Rademacher expansion of a Siegel modular form for $\mathcal{N} = 4$ counting

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based on work with Gabriel Cardoso and Suresh Nampuri, [2112.10023]

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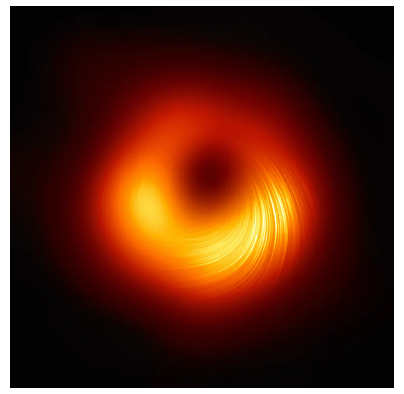
Introduction

Understanding the microscopic origin of Black Hole entropy remains a central question in Quantum Gravity.

String theory provides explicit counting formulas for a class of BPS configurations that appear as supersymmetric black holes in 4d supergravity.

$$S_{stat}(Q) = \ln d(Q) \qquad \leftrightarrow \qquad S_{BH}(Q) = \frac{A_{BH}(Q)}{4} + \ldots$$



Microscopic Macroscopic

Today: Study microscopic degeneracies of 1/4-BPS dyons in $\mathcal{N} = 4$ string theory.

Microscopic

The microscopic degeneracies of $1/4-$BPS dyons in $\mathcal{N} = 4$ supersymmetric String Theory are encoded in an automorphic form.

$$d(Q) = e^{S_{stat}(Q)} = \int d\sigma\, e^{-2\pi i \sigma Q} f(\sigma)$$

Compare

Macroscopic

Sen's Quantum Entropy Function gives a prescription for computing the degeneracy of extremal black hole states in terms of a path integral over string fields living on the near horizon geometry of the black hole

$$e^{S_{BH}(Q)} = \int D(\text{fields})\, e^{\hat{I}_E} \Bigg|_{AdS_2}^{finite}$$

Automorphic forms

Modular forms: uncannily effective in BPS black hole counting

$$f(\sigma) = \sum_{n \geq n_0} c(n) q^n \qquad q = e^{2\pi i \sigma}$$

Hardy-Ramanujan-Rademacher expansion: Modular symmetry so powerful that microscopic degeneracies have exact expression from finite set of data: the polar part.

$$c(n > 0) = \sum_{\gamma \geq 1} \sum_{\tilde{n} < 0} c(\tilde{n})\, Kl(n, \tilde{n}, \gamma) I(n, \tilde{n}, \gamma)$$

Phases **Bessel functions**

$c(\tilde{n} < 0)$ is the minimum amount of information to reconstruct $f(\sigma)$

Examples

Modular objects are generating functions for BPS black hole degeneracies

$$Z^{\mathcal{N}=4}_{1/2\,BPS} = \frac{1}{\eta^{24}(\sigma)} = \sum_{n\geq -1} d(n) q^n$$

[Dabholkar '05]

$$Z^{\mathcal{N}=8}_{1/8\,BPS} = \frac{\theta_1(\sigma,v)^2}{\eta^6(\sigma)} = \sum_{4n-\ell^2\geq -1} C(\Delta = 4n-\ell^2) q^n y^\ell$$

[Maldacena, Moore, Strominger '99]

Only one polar term $d(-1), C(-1)$

Hardy-Ramanujan-Rademacher expansion

$$d(n) = \sum_{\gamma=1}^{+\infty} \frac{2\pi}{\gamma\, n^{13/2}} Kl(n,-1,\gamma) I_{13}\left(\frac{4\pi\sqrt{n}}{\gamma}\right) \quad \text{for } n > 0$$

$$C(\Delta) = 2\pi \left(\frac{\pi}{2}\right)^{7/2} i^{5/2} \sum_{\gamma=1}^{\infty} \gamma^{-9/2} Kl_\gamma(\Delta) I_{7/2}\left(\frac{\pi\sqrt{\Delta}}{\gamma}\right) \quad \text{for } \Delta > 0$$

1/4-BPS generating function

The generating function for $1/4$-BPS dyonic degeneracies in $\mathcal{N} = 4, D = 4$ heterotic string theory is a modular form of the genus-2 modular group $Sp(2,\mathbb{Z})$. Φ_{10} Igusa cusp form.

$$\frac{1}{\Phi_{10}(\rho,\sigma,v)} = \sum_{\substack{m,n \geq -1 \\ m,n,\ell \in \mathbb{Z}}} (-1)^{\ell+1} d(m,n,\ell) e^{2\pi i(m\rho + n\sigma + \ell v)}$$

[Dijgkraaf, Verlinde, Verlinde '96]

What are the polar terms? $\Delta = 4mn - \ell^2 < 0$, bound states of $1/2$-BPS states

$SL(2,\mathbb{Z}) \subset Sp(2,\mathbb{Z})$, can we use the $Sp(2,\mathbb{Z})$ symmetries to specify the minimal amount of information needed to reconstruct $d(m,n,\ell)$? Can this give us insight into the saddle points for the Quantum Entropy Function path integral?

Results

Using symplectic symmetries, the sum over residues of $1/\Phi_{10}$ yields a fine-grained Rademacher expansion, with the the polar terms specified by the coefficients of $\eta^{-24}(\sigma)$.

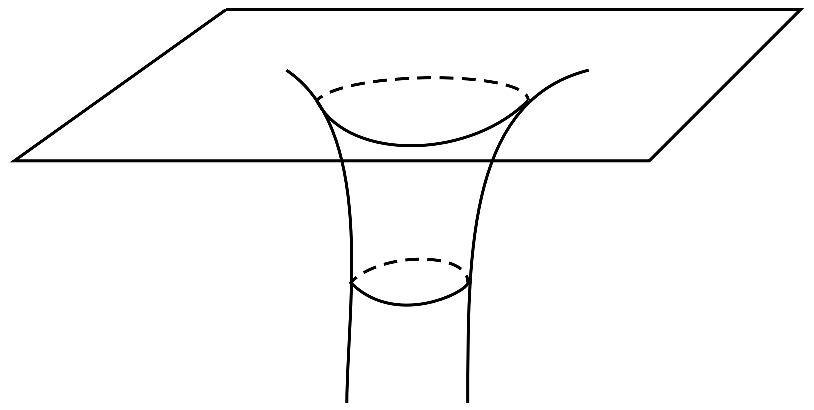
Answer parametrised by two sets of $SL(2,\mathbb{Z})$ matrices inside $Sp(2,\mathbb{Z})$. One set of matrices parametrizes the Rademacher expansion and the other the polar coefficients.

Structure of polar coefficients reproduces exactly the continued fraction structure found in previous work.

