Kavli IPMU 14/12/21

Affine Gaudin models, 4d Chern–Simons & non-ultralocality

2d Integrable field theories.

A field theory in 2d is classically integrable if its equations of motion take the Lax form:

$$\text{e.o.m.}(\{\phi_i\}) = 0 \iff \boxed{d\mathcal{L}(z) + \tfrac{1}{2}[\mathcal{L}(z), \mathcal{L}(z)] = 0}$$

Lax connection $\quad \mathcal{L}(z) = \mathscr{L}(\sigma,\tau,z)\, d\sigma + \mathscr{M}(\sigma,\tau,z)\, d\tau$

meromorphic dependence on z.

Examples:

$\mathbb{R}^2$ or $\mathbb{R}^2 \times S^1$

1) Sinh – Gordon. $\quad$ Field $\phi \in C^\infty(\Sigma, \mathbb{R})$.

- Action: $\quad S[\phi] = \int d\sigma\, d\tau \left(\frac{1}{2}(\partial_\tau \phi)^2 - \frac{1}{2}(\partial_\sigma \phi)^2 + \frac{m^2}{2}(e^{2\phi} + e^{-2\phi}) \right).$

- Equation of motion: $\quad \partial_\tau^2 \phi - \partial_\sigma^2 \phi - m^2(e^{2\phi} - e^{-2\phi}) = 0.$

- Lax connection:

$$\mathscr{L}(z) = \begin{pmatrix} \frac{1}{2}\partial_\tau \phi & \frac{m}{2}(z e^{\phi} + z^{-1} e^{-\phi}) \\ \frac{m}{2}(z e^{-\phi} + z^{-1} e^{\phi}) & -\frac{1}{2}\partial_\tau \phi \end{pmatrix},$$

$$\mathscr{M}(z) = \begin{pmatrix} \frac{1}{2}\partial_\sigma \phi & -\frac{m}{2}(z e^{\phi} - z^{-1} e^{-\phi}) \\ -\frac{m}{2}(z e^{-\phi} - z^{-1} e^{\phi}) & -\frac{1}{2}\partial_\sigma \phi \end{pmatrix}.$$

$$\hookrightarrow \quad d\mathcal{L}(z) + \frac{1}{2}[\mathcal{L}(z), \mathcal{L}(z)] \propto \begin{pmatrix} \text{e.o.m.} & 0 \\ 0 & \text{e.o.m.} \end{pmatrix} d\sigma \wedge d\tau .$$

2) <u>Principal chiral model.</u> Field $g \in C(G)$.

- Action: $$S[g] = \int d\sigma d\tau \, \langle g^{-1}\partial_\mu g, g^{-1}\partial^\mu g \rangle$$

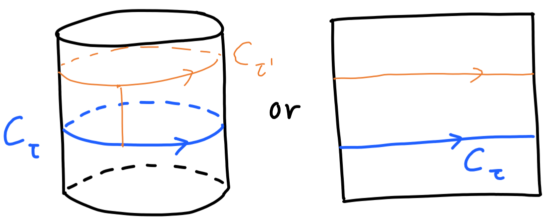
- Equation of motion: $$\partial_+ (\overbrace{g^{-1}\partial_- g}^{j_-}) - \partial_- (\overbrace{g^{-1}\partial_+ g}^{j_+}) = 0 .$$

- Lax connection: Let $j := g^{-1} dg \in \Omega^1(\Sigma, \mathfrak{g})$,

$$\mathcal{L}(z) = \frac{j - z * j}{1 - z^2} = \frac{j_+ d\sigma^+}{1 - z} + \frac{j_- d\sigma^-}{1 + z}$$

$$\hookrightarrow \quad d\mathcal{L}(z) + \frac{1}{2}[\mathcal{L}(z), \mathcal{L}(z)] = \frac{1}{1-z^2}\left(dj + \frac{1}{2}[j, j]\right) - \frac{z}{1-z^2} d * j .$$

Infinitely many integrals of motion:



$S^1 \times \mathbb{R}$ $\qquad$ $\mathbb{R}^2$

$$\partial_\tau \operatorname{tr}\Big(\underbrace{P \overleftarrow{\exp} \int_{C_\tau} \mathcal{L}(z)}_{T(z)} \Big) = 0$$

For a (finite-dimensional) Liouville integrable system, also require integrals of motion to be in involution:

$$F_i, \; i = 1, \dots, n \quad \text{s.t.} \quad \frac{dF_i}{dt} = \{H, F_i\} = 0 \quad \& \quad \{F_i, F_j\} = 0 \quad \forall i, j = 1, \dots, n.$$

Sufficient condition for involution of integrals of motion in field theory case:

$$\{\mathcal{L}_{\underline{1}}(z;\sigma), \mathcal{L}_{\underline{2}}(w,\sigma')\} = [r_{\underline{12}}(z,w), \mathcal{L}_{\underline{1}}(z;\sigma) + \mathcal{L}_{\underline{2}}(w,\sigma')]\, \underline{\delta(\sigma-\sigma')}$$

