacetime dimensions.

A similar story gives the action for the dimensional reduction of general relativity to two Lorentzian spacetime dimensions (cylindrical gravitational waves).

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Kac-Moody algebras

Kac-Moody algebras were introduced independently by Victor Kac and Robert Moody in 1968.

They come in families of increasingly complicated types:

finite

affine

hyperbolic

. . .

Finite Kac-Moody algebras are the same thing as ordinary finite dimensional Lie algebras.

Affine Kac-Moody algebras

Affine Kac-Moody algebras are ubiquitous in modern physics.

They appear in conformal field theory as current algebras.

They have a beautiful geometrical realization in terms of **loop algebras**: *every affine Kac-Moody algebra is (an extension of) the space of maps,*

$$S^1 \to \mathfrak{g}\,,$$

from the circle, S^1, into a finite dimensional Lie algebra, $\mathfrak{g}$.

Affine Kac-Moody algebras

The different applications of affine Kac-Moody algebras to physics correspond to different interpretations of the circle:

- *Conformal field theory* : S^1 is a loop in spacetime.
- *String theory* : S^1 is a loop on the string worldsheet.
- *The theory of integrable systems* : S^1 is a loop in twistor space.

Hyperbolic Kac-Moody algebras

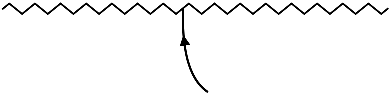
Hyperbolic Kac-Moody algebras have no known geometrical interpretation and their appearances in physics are very rare.

A better geometrical understanding of these algebras would probably inspire new physics applications *and vice versa*.

The places where hyperbolic Kac-Moody algebras *do* appear in physics are intriguing...

Spacelike singularities

The dynamics of gravity and supergravity becomes effectively one dimensional near spacelike singularities.



The near-singularity dynamics has a billiard description, the **BKL billiard**.

The symmetries of the BKL billiard models of different supergravity theories are Weyl subalgebras of hyperbolic Kac-Moody algebras.

There are hints that the full hyperbolic Kac-Moody algebras might emerge as one moves away from the exact BKL billiard limit.

It is striking that the mysteries of hyperbolic Kac-Moody algebras and the mysteries of curvature singularities might be related.

Billiards references

- V. Belinski and M. Henneaux, *The Cosmological Singularity*, Cambridge University Press (2017)

- A. Kleinschmidt and H. Nicolai, "Cosmological quantum billiards," *Proceedings, Foundations of Space and Time: Reflections on Quantum Gravity*, (2009) [arXiv:0912.0854]

- T. Damour, M. Henneaux, H. Nicolai, "Cosmological billiards," Class. Quant. Grav. **20**, R145 (2003), [arXiv:hep-th/0212256]

- T. Damour, M. Henneaux, H. Nicolai, "E_{10} and a 'small tension' expansion of M theory," Phys. Rev. Lett. **89**, 221601 (2002), [arXiv:hep-th/0207267]

Hyperbolic Kac-Moody algebras

The **rank** of a Kac-Moody algebra is the dimension of its Cartan subalgebra (the maximal commuting subalgebra).

There are infinitely many hyperbolic Kac-Moody algebras with rank 2.

There are 238 hyperbolic Kac-Moody algebras with ranks 3 through 10.

There are no hyperbolic Kac-Moody algebras with higher rank.

$\mathfrak{e}_{10}$ and $d = 11$ supergravity

The biggest hyperbolic Kac-Moody algebra is called $\mathfrak{e}_{10}$ (it contains the other rank 10 algebras as subalgebras).

$\mathfrak{e}_{10}$ is a symmetry of the dimensional reduction of $d = 11$ supergravity to one dimension (Kleinschmidt, Nicolai, ...).

There is no supersymmetry (and really no physics of any kind) in the definition of hyperbolic Kac-Moody algebras.

So it is striking that the biggest hyperbolic Kac-Moody algebra and the biggest supergravity are related.

Dimensionally reduced general relativity

The dimensional reduction of general relativity from four dimensions to a single null dimension appears to be governed by an infinite dimensional hyperbolic Kac-Moody symmetry.

This symmetry is an enhanced version of the Geroch group.

Context

The results in this section will appear in an upcoming work (RFP arXiv:21xxx.xxxx).