$Sp(2,\mathbb{Z})$ symmetry powerful enough to systematically encode all this data

Outline

Dyonic degeneracies

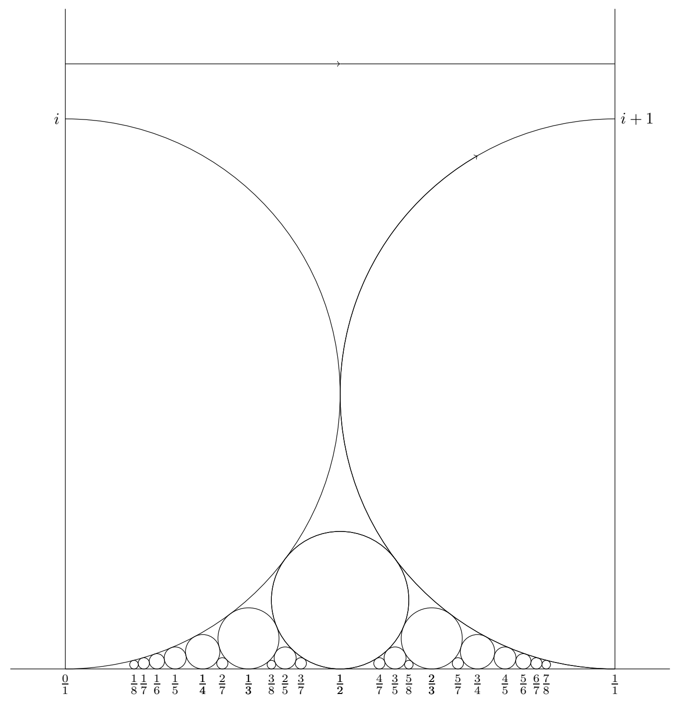


Siegel modular forms

Mock Jacobi forms

$$\frac{1}{\Phi_{10}} \qquad \psi_m^F$$

Rademacher expansion



Continued fractions

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

Dyonic degeneracies

Heterotic string theory on T^6, $\mathcal{N} = 4$ supersymmetry, $S-$duality group is $SL(2,\mathbb{Z})$

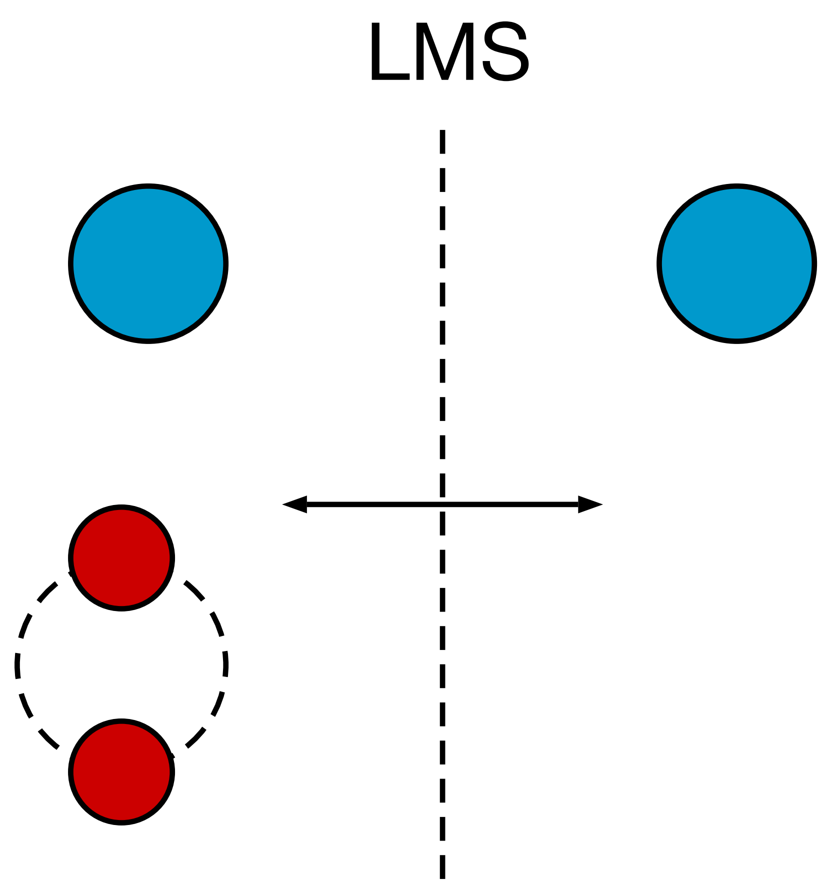
$1/4$-BPS states carry electric $\overrightarrow{Q}$ and magnetic $\overrightarrow{P}$ charges: **Dyons**

Degeneracies characterized by $m = P^2/2 \in \mathbb{Z}$, $n = Q^2/2 \in \mathbb{Z}$, $\ell = P \cdot Q \in \mathbb{Z}$

$$d(\overrightarrow{P}, \overrightarrow{Q}) = d(m, n, \ell)$$

$S-$duality invariant: $\Delta = Q^2P^2 - (Q \cdot P)^2 = 4mn - \ell^2$

Two types of state contribute to index, single-centered: immortal, and two-centered: can (dis)appear across Lines of Marginal Stability.



For immortal dyons, $\Delta > 0$, Area $\sim \sqrt{\Delta}$

Modular forms

A modular form $f(\tau)$ of weight w on $SL(2,\mathbb{Z})$ is a holomorphic function on $\mathbb{H}$ that transforms as

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^w f(\tau), \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

$$f(\tau) = \sum_{n\geq n_0}^{+\infty} a(n) q^n, \quad q = e^{2\pi i\tau}$$

growth of $f(\tau)$ as $\tau \to i\infty$ given by terms $a(n)q^n$ with $n \leq 0$.

Jacobi forms

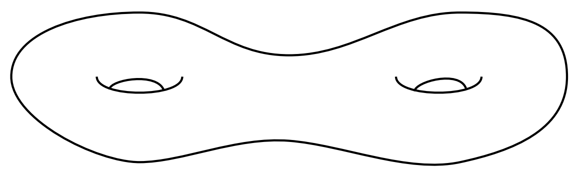
Jacobi forms $\psi_m(\sigma, v)$ of weight ω and index m,

$$\psi_m\left(\frac{a\sigma+b}{c\sigma+d}, \frac{v}{c\sigma+d}\right) = (c\sigma+d)^{\omega}\, e^{\frac{2\pi i m c v^2}{c\sigma+d}}\, \psi_m(\sigma, v)\,, \qquad G = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

$$\psi_m(\sigma, v+\lambda\sigma+\mu) = e^{-2\pi i m(\lambda^2\sigma+2\lambda v)}\, \psi_m(\sigma, v), \qquad \lambda, \mu \in \mathbb{Z}$$

Theta-expansion $\psi_m = \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} h_\ell(\sigma)\vartheta_{m,\ell}(\sigma, v)$, vector valued modular forms $h_\ell(\sigma)$ of weight $\omega - 1/2$

$$h_\ell\left(\frac{a\sigma+b}{c\sigma+d}\right) = (c\sigma+d)^{\omega-1/2} \sum_{j\in\mathbb{Z}/2m\mathbb{Z}} \rho_{\ell j}(G) h_j(\sigma)$$



Siegel modular forms

$\Phi(\Omega)$ a Siegel modular form of degree 2 and weight ω if

$$\Phi((A\Omega + B)(C\Omega + D)^{-1}) = \det(C\Omega + D)^{\omega}\Phi(\Omega)$$

$$\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix} \qquad \det \mathrm{Im}\Omega > 0 \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2, \mathbb{Z})$$