$\{T,T\} = [r_{12}, T_1 T_2]$

$$+ [s_{\underline{12}}(z,w), \mathcal{L}_{\underline{1}}(z;\sigma) - \mathcal{L}_{\underline{2}}(w,\sigma')]\, \underline{\delta(\sigma-\sigma')}$$

RTT = TTR

$$- 2\, s_{\underline{12}}(z,w)\, \underline{\delta'(\sigma-\sigma')}$$

skew-symmetric & symmetric parts of

$\varphi(z) = \frac{1}{z}$

$$R_{\underline{12}}(z,w) = \frac{I_a \otimes I^a}{w-z}\, \varphi(w)^{-1}$$

$\varphi(z) = 1$

[$\varphi(z)dz \neq 0$ is ultralocal case]

|| Difficult to quantise classical IFT when $\varphi(z)dz$ has zeroes.

↳ problem of non-ultralocality.

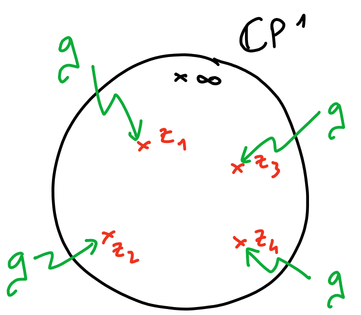
Q: What is the origin of the Lax connection and its non-ultralocal Poisson bracket?

Algebraic/Hamiltonian origin:

* Classical Gaudin models:

$\mathfrak{g}$ semisimple Lie algebra $/\mathbb{C}$, dual bases $\{I^a\}, \{I_a\}$.

Lax matrix



$$L(z) := \sum_{i=1}^{N} \frac{I_a \otimes I^{a(i)}}{z - z_i} \in \mathfrak{g} \otimes S(\mathfrak{g})^{\otimes N}$$

satisfies

$$\{L_{\underline{1}}(z), L_{\underline{2}}(w)\} = \Big[\underbrace{\frac{I_a \otimes I^a}{w-z}}_{r_{12}(z,w)}, L_{\underline{1}}(z) + L_{\underline{2}}(w)\Big]$$

Gaudin Hamiltonians:

$$H_i := \mathrm{res}_{z_i} \langle L(z), L(z) \rangle = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{I_a^{(i)} I^{a(j)}}{z_i - z_j}$$

Many finite dimensional integrable systems are representations of Gaudin models.

Example: Neumann model: $\mathfrak{g} = \mathfrak{sl}_2 = \langle E, H, F \rangle$,

$$\sum_{i=1}^{N} \frac{I_a \otimes I^{a(i)}}{z - z_i} + I_a \otimes I^{a(\infty)} \xrightarrow{\text{rep.}} \frac{1}{2} \begin{pmatrix} \sum_{i=1}^{N} \frac{x_i p_i}{z - z_i} & -\sum_{i=1}^{N} \frac{x_i^2}{z - z_i} \\ 1 + \sum_{i=1}^{N} \frac{p_i^2}{z - z_i} & -\sum_{i=1}^{N} \frac{x_i p_i}{z - z_i} \end{pmatrix}$$

$E^{(i)} \longrightarrow \frac{1}{2} p_i^2$, $H^{(i)} \longrightarrow x_i p_i$, $F^{(i)} \longrightarrow -\frac{1}{2} x_i^2$

$E^{(\infty)} \longrightarrow \frac{1}{2}$, $H^{(\infty)} \longrightarrow 0$, $F^{(\infty)} \longrightarrow 0$

$\{x_i, p_j\} = \delta_{ij}$

* Affine classical Gaudin models:

$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} k \oplus \mathbb{C} d$ untwisted affine KM algebra.

dual bases $\{I^{\tilde{a}}\} = \{I^a \otimes t^n, k, d\}, \{I_{\tilde{a}}\} = \{I_a \otimes t^{-n}, d, k\}$ $n \in \mathbb{Z}$

Idea: 2d IFTs with $\mathfrak{g}$-valued Lax connections as representations of Gaudin models associated with $\tilde{\mathfrak{g}}$:

[Feigin-Frenkel '07] [BV '17]

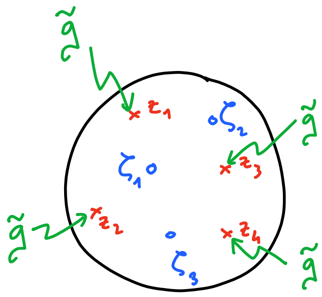
$$L(z) dz = \sum_{i=1}^{N} \frac{\sum_{\tilde{a}} I_{\tilde{a}} \otimes \boxed{I^{\tilde{a}(i)}}}{z - z_i} dz \xrightarrow{\text{rep.}} \omega \left(\partial_\sigma + \underline{\mathcal{L}(\sigma, z)} \right)$$