This work is inspired by and has some overlap with:

H. Nicolai, "*A hyperbolic Kac-Moody algebra from supergravity*," *Physics Letters B*, **276** (1992), no.3 333–340.

Hermann studied the dimensional reduction of $N = 1$ supergravity from $d = 4$ to $d = 1$ and argued for the emergence of a hyperbolic Kac-Moody algebra in that case.

The Ehlers group

The Geroch group is an affine Kac-Moody algebra. It is itself an enhanced version of the Ehlers group.

The **Ehlers group** is a hidden symmetry of the dimensional reduction of general relativity to three spacetime dimensions.

The Ehlers group is $\mathrm{SL}(2, \mathbb{R})$.

General relativity and Kac-Moody algebras

the hidden Kac-Moody symmetry is...	*when GR is reduced to...*
finite	$d = 3$
affine	$d = 2$
hyperbolic	$d = 1$

General relativity and Kac-Moody algebras

the hidden Kac-Moody symmetry is...	*when GR is reduced to...*
$\bullet$	$d = 3$
$\bullet\!\Longleftrightarrow\!\bullet$	$d = 2$
$\bullet\!\Longleftrightarrow\!\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet$	$d = 1$

The vierbein

Let's regard the **vierbein**, e_μ^a, as the basic field of general relativity.

The spacetime **metric** is

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$$

where η_{ab} is the Minkowski metric.

The vierbein is a 4×4 upper triangular matrix.

The vierbein

The vierbein contains 10 scalar degrees of freedom.

We can eliminate four degrees of freedom using a diffeomorphism.

So the vierbein can be parametrized by six scalar fields.

A convenient choice is (Nicolai 1992):

$$e_\mu^a = \begin{pmatrix} \Delta^{-1/2} & 0 & 0 & 0 \\ 0 & \Delta^{-1/2}\lambda & \Delta^{-1/2}\rho A & \Delta^{1/2}B_- \\ 0 & 0 & \Delta^{-1/2}\rho & \Delta^{1/2}B_2 \\ 0 & 0 & 0 & \Delta^{1/2} \end{pmatrix}.$$

The six scalar fields are $(\Delta, B_2, B_-, \rho, A, \lambda)$.

Dimensional reduction

For dimensional reduction, we assume the six scalar fields are functions of a single **null spacetime coordinate**, u

$$\Delta = \Delta(u), B_2 = B_2(u), \ \ldots$$

The spacetime coordinates are (u, v, x^2, x^3).

The Minkowski metric is

$$\eta_{ab} = \begin{pmatrix} 0 & -1/2 & 0 & 0 \\ -1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution space

Einstein's equations, $G_{\mu\nu} = 0$, are equivalent to three constraints:

$$\lambda^{-1}\rho^{-1}(\lambda'\rho' - \lambda\rho'') = \frac{1}{2}\Delta^{-2}\left(\Delta'^2 + \frac{\Delta^4 B_2'^2}{\rho^2}\right)$$

$$B'_- = A\, B'_2$$

$$A' = 0$$

Primes are u derivatives.

So solution space is parameterized by six functions $(\Delta(u), B_2(u), \dots)$ *subject to three constraints.*

Solution space

The equations of motion are almost trivial.

On the other hand, they are nonlinear differential equations. So, for example, the sum of two solutions is not again a solution.

Thus we can ask if there is a symmetry, analogous to the Geroch group, that parametrizes the solutions.

The Matzner-Misner group

There is fairly obvious $\mathrm{SL}(3, \mathbb{R})$ symmetry that maps solutions to solutions.

This is called the **Matzner-Misner group**.

The Ehlers group

To find more symmetries, we use the following trick.

First, we enlarge solution space by adding a new function, $B = B(u)$, and a new constraint,

$$B' = \frac{\Delta^2}{\rho} B_2'$$

Then we rewrite the complicated looking constraint from before in the simpler form

$$\lambda^{-1}\rho^{-1}(\lambda'\rho' - \lambda\rho'') = \frac{1}{2}\Delta^{-2}\left(\Delta'^2 + B'^2\right)$$

Now the form of the right hand side suggests there is an $\mathrm{SL}(2,\mathbb{R})$ symmetry. This will be the **Ehlers group**.

$\mathfrak{sl}(2,\mathbb{R})$

Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ and decompose

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h}$$

where $\mathfrak{h} = \mathfrak{so}(2)$ and $\mathfrak{k}$ is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$.

A convenient basis for $\mathfrak{g}$ is

$$\mathrm{Y}^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathrm{Y}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathrm{Y}^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Y^3 is a basis for $\mathfrak{h} = \mathfrak{so}(2)$ and Y^1 and Y^2 are a basis for $\mathfrak{k}$.