Fourier-Jacobi expansion into Jacobi forms $\psi_m(\sigma, v)$ of weight ω and index m,

$$\Phi(\Omega) = \sum_{m \geq m_0} \psi_m(\sigma, v) p^m \qquad p = e^{2\pi i \rho}$$

Siegel modular form

$$\frac{1}{\Phi_{10}(\rho,\sigma,v)} = \sum_{\substack{m,n\geq -1\\ m,n,\ell\in\mathbb{Z}}} (-1)^{\ell+1} d(m,n,\ell) e^{2\pi i(m\rho+n\sigma+\ell v)}$$

$$\Phi_{10}^{-1}((A\Omega+B)(C\Omega+D)^{-1}) = \det(C\Omega+D)^{-10}\Phi_{10}^{-1}(\Omega)$$

$\Phi_{10}^{-1}(\Omega)$ is meromorphic. Coefficients depend on choice of contour,

[Cheng,Verlinde '07]

[Sen '07]

$$d(m,n,\ell)_{imm} = \int_{\mathcal{C}_{imm}} d\sigma\, dv\, d\rho\, (-1)^{\ell+1} \frac{e^{-2\pi i(m\rho+n\sigma+\ell v)}}{\Phi_{10}(\rho,\sigma,v)}$$

$$\mathcal{C}_{imm}: \rho_2 = 2nK,\ \sigma_2 = 2mK,\ v_2 = -\ell K,\ \ K \gg 1$$

Poles

Poles labelled by 5 integers (n_2, n_1, j, m_2, m_1)

$$n_2(\rho\sigma - v^2) + jv + n_1\sigma - m_1\rho + m_2 = 0$$

Satisfying

$$m_1 n_1 + m_2 n_2 = \frac{1}{4}\left(1 - j^2\right)$$

Two types, $n_2 = 0$ linear poles and $n_2 \neq 0$ quadratic poles. Associated to two-centered and single-centered states. Linear poles parametrized by $SL(2,\mathbb{Z})$ matrices. Using $Sp(2,\mathbb{Z})$, can map any pole to 'simplest' pole given by $v = 0$, where

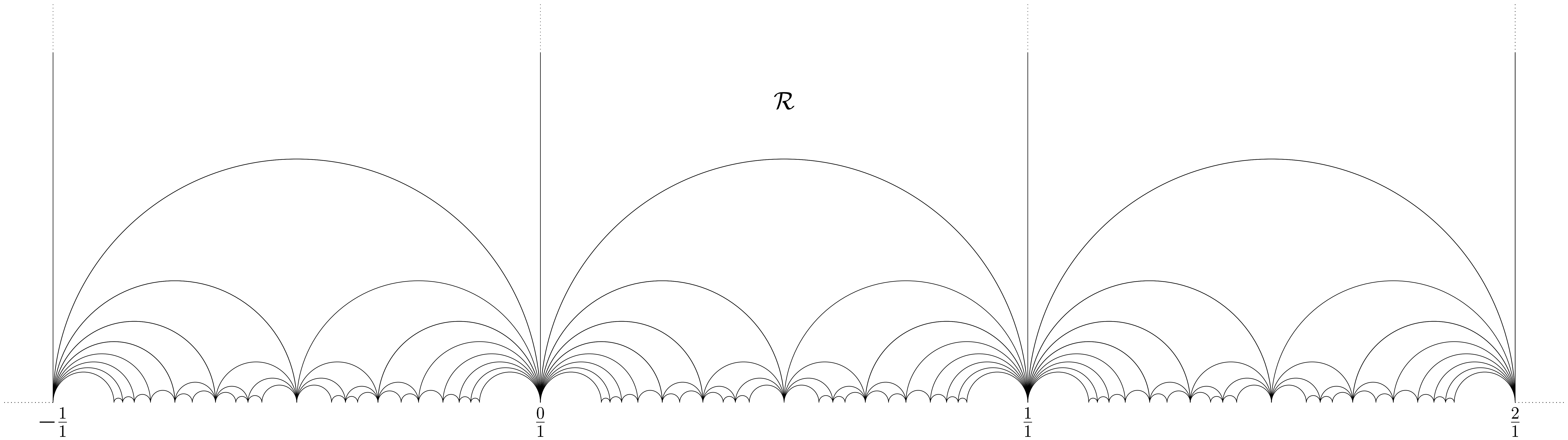
$$\frac{1}{\Phi_{10}(\rho,\sigma,v)} = -\frac{1}{4\pi^2}\frac{1}{v^2}\frac{1}{\eta^{24}(\rho)}\frac{1}{\eta^{24}(\sigma)} + \mathcal{O}(v^0)$$

Poles

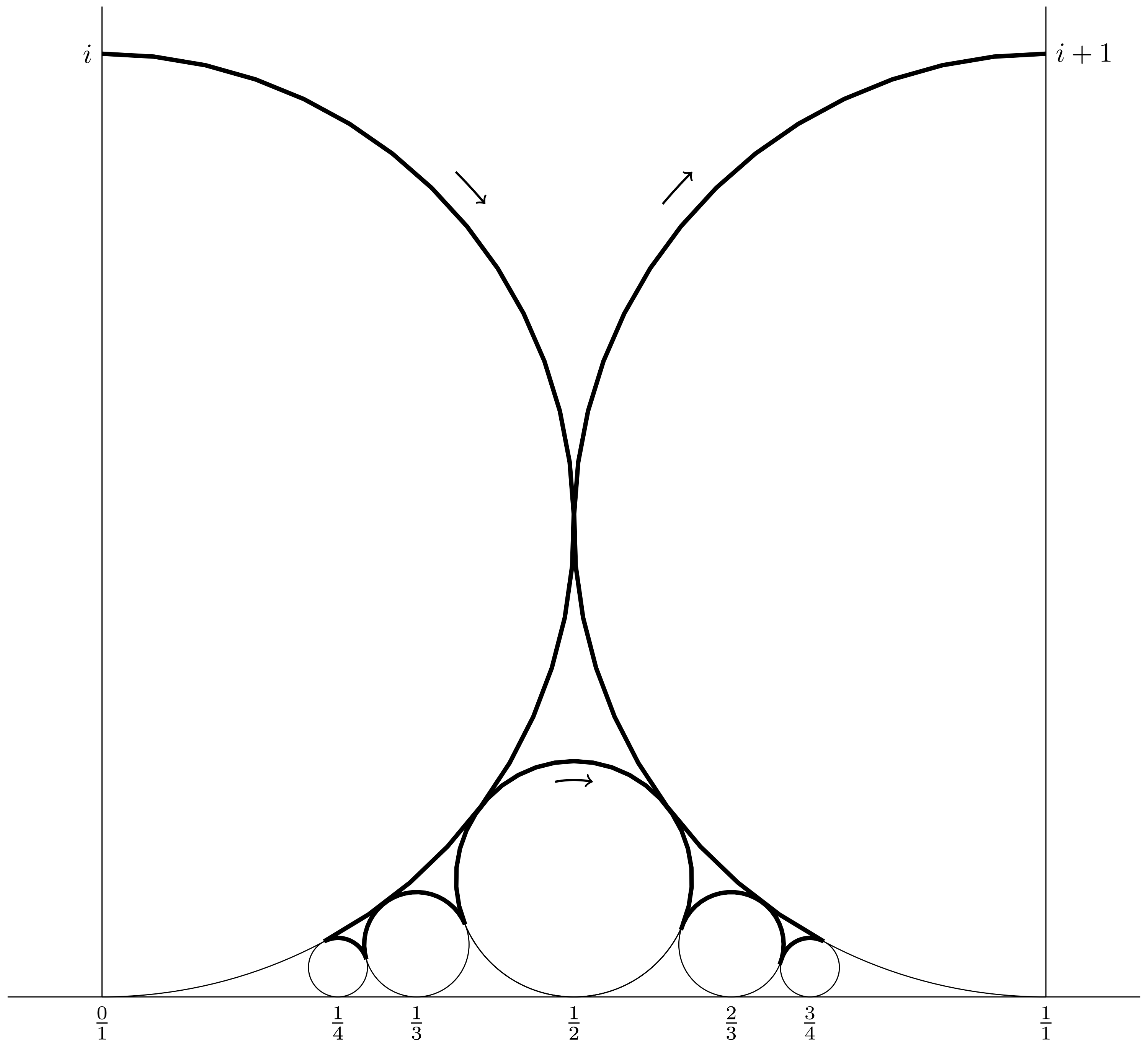
Linear poles parametrized by $SL(2,\mathbb{Z})$ matrices. Represent lines of marginal stability.