$\mathcal{L} = \mathcal{L} d\sigma + \mathcal{M} d\tau$

$$\sum I_{\tilde{a}} \otimes I^{\tilde{a}(i)} = k \otimes d^{(i)} + d \otimes k^{(i)} + \sum I_{a,-n} \otimes I_n^{a(i)}$$

$$\longrightarrow k_i\partial_\sigma + I_a \sum_{n\in\mathbb{Z}} e^{-in\sigma} I^{a(i)}_n$$

$\tilde{a}$ $\quad a \quad 0 \quad -i\partial_\sigma \quad n\in\mathbb{Z} \quad e^{-in\sigma} I_a$



$\mathbb{C}P^1 \setminus \{\zeta_j\}$

where $$\boxed{\omega = \sum_{i=1}^{N} \frac{k_i}{z-z_i}\, dz} = \frac{\prod_j (z-\zeta_j)}{\prod_i (z-z_i)}\, dz$$

$\varphi(z)$

$$\boxed{\mathcal{L}(\sigma, z) = \sum_j \frac{\mathcal{J}_j(\sigma)}{z-\zeta_j}}$$

Affine Gaudin Poisson algebra

$$\{L_1(z), L_2(w)\} = \Big[\underbrace{\frac{I_{\tilde{a}} \otimes I^{\tilde{a}}}{w-z}}, L_1(z) + L_2(w)\Big]$$

becomes non-ultralocal algebra $\displaystyle\sum_{n,m} \frac{I_a \otimes I^a\, e^{-in\sigma}\, e^{in\sigma'}}{w-z} = r_{12}(z,w)\,\delta(\sigma-\sigma')$

$$\{\mathcal{L}_1(z;\sigma), \mathcal{L}_2(w,\sigma')\} = [r_{12}(z,w), \mathcal{L}_1(z;\sigma) + \mathcal{L}_2(w,\sigma')]\,\delta(\sigma-\sigma')$$

$$+ [s_{12}(z,w), \mathcal{L}_1(z;\sigma) - \mathcal{L}_2(w,\sigma')]\,\delta(\sigma-\sigma')$$

$$- 2\, s_{12}(z,w)\,\delta'(\sigma-\sigma')$$

skew-symmetric & symmetric parts of

$$R_{12}(z,w) = \frac{I_a \otimes I^a}{w-z}\,\varphi(w)^{-1} \qquad \text{with } \varphi(z) = \sum_{i=1}^{N} \frac{k_i}{z-z_i}.$$

Theorem [Magro–Lacroix–BV '16]

For any $\mathfrak{g}$ of classical type with exponents E,

$\exists$ invariant homogeneous polynomials

Coxeter number of $\mathfrak{g}$

$$\tilde{P}_n : \mathfrak{g}^{\times(n+1)} \longrightarrow \mathbb{C}, \qquad n \in E + h\,\mathbb{Z}_{\geq 0} \qquad \tilde{\mathfrak{g}}$$

[Evans–Hassan–MacKay–Mountain '99] such that $\big(L(z;\sigma) := \varphi(z)\,\mathcal{L}(z;\sigma)\big)$

$$\left\{ \int_{S^1} \tilde{P}_m\big(L(z_0;\sigma)\big)\,d\sigma \,,\, \int_{S^1} \tilde{P}_n\big(L(w_0,\sigma')\big)\,d\sigma' \right\} = 0$$

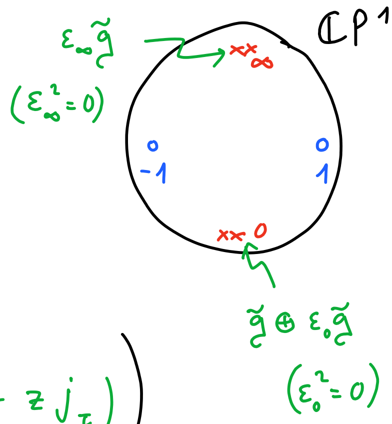
for all $m,n \in E + h\,\mathbb{Z}_{\geq 0}$ and z_0, w_0 with $\boxed{\varphi(z_0) = \varphi(w_0) = 0}$. $\square$

Example: Principal chiral model

$$\left(\frac{\overbrace{I_{\tilde a} \otimes I^{\tilde a (0)}_{[0]}}^{-j_\tau}}{z^1} + \frac{\overbrace{I_{\tilde a} \otimes I^{\tilde a (0)}_{[1]}}^{\partial_\sigma + j_\sigma}}{z^2} - \overbrace{I_{\tilde a} \otimes I^{\tilde a (\infty)}_{[1]}}^{\partial_\sigma} \right) dz$$

$j = g^{-1}\partial_\mu g$

$$\xrightarrow{\text{rep.}} \underbrace{\left(\frac{1}{z^2} - 1\right) dz}_{\omega_{PCM}} \left(\partial_\sigma + \underbrace{\frac{1}{1-z^2}\left(j_\sigma - z\, j_\tau\right)}_{\mathcal{L}_{PCM}(\sigma,\tau)} \right)$$