The Ehlers group

Define

$$V \equiv \begin{pmatrix} \Delta^{1/2} & B\Delta^{-1/2} \\ 0 & \Delta^{-1/2} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$$

Decompose

$$V^{-1}V' = P + Q$$

where $P \in \mathfrak{k}$ and $Q \in \mathfrak{h}$.

Now we can rewrite the complicated looking constraint from before as

$$\lambda^{-1}\rho^{-1}(\lambda'\rho' - \lambda\rho'') = \mathrm{Tr}(P^2)\,.$$

The **Ehlers group** is an action of $\mathrm{SL}(2,\mathbb{R})$ on V that preserves the constraints.

Infinite dimensional symmetry

The Ehlers group and the Matzner-Misner group do not commute.

Together, they generate an infinite dimensional symmetry, which is the **Geroch group in one dimension**.

We will argue that the infinite dimensional symmetry is a hyperbolic Kac-Moody algebra.

A problem

The Ehlers group acts directly on B and Δ.

To get the action of the Ehlers group on B_2 and B_-, we use the constraint equations

$$
\begin{aligned}
B_2' &= \frac{\rho}{\Delta^2} B' \\
B_-' &= A\, B_2'
\end{aligned}
$$

Now we need to confront the following **problem**: the action of the Ehlers group on B_2 and B_- is nonlocal.

By "nonlocal" we mean that the infinitesimal variations of B_2 and B_- involve integrals over u.

An infinite tower of new fields

To get local actions, we use a variant of our earlier trick: we enlarge solution space by adding new functions and new constraints.

In the present version of the trick, we need to add **an infinite number of new functions** and **an infinite number of new constraints**.

An infinite tower of new fields

The new functions ("dual potentials") can be organized into two infinite pyramids:

$$\begin{array}{cccc} & \varphi & & \\ & \varphi_0 \quad \varphi_1 & & \\ \varphi_{00} & \varphi_{01} & \varphi_{10} & \varphi_{11} \\ & \ldots, & & \end{array}$$

$$\begin{array}{cccc} & \eta & & \\ & \eta_0 \quad \eta_1 & & \\ \eta_{00} & \eta_{01} & \eta_{10} & \eta_{11} \\ & \ldots. & & \end{array}$$

Each of these functions is a function of u: $\varphi = \varphi(u), \varphi_0 = \varphi_0(u), \ldots$.

$\mathfrak{sl}(3, \mathbb{R})$

These six matrices generate $\mathfrak{sl}(3, \mathbb{R})$:

$$E_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad H_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad F_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$E_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad H_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad F_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The action of the Matzner-Misner algebra

Let g be one of the six generators of $\mathfrak{sl}(3, \mathbb{R})$.

Act with g on the vierbein, e_μ^α, by

$$e \to -ge + eh(g)$$

where $h(g)$ is a Lorentz transformation.

Choose $h(g)$ to restore the upper triangular form of the vierbein.

This defines the action of the **Matzner-Misner** algebra on the six basic fields $(\Delta, B_2, B_-, \rho, A, \lambda)$.

The action of the Matzner-Misner algebra

The action of the **Matzner-Misner** algebra on the six basic fields is:

$$
\begin{aligned}
e_0 =& -\delta_{B_2} + \dots \\
h_0 =& 2\Delta\delta_\Delta - 2B_2\delta_{B_2} - B_-\delta_{B_-} + A\delta_A + 2\lambda\delta_\lambda + \dots \\
f_0 =& -2\Delta B_2\delta_\Delta + (B_2^2 - \rho^2\Delta^{-2})\delta_{B_2} + (B_-B_2 - \rho^2\Delta^{-2}A)\delta_{B_-} + (B_- - B_2A)\delta_A \\
& -2B_2\lambda\delta_\lambda + \dots \\
e_{-1} =& -B_2\delta_{B_-} - \delta_A + \dots \\
h_{-1} =& B_2\delta_{B_2} - B_-\delta_{B_-} + \rho\delta_\rho - 2A\delta_A - \lambda\delta_\lambda + \dots \\
f_{-1} =& -B_-\delta_{B_2} - A\rho\delta_\rho + A^2\delta_A + A\lambda\delta_\lambda + \dots
\end{aligned}
$$

The action of the Matzner-Misner algebra on B

To fix the action of the Matzner-Misner algebra on B, we make the assignments:

$$
\begin{aligned}
e_0(B) &= 0 \\
h_0(B) &= 2B \\
f_0(B) &= -(\varphi + 2B_2 B + \rho) \\
e_{-1}(B) &= 0 \\
h_{-1}(B) &= 0 \\
f_{-1}(B) &= 0
\end{aligned}
$$

These are the simplest assignments that are compatible with the constraint equations.