$$jv + n_1\sigma - m_1\rho + m_2 = 0 \qquad 4m_1n_1 + j^2 = 1$$

Can represent them in the $\left(-v_2/\sigma_2, \sqrt{\rho_2\sigma_2 - v_2^2}/\sigma_2\right)$ plane as Farey arcs



Rademacher expansion



$$\frac{1}{\eta^{24}(\rho)} = \sum_{n=-1}^{\infty} d(n)\, e^{2\pi i n \rho} \qquad d(n) = \int_z^{z+1} d\rho\, e^{-2\pi i n \rho} \frac{1}{\eta^{24}(\rho)}$$

Deform contour along Ford circles, use modular symmetry to find behaviour near rational points $-\delta/\gamma$

$$\eta^{-24}(\rho) = (\gamma\rho+\delta)^{12}\eta^{-24}\left(\frac{\alpha\rho+\beta}{\gamma\rho+\delta}\right) \qquad d(n) = \frac{2\pi}{n^{\frac{13}{2}}} \sum_{\gamma>0} \frac{K(-1,n,\gamma)}{\gamma} I_{13}\left(\frac{4\pi\sqrt{n}}{\gamma}\right)$$

where $Kl(n,m,\gamma)$ is the classical Kloosterman sum $\quad Kl(n,m,\gamma) = \sum_{\substack{\delta \in \mathbb{Z}/\gamma\mathbb{Z} \\ \alpha\delta \equiv 1 \mod \gamma}} e^{2\pi i \left(n\frac{\delta}{\gamma} + m\frac{\alpha}{\gamma}\right)}$

and $I_\rho(z)$ the Bessel function of index ρ $\quad I_\rho(z) = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{dt}{t^{\rho+1}} e^{t+\frac{z^2}{4t}}$

Rademacher expansion

Jacobi form of weight $\omega < 0$ and index m for $\Delta > 0$

$$C_\ell(\Delta) = i^{-\omega+\frac{1}{2}} \sum_{\gamma=1}^{\infty} \left(\frac{\gamma}{2\pi}\right)^{\omega-\frac{5}{2}} \sum_{\substack{\tilde{\Delta}<0 \\ \tilde{\ell}\in\mathbb{Z}/2m\mathbb{Z}}} C_{\tilde{\ell}}(\tilde{\Delta}) Kl(\frac{\Delta}{4m}, \frac{\tilde{\Delta}}{4m}, \gamma)_{\tilde{\ell}\ell} \left|\frac{\tilde{\Delta}}{4m}\right|^{\frac{3}{2}-\omega} I_{\frac{3}{2}-\omega}\left(\frac{\pi}{\gamma}\sqrt{|\tilde{\Delta}|\Delta}\right)$$

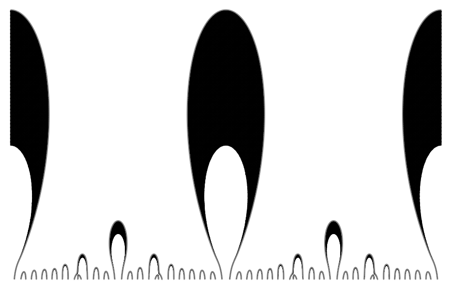
$C_{\tilde{\ell}}(\tilde{\Delta})$ with $\tilde{\Delta} < 0$, the polar coefficients, are the only input of the formula

$$Kl(\frac{\Delta}{4m}, \frac{\tilde{\Delta}}{4m}; \gamma, \psi)_{\ell\tilde{\ell}} = \sum_{\substack{0\leq-\delta<\gamma \\ (\delta,\gamma)=1, \alpha\delta\equiv 1 \bmod \gamma}} e^{2\pi i\left(\frac{\alpha}{\gamma}\frac{\tilde{\Delta}}{4m} + \frac{\delta}{\gamma}\frac{\Delta}{4m}\right)} \psi(\Gamma)_{\tilde{\ell}\ell}$$

Generalized Kloosterman sum

$$\psi(\Gamma)_{\ell j} = \frac{1}{\sqrt{2m\gamma i}} \sum_{T\in\mathbb{Z}/\gamma\mathbb{Z}} e^{2\pi i\left(\frac{\alpha}{\gamma}\frac{(\ell-2mT)^2}{4m} - \frac{j(\ell-2mT)}{2m\gamma} + \frac{\delta}{\gamma}\frac{j^2}{4m}\right)}$$

Multiplier system



Mock Jacobi forms

[Ramanujan ’1920]

[Zwegers ’2001]

Fourier-Jacobi expansion $$\frac{1}{\Phi_{10}(\rho,\sigma,v)} = \sum_{m\geq -1} \psi_m(\sigma,v)e^{2\pi i m\rho}$$

$$\psi_m(\sigma,v) = \psi_m^F(\sigma,v) + \psi_m^P(\sigma,v)$$

[Dabholkar, Murthy, Zagier ’12]

Split into **mock** Jacobi forms: a finite and a polar part.

$$\psi_m^F(\sigma,v) = \sum_{n,\ell} c_m^F(n,\ell)q^n y^\ell$$ has no poles in (σ,v) **Immortal**

Modularity can be restored at the expense of **holomorphicity**.

$$d_{imm}(m,n,\ell) = (-1)^{\ell+1}c_m^F(n,\ell)$$ (for $n \geq m$)

Mock Jacobi forms

$$\psi_m^F(\sigma, v) = \sum_{n,\ell} c_m^F(n,\ell) q^n y^\ell$$ transforms anomalously