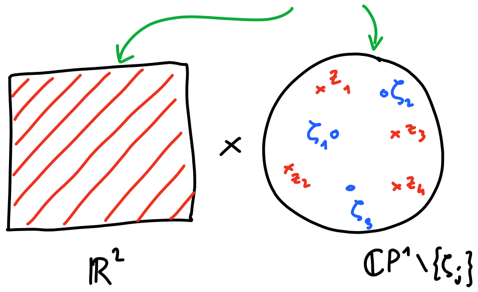


Geometric / Lagrangian origin:

Start from 4d Chern-Simons action:

$$S_\omega(A) = \frac{i}{4\pi} \int_{\Sigma \times C =: X} \omega \wedge CS(A)$$

[Costello '13]
[Costello-Witten-Yamazaki '17,'18]
[Costello-Yamazaki '19]



- $\omega = \varphi(z)\,dz = \sum_{i=1}^{N} \frac{k_i}{z - z_i}\, dz$
- $A = A_\sigma d\sigma + A_\tau d\tau + A_{\bar z} d\bar z \in \Omega^1(X, \mathfrak{g})$
- $CS(A) = \langle A, dA + \frac{1}{3}[A,A] \rangle$

Remark: ω is singular on surface defects:

$(\underline{z} = \{\text{poles of } \omega\})$ $\quad D = \bigsqcup \Sigma_x \,, \quad \Sigma_x := \Sigma \times \{x\}.$

$x\in\mathfrak{z}$

Nevertheless, $\omega\wedge CS(A)$ is locally integrable near D.

Behaviour of $S_\omega(A)$ under (finite) gauge transformations

$$A \xmapsto{g\in C^\infty(X,G)} {}^gA := -dg\,g^{-1} + gAg^{-1} \quad ?$$

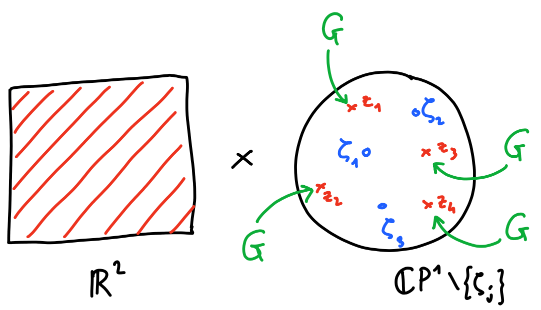
$$S_\omega({}^gA) = S_\omega(A) + \frac{i}{4\pi}\int_X \omega\wedge d\langle g^{-1}dg, A\rangle + \frac{i}{24\pi}\int_X \omega\wedge\langle g^{-1}dg,[g^{-1}dg, g^{-1}dg]\rangle$$

Define defect (Lie) group

$$G^{\mathfrak{z}} := \prod_{x\in\mathfrak{z}} G$$

and defect (Lie) algebra

$$\mathfrak{g}^{\mathfrak{z}} := \prod_{x\in\mathfrak{z}} \mathfrak{g}$$



with bilinear form $\langle\!\langle\cdot,\cdot\rangle\!\rangle_\omega : \mathfrak{g}^{\mathfrak{z}}\times\mathfrak{g}^{\mathfrak{z}} \longrightarrow \mathbb{C}$:

$$\langle\!\langle X, Y\rangle\!\rangle_\omega = \sum_{x\in\mathfrak{z}} k_x \langle X_x, Y_x\rangle \qquad X=(X_x), Y=(Y_x)\in G^{\mathfrak{z}}.$$

Consider embedding

$$\iota : D = \bigsqcup_{x\in\mathfrak{z}} \Sigma_x \hookrightarrow X.$$

<u>Note:</u> Pull back of $g\in C^\infty(X,G)$ & $A\in\Omega^1(X,\mathfrak{g})$ are:

- $\iota^* g \in C^\infty(D, G) = C^\infty\big(\bigsqcup_{x \in \underline{z}} \Sigma_x, G\big)$

$$\cong \prod_{x \in \underline{z}} C^\infty(\Sigma_x, G) \cong C^\infty(\Sigma, G^{\underline{z}})$$

- $\iota^* A \in \Omega^1(D, \mathfrak{g}) \cong \Omega^1(\Sigma, \mathfrak{g}^{\underline{z}})$.

$$S_\omega({}^g A) = S_\omega(A) - \frac{1}{2} \int_\Sigma \langle\!\langle (\iota^* g)^{-1} d_\Sigma (\iota^* g), \iota^* A \rangle\!\rangle_\omega$$

$$+ \frac{1}{12} \int_{\Sigma \times [0,1]} \langle\!\langle \hat{g}^{-1} d\hat{g}, [\hat{g}^{-1} d\hat{g}, \hat{g}^{-1} d\hat{g}] \rangle\!\rangle_\omega$$

[Localised on or "near" defect D.]