Later we will also need to fix the action of the Matzner-Misner algebra on the infinite tower of dual potentials $(\varphi, \varphi_0, \dots, \eta, \eta_0, \dots)$.

The action of the Ehlers algebra

There is a similar story for the Ehlers algebra. Recall

$$V = \begin{pmatrix} \Delta^{1/2} & B\Delta^{-1/2} \\ 0 & \Delta^{-1/2} \end{pmatrix}.$$

Let $g \in \mathfrak{sl}(2, \mathbb{R})$ act on V by

$$V \to -gV + Vh(g)\,,$$

where $h(g)$ is a compensating $\mathfrak{so}(2)$ transformation. We choose $h(g)$ to restore the upper triangular gauge of V.

This fixes the action of the Ehlers algebra on Δ and B.

The action of the Ehlers algebra

We extend the action of the Ehlers algebra to the other basic fields by making the simplest possible assignments that are compatible with the constraint equations.

This gives:

$$e_1 = -\,\delta_B + \dots$$

$$h_1 = -\,2\Delta\delta_\Delta + 2B_2\delta_{B_2} + 2B_-\delta_{B_-} - 2B\delta_B + \dots$$

$$f_1 = 2\Delta B\delta_\Delta + \varphi\delta_{B_2} + \eta\delta_{B_-} + (B^2 - \Delta^2)\delta_B + \dots$$

We still need to fix the action of the algebra on the dual potentials.

The dual potentials

The dual potentials are the two infinite pyramids of new fields:

$$
\begin{gathered}
\varphi \\
\varphi_0 \quad \varphi_1 \\
\varphi_{00} \quad \varphi_{01} \quad \varphi_{10} \quad \varphi_{11} \\
\ldots,
\end{gathered}
$$

$$
\begin{gathered}
\eta \\
\eta_0 \quad \eta_1 \\
\eta_{00} \quad \eta_{01} \quad \eta_{10} \quad \eta_{11} \\
\ldots.
\end{gathered}
$$

Let φ_I stand for any function in the first list and let η_I stand for any function in the second list.

The dual potentials

The dual potentials are linked to the basic fields by

$$\begin{aligned}\varphi &= f_1(B_2)\\ \eta &= f_1(B_-)\end{aligned}$$

Further define

$$\begin{aligned}\varphi_{I0} &= f_0(\varphi_I)\\ \eta_{I0} &= f_0(\eta_I)\\ \varphi_{I1} &= f_1(\varphi_I)\\ \eta_{I1} &= f_1(\eta_I)\end{aligned}$$

So φ_I is string of f_0's and f_1's acting on B_2.

And η_I is string of f_0's and f_1's acting on B_-.

The action of h_1 on the dual potentials

We need to fix the action of the algebra on the dual potentials.

Let's just discuss an example.

Let's compute

$$h_1(\varphi_I) = h_1 f_{i_n} \cdots f_{i_1} f_1 B_2$$

The Cartan matrix

The hyperbolic Kac-Moody algebra we are interested in has the Cartan matrix

$$A_{ij} = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

The definition of the hyperbolic Kac-Moody algebra fixes

$$[h_i, f_j] = -A_{ij} f_j$$

The dual potentials

So to compute

$$h_1(\varphi_I) = h_1 f_{i_n} \cdots f_{i_1} f_1 B_2$$

we use the commutation relations to move h_1 to the right.

This gives

$$h_1(\varphi_I) = -2(n_1^I - n_0^I)\varphi_I$$

where n_1^I is the number of f_1's and n_0^I is the number of f_0's in the subscript I.

f_{-1}

The hyperbolic Kac-Moody algebra has nine generators:

$$e_1\,, h_1\,, f_1\,, e_0\,, h_0\,, f_0\,, e_{-1}\,, h_{-1}\,, f_{-1}$$

I have closed form expressions for the action of the first eight generators on all of the dual potentials.

However, the ninth generator, f_{-1}, presents special difficulties.