[Dabholkar, Murthy, Zagier ’12]

$$\psi_m^F(\sigma, v) = (c\sigma + d)^{10} e^{-2\pi i m \frac{cv^2}{c\sigma + d}} \psi_m^F\left(\frac{a\sigma + b}{c\sigma + d}, \frac{v}{c\sigma + d}\right)$$

$$-\frac{d(m)}{\eta^{24}\left(\frac{a\sigma+b}{c\sigma+d}\right)}\sqrt{\frac{m}{8\pi^2}}\, i^{1/2}(c\sigma+d)^{21/2} \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} \vartheta_{m,\ell}(\sigma, v) \int_{-a/c}^{i\infty} \left(z + \frac{a\sigma+b}{c\sigma+d}\right)^{-3/2} \sum_{j\mathbb{Z}/2m\mathbb{Z}} \psi(\gamma)_{\ell j} \overline{\vartheta_{m,j}^0(-\bar{z})} dz$$

will have a modified Rademacher expansion

Generalized Rademacher expansion

$$c_m^{\mathrm{F}}(n,\ell) = 2\pi \sum_{k=1}^{\infty} \sum_{\substack{\widetilde{\ell}\in\mathbb{Z}/2m\mathbb{Z} \\ 4m\widetilde{n}-\widetilde{\ell}^2<0}} c_m^{\mathrm{F}}(\widetilde{n},\widetilde{\ell}) \frac{Kl\left(\frac{\Delta}{4m},\frac{\widetilde{\Delta}}{4m};k,\psi\right)_{\ell\widetilde{\ell}}}{k} \left(\frac{|\widetilde{\Delta}|}{\Delta}\right)^{23/4} I_{23/2}\left(\frac{\pi}{mk}\sqrt{|\widetilde{\Delta}|\Delta}\right)$$

$4mn - \ell^2 > 0$

$$+\sqrt{2m}\sum_{k=1}^{\infty} \frac{Kl\left(\frac{\Delta}{4m},-1;k,\psi\right)_{\ell 0}}{\sqrt{k}} \left(\frac{4m}{\Delta}\right)^{6} I_{12}\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta}\right) d(m) \qquad \text{(A.12)}$$

$$-\frac{1}{2\pi}\sum_{k=1}^{\infty} \sum_{\substack{j\in\mathbb{Z}/2m\mathbb{Z} \\ g\in\mathbb{Z}/2mk\mathbb{Z} \\ g\equiv j (\mathrm{mod}\ 2m)}} \frac{Kl\left(\frac{\Delta}{4m},-1-\frac{g^2}{4m};k,\psi\right)_{\ell j}}{k^2} \left(\frac{4m}{\Delta}\right)^{25/4} \times$$

$$d(m)\times \int_{-1/\sqrt{m}}^{+1/\sqrt{m}} f_{k,g,m}(u)\, I_{25/2}\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta(1-mu^2)}\right)(1-mu^2)^{25/4}\,\mathrm{d}u\,,$$

[Ferrari, Reys, ’17]

[Bringmann, Manschot ’10]

computes the coefficients $c_m^F(n,\ell)$ with $\Delta > 0$ in terms of $c_m^F(\tilde{n},\tilde{\ell})$ with $\Delta < 0$.

Polar coefficients

Study states in $\mathcal{N} = 4$ string theory with discriminant $\tilde{\Delta} = 4m\tilde{n} - \tilde{\ell}^2 < 0$.
They are bound states of two $1/2$-BPS states: Study their decays. [Sen '11]

Show there is a finite, computable set $W(m, \tilde{n}, \tilde{\ell}) \subset SL(2,\mathbb{Z})$ such that

$$d(m, \tilde{n}, \tilde{\ell})_{\tilde{\Delta}<0} = \sum_{\substack{\gamma \in W \\ W \subset SL(2,\mathbb{Z})}} (-1)^{\ell_\gamma + 1} \ell_\gamma d(m_\gamma) d(n_\gamma)$$

$$(m, \tilde{n}, \tilde{\ell}) \underset{\gamma}{\leftrightarrow} (m_\gamma, n_\gamma, \ell_\gamma)$$

[Chowdhury, Kidambi, Murthy, Reys, Wrase '19]

The set W is generated by the continued fraction of $\tilde{\ell}/2m$

$$\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2,\mathbb{Z})$$

$$\frac{\tilde{\ell}}{2m} = [a_0; a_1, \ldots, a_r] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_r}}}$$

$$0 \leq \frac{\tilde{\ell}}{2m} - \frac{q}{s} \leq \frac{1}{rs}$$

[Cardoso, Nampuri, MR '20]

Rademacher from Siegel

$$d(m,n,\ell) = \int_{\mathcal{C}} d\sigma\, dv\, d\rho\, (-1)^{\ell+1} \frac{e^{-2\pi i(m\rho+n\sigma+\ell v)}}{\Phi_{10}(\rho,\sigma,v)}$$

$\mathcal{C} : \rho_2/\sigma_2 \gg 1,\ v_2/\sigma_2 = -\ell/2m,\ 0 \le \ell < 2m$ $\qquad$ $0 \le \rho_1, \sigma_1, v_1 \le 1$

We want to obtain the Rademacher expansion as a sum over residues of $1/\Phi_{10}$ by writing the ρ integral as a sum over residues. This restricts the σ, v contours to take values over the pole in the Siegel upper-half plane.

$$d(m,n,\ell)|_{\Delta\ge 0} = (-1)^{\ell+1} \sum_{\substack{D \\ n_2\neq 0}} \left(\int_{\Gamma_\sigma(D)} d\sigma \int_{\Gamma_v(D)} dv\, \mathrm{Res}\left(\frac{1}{\Phi_{10}(\rho,\sigma,v)}\, e^{-2\pi i(m\rho+n\sigma+\ell v)} \right) \right)$$

Pole transformations

Bring any pole (n_2, n_1, j, m_2, m_1) to a $v' = 0$, $(m_2', m_1', j', n_2', n_1') = (0,0,1,0,0)$ pole.

Two $SL(2,\mathbb{Z}) \subset Sp(2,\mathbb{Z})$ transformations needed,

[Murthy, Pioline '09]

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \delta & 0 & -\beta \\ 0 & 0 & 1 & 0 \\ 0 & -\gamma & 0 & \alpha \end{pmatrix}, \begin{pmatrix} d & b & 0 & 0 \\ c & a & 0 & 0 \\ 0 & 0 & a & -c \\ 0 & 0 & -b & d \end{pmatrix} \in Sp(2,\mathbb{Z})$$

$$n_2 = \gcd(n_2, -m_1)\gamma$$
$$-m_1 = \gcd(n_2, -m_1)\delta$$

First one takes $n_2 \neq 0$ to $n_2' = 0$. Second takes general $n_2' = 0$ to $\tilde{v} = \Sigma \in \mathbb{Z}$

$$\Gamma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, G = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \Sigma \in \mathbb{Z}$$

Constrained to $a, -c, \gamma > 0$, $\alpha \in \mathbb{Z}/\gamma\mathbb{Z}$, $\Sigma \in \mathbb{Z}/(-ac)\mathbb{Z}$,
these transformations take (n_2, n_1, j, m_2, m_1) to $(0,0,1,0,0)$ pole