$\hat{g} \in C^\infty(\Sigma \times [0,1], G^{\underline{z}})$

s.t. $\hat{g}|_0 = \iota^* g$, $\hat{g}|_1 = e$.

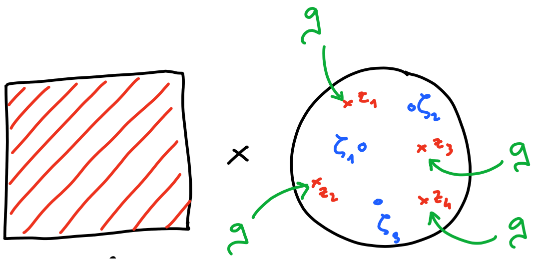
Let $\mathfrak{k} \subset \mathfrak{g}^{\underline{z}}$ be an isotropic Lie subalgebra.

$$\langle\!\langle X, Y \rangle\!\rangle_\omega = 0 \qquad \forall X, Y \in \mathfrak{k}.$$

Let $K \subset G^{\underline{z}}$ corresponding connected Lie subgroup.

Consider bulk fields $A \in \Omega^1(X, \mathfrak{g})$ and gauge transformations $g \in C^\infty(X, G)$ satisfying

- $\iota^* A \in \Omega^1(\Sigma, \mathfrak{k}) \subset \Omega^1(\Sigma, \mathfrak{g}^{\underline{z}})$
- $\iota^* g \in C^\infty(\Sigma, K) \subset C^\infty(\Sigma, G^{\underline{z}})$



$\iota^* A = (A|_x)_{x \in \underline{z}} \in \Omega^1(\Sigma, \mathfrak{k})$

$A|_x \in \Omega^1(\Sigma, \mathfrak{g})$

$\mathbb{R}^2$ $\mathbb{C}P^1 \setminus \{\zeta_i\}$ non-local boundary condition on $\mathbb{C}P^1$.

<u>Theorem</u> [Benini-Schenkel-BV '20]

$S_\omega(A)$ is gauge invariant.

From 4d CS to 2d IFTs

Want to turn <u>4d CS gauge field</u>

$$A = A_\sigma(\sigma,\tau,z)d\sigma + A_\tau(\sigma,\tau,z)d\tau + A_{\bar z}(\sigma,\tau,z)d\bar z$$

into the <u>2d IFT Lax connection</u> $\times$

$$\mathcal{L}(z) = \mathcal{L}(\sigma,\tau,z)d\sigma + \mathcal{M}(\sigma,\tau,z)d\tau$$

meromorphic dependence

$\rightarrow$ <u>Move to Hamiltonian formalism:</u> [BV '19]

$$A = A_\sigma(\sigma,z)d\sigma + A_\tau(\sigma,z)d\tau + A_{\bar z}(\sigma,z)d\bar z$$

$$\Pi = \Pi_\sigma(\sigma,z)d\sigma + \Pi_\tau(\sigma,z)d\tau + \Pi_{\bar z}(\sigma,z)d\bar z$$

$\{\Pi_i, A_j\} = \delta_{ij}\ \delta(\sigma-\sigma')\ \delta(z-z')$

- Primary constraints:

$$\Pi_\tau \approx 0, \quad \underbrace{A_{\bar z} - \frac{4\pi}{i\varphi}\Pi_\sigma \approx 0, \quad \Pi_{\bar z} + \frac{4\pi}{i\varphi}A_\sigma \approx 0}_{\text{second class} \dashrightarrow \text{Dirac bracket}}.$$

- Secondary constraints:

$$\gamma := \{H, \Pi_\tau\}^* \approx 0$$

first class - generates infinitesimal gauge transformations on A.

$\varepsilon \in \mathfrak{g}$ $\quad \frac{1}{2\pi}\{\langle\langle \varepsilon, \gamma\rangle\rangle, A_i\}^* = [\varepsilon, A_i] - \partial_i \varepsilon = \delta_\varepsilon A_i \quad i = \sigma, \bar{z}.$

$$\boxed{H = \langle\langle A_\tau, \gamma\rangle\rangle}$$

$$\langle\langle X, Y\rangle\rangle := \int_{S^1\times\mathbb{C}P^1} d\sigma\wedge dz\wedge d\bar{z}\, \langle X, Y\rangle$$

- No tertiary constraints: $\{H, \gamma\}^* \approx 0$.