This is because the Cartan matrix of the hyperbolic Kac-Moody algebra does not fix $[f_{-1}, f_0]$.

f_{-1}

I have checked that the constraints imposed by hyperbolic Kac-Moody symmetry are strong enough to uniquely fix the action of f_{-1} on the first few dual potentials:

$$
\begin{aligned}
f_{-1}(\varphi_1) &= -\eta_1 \\
f_{-1}(\varphi_0) &= -2(\eta_0 + B_2\eta - B_-\varphi) \\
f_{-1}(\eta_1) &= 0
\end{aligned}
$$

But I do not have a proof that the action of f_{-1} can be consistently extended to the full infinite tower of dual potentials.

This is the main technical open problem in the present work.

New constraint equations

The basic fields satisfy four constraint equations:

$$
\begin{aligned}
\lambda^{-1}\rho^{-1}(\lambda'\rho' - \lambda\rho'') &= \frac{1}{2}\Delta^{-2}\left(\Delta'^2 + \frac{\Delta^4 B_2'^2}{\rho^2}\right) \\
B_-' &= A\, B_2' \\
A' &= 0 \\
B' &= \frac{\Delta^2}{\rho} B_2'
\end{aligned}
$$

We get an infinite tower of **new constraint equations** by repeatedly acting on the four basic constraints with f_0 and f_1.

The new constraints are constraints on the dual potentials, φ_I and η_I.

Further constraint equations

The definition of the hyperbolic Kac-Moody algebra requires the **quadrilinear relations**:

$$[f_1, [f_1, [f_1, f_0]]] = 0$$
$$[f_0, [f_0, [f_0, f_1]]] = 0$$

To impose these relations, we need to impose **an infinite tower of further constraints** on the dual potentials.

Here is the first one:

$$[f_1, [f_1, [f_1, f_0]]](\Delta) = 4\Delta\varphi_{11} = 0\,.$$

So we need to impose $\varphi_{11} = 0$ as a further constraint on solution space.

Dual potentials

We started with two infinite pyramids of dual potentials:

$$\begin{array}{c} \varphi \\ \varphi_0 \quad \varphi_1 \\ \varphi_{00} \quad \varphi_{01} \quad \varphi_{10} \quad \varphi_{11} \\ \cdots \end{array}$$

$$\begin{array}{c} \eta \\ \eta_0 \quad \eta_1 \\ \eta_{00} \quad \eta_{01} \quad \eta_{10} \quad \eta_{11} \\ \cdots \end{array}$$

But now we see that the dual potentials are not all independent.

What does a hyperbolic Kac-Moody algebra look like?

A conjecture is that if we imposed all the constraints coming from the quadrilinear relations, solution space would be reduced to a **coadjoint orbit** of the hyperbolic Kac-Moody algebra.

A **coadjoint orbit** of a Lie group is a very symmetric symplectic manifold on which the Lie group acts. Coadjoint orbits are the classical analogue of irreducible representations.

Is it possible to use the rich geometrical structure of general relativity to understand the coadjoint orbits of the hyperbolic Kac-Moody algebra?

Imaginary roots

Let V_n be the vector space generated by repeated commutators for which e_1 appears n times, e_0 appears n times, and e_{-1} appears once.

These vectors are called **imaginary roots of affine level 1**.

It follow from results of Feingold and Frenkel (1983) that

$$\dim V_n = p(n)$$

where $p(n)$ is the number of partitions of n.

Imaginary roots

We can realize the imaginary roots in terms of their action on solution space.

For example, here is a basis for V_3:

$$\begin{gathered}\delta_{\eta_{0101}} + \dots \\ \delta_{\eta_{0110}} + \delta_{\eta_{1010}} \dots \\ \delta_{\eta_{1001}} + \dots\end{gathered}$$

In this case, $\dim V_3 = p(3) = 3$.

There seems to be an intriguing link to the representation theory of S_3.

Imaginary roots

Frenkel's conjecture is an upper bound on imaginary root multiplicities.

Some imaginary root multiplicities do not saturate the bound.

Is it possible to use the geometrical structure of general relativity to understand these "gaps in the spectrum"?

Open Problems

- Study the quantum version of gravitational pulse wave scattering. Is there an exact S matrix?

- Extend the theory of the Geroch group to include a cosmological constant. (Twistor Chern-Simons theory might prove useful here.)

- In anti de Sitter space, use integrability to study the interaction of cylindrical black holes with cylindrical gravitational waves.

- Use the rich geometrical structure of general relativity to understand the geometry of hyperbolic Kac-Moody algebras.