Residues

Use $Sp(2,\mathbb{Z})$ covariance, $\dfrac{1}{\Phi_{10}(\rho,\sigma,v)} = (\gamma\sigma+\delta)^{10}\dfrac{1}{\Phi_{10}(\rho',\sigma',v')}$

and behaviour $\dfrac{1}{\Phi_{10}(\rho',\sigma',v')} \xrightarrow[v'\to 0]{} -\dfrac{1}{4\pi^2}\dfrac{1}{v'^2}\dfrac{1}{\eta^{24}(\rho')}\dfrac{1}{\eta^{24}(\sigma')}$ to compute residue w.r.t. ρ,

$$(-1)^{\ell+1}\frac{(\gamma\sigma+\delta)^{10}}{ac}\left(\frac{m}{ac}+\frac{a}{c}E_2(\rho'_*)+\frac{c}{a}E_2(\sigma'_*)\right)\frac{1}{\eta^{24}(\rho'_*)}\frac{1}{\eta^{24}(\sigma'_*)}e^{-2\pi i(m\Lambda+n\sigma+\ell v)}$$

Fourier expansion

$$(-1)^{\ell+1}\frac{(\gamma\sigma+\delta)^{10}}{ac}\left(\frac{m}{ac}+\frac{a}{c}E_2(\rho_*'')+\frac{c}{a}E_2(\sigma_*'')\right)\frac{1}{\eta^{24}(\rho_*'')}\frac{1}{\eta^{24}(\sigma_*'')}e^{-2\pi i(m\Lambda+n\sigma+\ell v)}$$

Fourier expanding,

$$(-1)^{\ell}\sum_{P'}\sum_{M,N\geq -1}(\gamma\sigma+\delta)^{10}\boxed{Ld(M)d(N)}\exp\left(-2\pi i\left[-\tilde{n}\left(\frac{\alpha\sigma+\beta}{\gamma\sigma+\delta}\right)-\tilde{\ell}\frac{v}{\gamma\sigma+\delta}+m\frac{\gamma v^2}{\gamma\sigma+\delta}+\ell v+n\sigma\right]\right)$$

where $L=-\frac{m}{ac}+\frac{a}{c}M+\frac{c}{a}N$, $\tilde{\ell}=-\frac{ad+bc}{ac}m+\frac{a}{c}M-\frac{c}{a}N$, $\tilde{n}=\frac{bd}{ac}m-\frac{b}{c}M+\frac{d}{a}N$

Sum over $\Sigma\in\mathbb{Z}/(-ac)\mathbb{Z}$ forces $L,\tilde{\ell},\tilde{n}\in\mathbb{Z}$.

$$(m,\tilde{n},\tilde{\ell})\leftarrow\begin{pmatrix}a & b\\ c & d\end{pmatrix}\rightarrow(M,N,L),\quad \tilde{\Delta}=4MN-L^2=4m\tilde{n}-\tilde{\ell}^2$$

v integral

Condition det Im$(\Omega) > 0$ for poles imposes restricted v contour

$$v : -\frac{\ell}{2m\gamma}(\gamma\sigma+\delta) - \frac{b}{\gamma a} \quad \rightarrow \quad -\frac{\ell}{2m\gamma}(\gamma\sigma+\delta) - \frac{b}{\gamma a} - \frac{1}{ac\gamma}$$

yields $\qquad b \in \mathbb{Z}/a\gamma\mathbb{Z}$

$$(-1)^{\ell} \sum_{P'} (\gamma\sigma+\delta)^{10} L d(M) d(N) e^{2\pi i \left(\frac{\alpha}{\gamma}\frac{\tilde{\Delta}}{4m} + \frac{\delta}{\gamma}\frac{\Delta}{4m}\right)}$$

$$e^{2\pi i \left(\frac{\alpha}{\gamma}\frac{\tilde{\ell}^2}{4m} - \frac{\ell\tilde{\ell}}{2m\gamma} + \frac{\delta}{\gamma}\frac{\ell^2}{4m}\right)} e^{\left(-2\pi i\left[\frac{\tilde{\Delta}}{4m}\frac{1}{\gamma}\frac{1}{\gamma\sigma+\delta} + \frac{\Delta}{4m}\frac{\gamma\sigma+\delta}{\gamma}\right]\right)}$$

$$\frac{1}{2}\frac{\sqrt{\gamma\sigma+\delta}}{\sqrt{2m\gamma i}}\left(\mathrm{Erf}\left[\sqrt{\frac{2\pi i m\gamma}{\gamma\sigma+\delta}}\left(-\frac{b}{a\gamma} - \frac{1}{ac\gamma} - \frac{\tilde{\ell}}{2m\gamma}\right)\right] - \mathrm{Erf}\left[\sqrt{\frac{2\pi i m\gamma}{\gamma\sigma+\delta}}\left(-\frac{b}{a\gamma} - \frac{\tilde{\ell}}{2m\gamma}\right)\right]\right).$$

σ integral

Writing condition for v as

$$v_1 = \frac{v_2}{\gamma\sigma_2}(\gamma\sigma_1 + \delta) - \frac{b}{\gamma a} + x \qquad x \in (0, -1/ac)$$

the contour of integration will be on the pole if

$$(\rho_2\sigma_2 - v_2^2)\left(\sigma_1 + \frac{\delta}{\gamma}\right)^2 + \sigma_2^2\left(x + \frac{1}{2ac\gamma}\right)^2 = \sigma_2^2\left[\frac{1}{4(ac\gamma)^2} - (\rho_2\sigma_2 - v_2^2)\right]$$

> 0

which describes an ellipse in (σ_1, x).

Pole crossed as $\sigma_2 \to 0$. This fixes $\sigma_1 = -\dfrac{\delta}{\gamma}$: Ford circle in σ plane

Continued fraction condition

$$\mathrm{Erf}(x) = 1 - \mathrm{Erfc}(x) = 1 - \frac{1}{\sqrt{\pi}}\frac{e^{-x^2}}{x} + \frac{1}{2\sqrt{\pi}}\int_{x^2}^{\infty} t^{-3/2}e^{-t}dt, \quad \text{for } \mathrm{Re}(x) > 0.$$

$$\mathrm{Erf}\left[\sqrt{\frac{2\pi i m}{\gamma(\gamma\sigma+\delta)}}\left(\frac{b}{a}+\frac{\tilde{\ell}}{2m}\right)\right] + \mathrm{Erf}\left[\sqrt{\frac{2\pi i m}{\gamma(\gamma\sigma+\delta)}}\left(-\frac{b}{a}-\frac{1}{ac}-\frac{\tilde{\ell}}{2m}\right)\right].$$

Both Erf functions will have positive real part iff

$$0 < \frac{b}{a} + \frac{\tilde{\ell}}{2m} < -\frac{1}{ac}$$

Continued fraction condition!