- Gauge fix $\gamma \simeq 0$ using condition $\underline{A_{\bar{z}} \approx 0}$:

Dirac bracket

$$\{A_{\sigma\underline{1}}(\sigma, z), A_{\sigma\underline{2}}(\sigma', w)\}^{\star} = [r_{\underline{12}}(z,w), A_{\sigma\underline{1}}(\sigma,z) + A_{\sigma\underline{2}}(\sigma,w)]\,\delta(\sigma-\sigma')$$
$$+ [s_{\underline{12}}(z,w), A_{\sigma\underline{1}}(\sigma,z) - A_{\sigma\underline{2}}(\sigma,w)]\,\delta(\sigma-\sigma')$$
$$- 2\, s_{\underline{12}}(z,w)\, \delta'(\sigma-\sigma')$$

skew-symmetric & symmetric parts of

$$R_{\underline{12}}(z,w) = \frac{C_{\underline{12}}}{w-z}\varphi(w)^{-1} \qquad \omega = \varphi(z)\, dz$$

Moreover, $\gamma = 0 \Rightarrow A_\sigma(\sigma, z)$ is meromorphic in z with poles at the zeroes of $\varphi(z)$.

Morally,

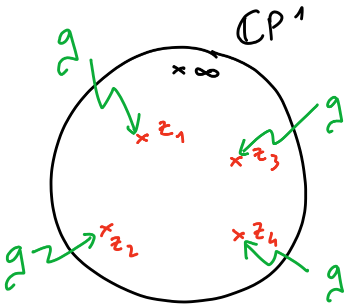
$$\left[\begin{array}{c}\text{Gauged fixed version}\\ \text{of 4d CS theory for } \omega = \varphi(z)dz\\ \text{in gauge } A_{\bar{z}} \approx 0\end{array}\right] \Longleftrightarrow \left[\begin{array}{c}\text{Affine Gaudin model}\\ \text{with twist function } \varphi(z)\end{array}\right]$$

* Quantum Gaudin model:

- $\mathfrak{g}$ – finite-dimensional semisimple Lie algebra $/\mathbb{C}$.
- $\{I^a\}, \{I_a\}$ – dual basis w.r.t. $\langle\cdot,\cdot\rangle : \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathbb{C}$.

• $\{z_i\}_{i=1}^N \subset \mathbb{C}$.

Lax matrix



$$L(z) := \sum_{i=1}^N \frac{I_a \otimes I^{a(z_i)}}{z - z_i} \in \mathfrak{g} \otimes \underbrace{U(\mathfrak{g})^{\otimes N}}$$

satisfies

$$\big[L_{\underline{1}}(z), L_{\underline{2}}(w)\big] = \left[\frac{I_a \otimes I^a}{w - z}, L_{\underline{1}}(z) + L_{\underline{2}}(w)\right].$$

Quadratic Gaudin Hamiltonians:

$$H_i := \operatorname{res}_{z_i} P_1(L(z)) = \sum_{j \neq i} \frac{I_a^{(z_i)} I^{a(z_j)}}{z_i - z_j}$$

Theorem [Feigin–Frenkel–Reshetikhin '94][Rybnikov '06]

$\exists$ unique large commutative subalgebra

$$\{H_i\}_{i=1}^N \subset \mathcal{Z}_{(z_i)}(\mathfrak{g}) \subset U(\mathfrak{g})^{\otimes N}$$

Gaudin / Bethe algebra

quantising Poisson commutative subalgebra $\mathcal{C}_{(z_i)}(\mathfrak{g}) \subset S(\mathfrak{g})^{\otimes N}$. $\square$

$\mathfrak{g}$-modules

Q: What is joint spectrum of $\mathcal{Z}_{(z_i)}(\mathfrak{g})$ on $\mathcal{H} = \bigotimes_{i=1}^N M_i$?

Theorem [E. Frenkel '04]

regular singularities

$$\mathcal{Z}_{(z_i)}(\mathfrak{g}) \cong \operatorname{Fun}\Big(\operatorname{Op}^{RS}_{{}^L\mathfrak{g}}(\mathbb{CP}^1)_{(z_i),\infty}\Big). \qquad \square$$

Here ${}^L\mathfrak{g}$ is Langlands dual Lie algebra, with

- Cartan decomposition $\quad {}^L\mathfrak{g} = {}^L\mathfrak{n}_- \oplus \overbrace{{}^L\mathfrak{h} \oplus {}^L\mathfrak{n}_+}^{{}^L\mathfrak{b}_+}$
- Chevalley generators $\quad \check{e}_i,\ \alpha_i,\ \check{f}_i \qquad i = 1, \dots, \ell \qquad ({}^L\mathfrak{h} \equiv \mathfrak{h}^*)$

Space of ${}^L\mathfrak{g}$-opers:

$$\mathrm{Op}_{{}^L\mathfrak{g}}(\mathbb{CP}^1) := \left(\partial_z + \sum_{i=1}^{\ell} \check{f}_i + {}^L\mathfrak{b}_+(\mathcal{M})\right) \Big/ {}^L N_+(\mathcal{M})$$

rational functions on $\mathbb{CP}^1$

Example: $\quad \mathfrak{g} = \mathfrak{sl}_2$.