The integral over σ along Ford circles imposes $\tilde{\Delta} < 0$. Sum over a, c, M, N becomes

$$c_m^F(\tilde{n}, \tilde{\ell})$$

Bessel 23/2

Final answer is then

$$(-1)^{\ell+1} 2\pi \sum_{\substack{\gamma>0,\tilde{\Delta}<0 \\ \tilde{\ell}\in\mathbb{Z}/2m\mathbb{Z}}} c_m^F(\tilde{n},\tilde{\ell}) \frac{Kl(\frac{\Delta}{4m},\frac{\tilde{\Delta}}{4m},\gamma,\psi)_{\ell\tilde{\ell}}}{\gamma} \left(\frac{|\tilde{\Delta}|}{\Delta}\right)^{23/4} I_{23/2}\left(\frac{\pi}{\gamma m}\sqrt{\Delta|\tilde{\Delta}|}\right)$$

compare with

[Ferrari, Reys, '17]

$$\begin{aligned} c_m^{\mathrm{F}}(n,\ell) &= 2\pi \sum_{k=1}^{\infty} \sum_{\substack{\tilde{\ell}\in\mathbb{Z}/2m\mathbb{Z} \\ 4m\tilde{n}-\tilde{\ell}^2<0}} c_m^{\mathrm{F}}(\tilde{n},\tilde{\ell}) \frac{Kl(\frac{\Delta}{4m},\frac{\tilde{\Delta}}{4m};k,\psi)_{\ell\tilde{\ell}}}{k} \left(\frac{|\tilde{\Delta}|}{\Delta}\right)^{23/4} I_{23/2}\left(\frac{\pi}{mk}\sqrt{|\tilde{\Delta}|\Delta}\right) \\ &+ \sqrt{2m} \sum_{k=1}^{\infty} \frac{Kl(\frac{\Delta}{4m},-1;k,\psi)_{\ell 0}}{\sqrt{k}} \left(\frac{4m}{\Delta}\right)^{6} I_{12}\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta}\right) d(m) \qquad \text{(A.12)} \\ &- \frac{1}{2\pi} \sum_{k=1}^{\infty} d(m) \sum_{\substack{j\in\mathbb{Z}/2m\mathbb{Z} \\ g\in\mathbb{Z}/2mk\mathbb{Z} \\ g\equiv j (\mathrm{mod}\ 2m)}} \frac{Kl(\frac{\Delta}{4m},-1-\frac{g^2}{4m};k,\psi)_{\ell j}}{k^2} \left(\frac{4m}{\Delta}\right)^{25/4} \times \\ &\times \int_{-1/\sqrt{m}}^{+1/\sqrt{m}} f_{k,g,m}(u)\, I_{25/2}\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta(1-mu^2)}\right)(1-mu^2)^{25/4}\, du\,, \end{aligned}$$

First term!

Mock part

Other terms will come from

$$\mathrm{Erf}(x) = 1 - \mathrm{Erfc}(x) = 1 - \frac{1}{\sqrt{\pi}}\frac{e^{-x^2}}{x} + \frac{1}{2\sqrt{\pi}}\int_{x^2}^{\infty} t^{-3/2}e^{-t}dt, \quad \text{for } \mathrm{Re}(x) > 0.$$

$$\begin{aligned} c_m^{\mathrm{F}}(n,\ell) \;=\; & 2\pi \sum_{k=1}^{\infty} \sum_{\substack{\widetilde{\ell}\in\mathbb{Z}/2m\mathbb{Z} \\ 4m\widetilde{n}-\widetilde{\ell}^2<0}} c_m^{\mathrm{F}}(\widetilde{n},\widetilde{\ell}) \frac{Kl\big(\frac{\Delta}{4m},\frac{\widetilde{\Delta}}{4m};k,\psi\big)_{\ell\widetilde{\ell}}}{k} \left(\frac{|\widetilde{\Delta}|}{\Delta}\right)^{23/4} I_{23/2}\left(\frac{\pi}{mk}\sqrt{|\widetilde{\Delta}|\Delta}\right) \\ & + \sqrt{2m} \sum_{k=1}^{\infty} \frac{Kl\big(\frac{\Delta}{4m},-1;k,\psi\big)_{\ell 0}}{\sqrt{k}} \left(\frac{4m}{\Delta}\right)^{6} I_{12}\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta}\right) d(m) \qquad \text{(A.12)} \\ & - \frac{1}{2\pi} \sum_{k=1}^{\infty} d(m) \sum_{\substack{j\in\mathbb{Z}/2m\mathbb{Z} \\ g\in\mathbb{Z}/2mk\mathbb{Z} \\ g\equiv j (\text{mod } 2m)}} \frac{Kl\big(\frac{\Delta}{4m},-1-\frac{g^2}{4m};k,\psi\big)_{\ell j}}{k^2} \left(\frac{4m}{\Delta}\right)^{25/4} \times \\ & \times \int_{-1/\sqrt{m}}^{+1/\sqrt{m}} f_{k,g,m}(u)\, I_{25/2}\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta(1-mu^2)}\right)(1-mu^2)^{25/4}\, \mathrm{d}u\,, \end{aligned}$$

Mock part

Other terms will come from

$$(-1)^{\ell+1}\sum_{P''''} Kl(\frac{\Delta}{4m},\frac{\tilde{\Delta}}{4m};\gamma,\psi)_{\ell\tilde{\ell}}\int d\tilde{\sigma}\ \frac{\tilde{\sigma}^{11}}{\gamma^{25/2}}\,d(M)d(N)e^{\left(-2\pi i\left[\frac{\tilde{\Delta}}{4m}\frac{1}{\tilde{\sigma}}+\frac{\Delta}{4m}\frac{\tilde{\sigma}}{\gamma^2}\right]\right)}\frac{L}{\left(\frac{\tilde{\ell}}{2m}+\frac{b}{a}\right)}$$

$$\left(-\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{\frac{2\pi im}{\tilde{\sigma}}}}e^{-\frac{2\pi im}{\tilde{\sigma}}\left(\frac{\tilde{\ell}}{2m}+\frac{b}{a}\right)^2}-\frac{1}{2\sqrt{\pi}}\frac{1}{\sqrt{2\pi im}}e^{-\frac{2\pi im}{\tilde{\sigma}}\left(\frac{\tilde{\ell}}{2m}+\frac{b}{a}\right)^2}\int_0^{i\infty}\left(\frac{1}{\tilde{\sigma}}-z\right)^{-3/2}e^{2\pi im\left(\frac{\tilde{\ell}}{2m}+\frac{b}{a}\right)^2 z}dz\right)$$

there will be huge cancellations (with some regularization needed) giving at the end the right answer.