$$\mathrm{Op}_{\mathfrak{sl}_2}(\mathbb{CP}^1) = \left\{ \partial_z + \begin{pmatrix} a(z) & b(z) \\ 1 & -a(z) \end{pmatrix} \right\} \Big/ \left\{ \begin{pmatrix} 1 & A(z) \\ 0 & 1 \end{pmatrix} \text{ gauge transf.} \right\}$$

$$\cong \partial_z + \begin{pmatrix} 0 & v(z) \\ 1 & 0 \end{pmatrix}$$

Drinfel'd-Sokolov gauge

$v(z) = b(z) + a(z)^2 + \partial_z a(z)$

Eigenvalue of quadratic Hamiltonian $H_i = \mathrm{res}_{z_i} P_1(L(z))$ is given by $\mathrm{res}_{z_i} v(z)$. $\square$

In general, an oper $[\nabla] \in \mathrm{Op}^{RS}_{{}^L\mathfrak{g}}(\mathbb{CP}^1)_{(z_i), \infty}$ has a unique canonical form (Drinfel'd-Sokolov gauge)

$$\nabla' = \partial_z + \sum_{i=1}^{\ell} \check{f}_i + \sum_{n \in E} v_n(z)\, \bar{p}_n$$

some element of degree n in principal gradation of $\mathfrak{g}$.

$\hookrightarrow$ $v_n(z)$ gives eigenvalue of quantisation of $P_n(L(z)) \in S(\mathfrak{g})^{\otimes N}$.

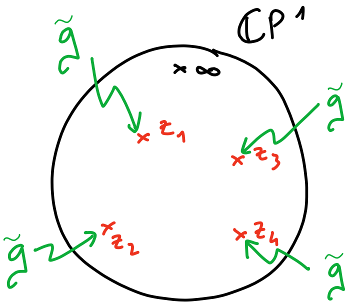
ad-invariant polynomial $P_n : \mathfrak{g}^{\times(n+1)} \to \mathbb{C}$

* Affine quantum Gaudin model(s):

- $\tilde{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}k \oplus \mathbb{C}d$ — untwisted affine KM algebra.
- $\{I^{\tilde{a}}\} := \{I^a \otimes t^n, k, d\}$, $\{I_{\tilde{a}}\} := \{I_a \otimes t^{-n}, d, k\}$

  — dual basis w.r.t. $(\cdot|\cdot) : \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \longrightarrow \mathbb{C}$.
- $\{z_i\}_{i=1}^N \subset \mathbb{C}$.

Lax matrix



$$L(z) := \sum_{i=1}^N \frac{I_{\tilde{a}} \otimes I^{\tilde{a}(z_i)}}{z - z_i} \in \tilde{\mathfrak{g}} \tilde{\otimes} \tilde{U}(\tilde{\mathfrak{g}}^{\oplus N})$$

satisfies

$$\{L_{\underline{1}}(z), L_{\underline{2}}(w)\} = \left[\frac{I_{\tilde{a}} \otimes I^{\tilde{a}}}{w - z}, L_{\underline{1}}(z) + L_{\underline{2}}(w) \right].$$

Quadratic affine Gaudin Hamiltonians:

$$H_i := \sum_{\substack{j=1 \\ \underline{\underline{j \neq i}}}}^N \frac{I_{\tilde{a}}^{(z_i)} I^{\tilde{a}(z_j)}}{z_i - z_j}$$

$\underline{Q}$: What are higher "local" and "non-local" Hamiltonians?

i.e. What is affine Gaudin algebra?

Can define notion of affine ${}^L\tilde{\mathfrak{g}}$-opers:

Let ${}^L\tilde{\mathfrak{g}}$ be the Langlands dual of $\tilde{\mathfrak{g}}$, with

- Cartan decomposition ${}^L\tilde{\mathfrak{g}} = {}^L\tilde{\mathfrak{n}}_- \oplus \underbrace{{}^L\tilde{\mathfrak{h}} \oplus {}^L\tilde{\mathfrak{n}}_+}_{{}^L\tilde{\mathfrak{b}}_+}$
- Chevalley generators $\check{e}_i, \alpha_i, \check{f}_i$ $\quad i = 0, 1, \dots, \ell$ $\quad ({}^L\tilde{\mathfrak{h}} = \tilde{\mathfrak{h}}^*)$

Key difference with finite case: $\rho := \sum_{i=0}^{\ell} \Lambda_i \notin [{}^L\tilde{\mathfrak{g}}, {}^L\tilde{\mathfrak{g}}]$

Space of ${}^L\tilde{\mathfrak{g}}$-opers:

$$\mathrm{Op}_{{}^L\tilde{\mathfrak{g}}}(\mathbb{C}P^1)^{\varphi} := \Big(\partial_z + \sum_{i=0}^{\ell} \check{f}_i - \frac{\varphi(z)}{h^\vee}\rho + \underbrace{[{}^L\tilde{\mathfrak{b}}_{+}, {}^L\tilde{\mathfrak{b}}_{+}](\mathcal{M})}_{{}^L\tilde{\mathfrak{b}}_{+}(\mathcal{M})}\Big) \Big/ {}^L\tilde{N}_{+}(\mathcal{M})$$

rational functions on $\mathbb{C}P^1$

Let $\tilde{E}$ be (multi)set of exponents of $\tilde{\mathfrak{g}}$.

Theorem [Lacroix–BV–Young '18]

Any $[\nabla] \in \mathrm{Op}_{{}^L\tilde{\mathfrak{g}}}(\mathbb{C}P^1)^{\varphi}$ has a quasi-canonical representative

$$\nabla' = \partial_z + \sum_{i=0}^{\ell} \check{f}_i - \frac{\varphi(z)}{h^\vee}\rho + \sum_{n \in \tilde{E}_{>0}} v_n(z)\, \tilde{p}_n$$

span inf-dim. abelian Lie algebra.

which is unique only up to

$$P(z)^{-n/h^\vee} v_n(z) \longmapsto P(z)^{-n/h^\vee} v_n(z) - \partial_z\big(P(z)^{-n/h^\vee} a_n(z)\big)$$

for $n \in \tilde{E}_{>1}$ where $\boxed{\varphi(z) = \partial_z \log P(z)}$. $\square$

e.g. if $\varphi(z) = \sum_{i=1}^{N} \frac{k_i}{z - z_i}$ then $P(z) = \prod_{i=1}^{N} (z - z_i)^{k_i}$.

Corollary [Lacroix–BV–Young '18]

For any contour γ with uni-valued branch of $P(z)^{1/h^\vee}$,

$$[\nabla] \longmapsto \int_\gamma P(z)^{-n/h^\vee} v_n(z)\, dz$$

is a well defined function on $\mathrm{Op}_{{}^L\tilde{\mathfrak{g}}}(\mathbb{C}P^1)^{\varphi}$. $\square$

Conjecture [Lacroix–BV–Young '18]

$\exists$ rational $\tilde{S}_n(z) \in \tilde{U}_{\underline{k}}(\hat{g})^{\otimes N}$ for every $n \in \underline{\tilde{E}_{>0}}$ such that

(i) $$\left[\int_\gamma \underbrace{P(z)^{-m/h^\vee}} \tilde{S}_m(z)\, dz \,,\, \int_\eta P(z)^{-n/h^\vee} \tilde{S}_n(z)\, dz\right] = 0$$

contours along which $P(z)^{1/h^\vee}$ is single valued.

(ii) Eigenvalue of $\int_\gamma P(z)^{-n/h^\vee} \tilde{S}_n(z) dz$ is $\int_\gamma P(z)^{-n/h^\vee} v_n(z)\, dz$

(iii) In semiclassical limit

$$\int_\gamma P(z)^{-n/h^\vee \hbar} \tilde{S}_n(z)\, dz \xrightarrow[\hbar \to 0]{} \int_{S^1} \tilde{P}_n(L(z_0;\sigma))\, d\sigma$$

$\varphi(z_0) = \partial_z \log P(z)\big|_{z=z_0}$

for all $n \in \tilde{E}_{>0} = E + h\mathbb{Z}_{\geq 0}$. $\square$

For $g = \mathfrak{sl}_M$, conjecture holds for $n = 1, 2$ with:

$$\tilde{S}_1(z) := \int_{S^1} :\tilde{P}_1(L(z,\sigma)): \, d\sigma \qquad (\tilde{P}_1 \propto \langle\cdot,\cdot\rangle)$$

$$\tilde{S}_2(z) := \int_{S^1} :\tilde{P}_2(L(z,\sigma)): \, d\sigma \qquad \left(\tilde{P}_2(X,Y,Z) = \mathrm{tr}(XYZ + YXZ)\right)$$

Conclusions & outlook:

- 4d CS & Affine Gaudin models both offer different perspectives on old problem of non-ultralocality.

- Establish precise correspondence:

$$\left\{ \begin{array}{c} \text{b.c. on surface defects} \\ \text{in 4d CS theory} \end{array} \right\} \overset{?}{\longleftrightarrow} \left\{ \begin{array}{c} \text{representations of } \tilde{\mathfrak{g}}^{\oplus N} \\ \text{in affine Gaudin model} \end{array} \right\}$$

- "Naive" affine extension of Gaudin/oper correspondence provides ODE/IM correspondence for Local IM:

ODE ($^L\tilde{\mathfrak{g}}$-oper) ⟶ Eigenvalues of Local IM (of $\tilde{\mathfrak{g}}$-Gaudin model)

gauge transformation ↓

quasi-canonical representative

- What about non-local IM?

Obtained from the quantized monodromy matrix or Kondo line operator.

Renormalized $P\overleftarrow{\exp}$ of affine Gaudin Lax matrix.

[Bazhanov - Lukyanov - Zamolodchikov, ...]

[Gaiotto - Lee - Wu '20]

[Gaiotto - Lee - BV - Wu '20]

- Quantization

  * Relationship between quantum 4d CS & 2d QIFT?
  * Quantum 4d CS vs. affine quantum Gaudin model?
  * ...