Cancellations

$$(-1)^{\ell+1}\sum_{P''''} Kl(\frac{\Delta}{4m},\frac{\tilde{\Delta}}{4m};\gamma,\psi)_{\ell\tilde{\ell}}\int d\tilde{\sigma}\;\frac{\tilde{\sigma}^{11}}{\gamma^{25/2}}\,d(M)d(N)e^{\left(-2\pi i\left[\frac{\tilde{\Delta}}{4m}\frac{1}{\tilde{\sigma}}+\frac{\Delta}{4m}\frac{\tilde{\sigma}}{\gamma^2}\right]\right)}\frac{L}{\left(\frac{\tilde{\ell}}{2m}+\frac{b}{a}\right)}$$

$$\left(-\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{\frac{2\pi i m}{\tilde{\sigma}}}}e^{-\frac{2\pi i m}{\tilde{\sigma}}\left(\frac{\tilde{\ell}}{2m}+\frac{b}{a}\right)^2}-\frac{1}{2\sqrt{\pi}}\frac{1}{\sqrt{2\pi i m}}e^{-\frac{2\pi i m}{\tilde{\sigma}}\left(\frac{\tilde{\ell}}{2m}+\frac{b}{a}\right)^2}\int_0^{i\infty}\left(\frac{1}{\tilde{\sigma}}-z\right)^{-3/2}e^{2\pi i m\left(\frac{\tilde{\ell}}{2m}+\frac{b}{a}\right)^2 z}dz\right)$$

Look at term $\frac{L}{\left(\frac{b}{a}+\frac{\tilde{\ell}}{2m}\right)}=2m\frac{\frac{m-a^2M}{c}+c}{\frac{m-a^2M}{c}-c}$, sum over $c \mid m-a^2M$

will come in pairs with opposite sign $\pm\frac{L}{\left(\frac{b}{a}+\frac{\tilde{\ell}}{2m}\right)}$

Result only non-vanishing for $N=-1,\; m-a^2M=0$

$$\implies \quad M=m, \quad a=1$$

Shadow

Term with integral:

$$(-1)^{\ell+1}\frac{\sqrt{2m}\,d(m)}{4\sqrt{\pi^2 i}}\sum_{\gamma=1}^{\infty}\sum_{j\in\mathbb{Z}/2m\mathbb{Z}}\sum_{\substack{c<0\\ c=j \mod 2m}}\mathrm{Kl}\Big(\frac{\Delta}{4m},-1-\frac{c^2}{4m};\gamma,\psi\Big)_{\ell j}$$

$$\int_{\tilde{\Gamma}} d\tilde{\sigma}\,\frac{\tilde{\sigma}^{21/2}}{\gamma^{25/2}}\,e^{\left(2\pi i\left[\frac{1}{\tilde{\sigma}}-\frac{\Delta}{4m}\frac{\tilde{\sigma}}{\gamma^2}\right]\right)}\int_0^{i\infty}\left(\frac{1}{\tilde{\sigma}}-z\right)^{-3/2}e^{2\pi i\frac{c^2}{4m}z}dz\ .$$

$$\vartheta_{m,j}\left(z-\frac{\alpha}{\gamma};0\right)=\sum_{c=j\mathrm{mod}\ 2m}e^{-2\pi i\frac{\alpha}{\gamma}\frac{c^2}{4m}}e^{2\pi i\frac{c^2}{4m}z}$$

Final result

$$d(m,n,\ell) =$$

$$(-1)^{\ell+1}\sum_{\gamma=1}^{+\infty}\sum_{\tilde\ell\in\mathbb{Z}/2m\mathbb{Z}}\left(2\pi\sum_{\substack{\tilde n\geq -1,\\ \tilde\Delta<0}} c_m^F(\tilde n,\tilde\ell)\frac{\mathrm{Kl}(\frac{\Delta}{4m},\frac{\tilde\Delta}{4m},\gamma,\psi)_{\ell\tilde\ell}}{\gamma}\left(\frac{|\tilde\Delta|}{\Delta}\right)^{23/4} I_{23/2}\left(\frac{\pi}{\gamma m}\sqrt{\Delta|\tilde\Delta|}\right)\right.$$

$$-\delta_{\tilde\ell,0}\sqrt{2m}\,d(m)\frac{\mathrm{Kl}(\frac{\Delta}{4m},-1;\gamma,\psi)_{\ell 0}}{\sqrt{\gamma}}\left(\frac{4m}{\Delta}\right)^{6} I_{12}\left(\frac{2\pi}{\gamma}\sqrt{\frac{\Delta}{m}}\right)$$

$$+\frac{1}{2\pi}d(m)\sum_{\substack{g\in\mathbb{Z}/2m\gamma\mathbb{Z}\\ g=\tilde\ell \bmod 2m}}\frac{\mathrm{Kl}(\frac{\Delta}{4m},-1-\frac{g^2}{4m};\gamma,\psi)_{\ell\tilde\ell}}{\gamma^2}$$

$$\left.\left(\frac{4m}{\Delta}\right)^{25/4}\int_{-1/\sqrt{m}}^{1/\sqrt{m}} dx' f_{\gamma,g,m}(x')(1-mx'^2)^{25/4} I_{25/2}\left(\frac{2\pi}{\gamma\sqrt{m}}\sqrt{\Delta(1-mx'^2)}\right)\right)$$

A special case, $m = 0$

For $m = 0$,

$$\psi_0^F(\sigma, v) = 2\frac{E_2(\sigma)}{\eta^{24}(\sigma)},$$

for $\ell = 0$ and $n \geq 0$, the immortal degeneracies $d_{imm}(0,n,0)$ are given by the Fourier coefficients of a quasi-modular form.

Using
$$\frac{E_2(\sigma)}{\eta^{24}(\sigma)} = (\gamma\sigma+\delta)^{10}\frac{E_2\left(\frac{\alpha\sigma+\beta}{\gamma\sigma+\delta}\right)}{\eta^{24}\left(\frac{\alpha\sigma+\beta}{\gamma\sigma+\delta}\right)} - \frac{6\gamma}{\pi i}\frac{(\gamma\sigma+\delta)^{11}}{\eta^{24}\left(\frac{\alpha\sigma+\beta}{\gamma\sigma+\delta}\right)}$$

one can obtain a Rademacher expansion for the coefficients

$$d_{imm}(0,n,0) = \sum_{\gamma=1}^{+\infty} Kl(n,-1,\gamma)\left(-\frac{24}{n^6}I_{12}\left(\frac{4\pi\sqrt{n}}{\gamma}\right) + \frac{4\pi}{\gamma\, n^{11/2}}I_{11}\left(\frac{4\pi\sqrt{n}}{\gamma}\right)\right)$$

$d_{imm}(0,n,0)$ from Siegel

Performing the same sum over poles of $1/\Phi_{10}$ fixing $m = \ell = 0$, the exponent has a much simpler form. No term quadratic in v.

$$(-1)\sum_{P'}(\gamma\sigma+\delta)^{10}Ld(M)d(N)\exp\left(-2\pi i\left[-\tilde{n}\left(\frac{\alpha\sigma+\beta}{\gamma\sigma+\delta}\right)-\tilde{\ell}\frac{v}{\gamma\sigma+\delta}+n\sigma\right]\right)$$

When there is no v dependence we can integrate it out. Integrate over σ along Ford circles and obtain

$$\sum_{\gamma=1}^{+\infty}Kl(n,-1,\gamma)\frac{4\pi}{\gamma\, n^{11/2}}I_{11}\left(\frac{4\pi\sqrt{n}}{\gamma}\right)$$

$d_{imm}(0,n,0)$ from Siegel

For the linear term in v,

$$\frac{(-1)}{2\pi i}\sum_{P''}\sum_{\substack{N>-1\\M=-1}}(\gamma\sigma+\delta)^{11}\frac{L}{\tilde{\ell}}d(N)e^{-2\pi i\left(-\tilde{n}\frac{\alpha}{\gamma}+T\tilde{\ell}\frac{\alpha}{\gamma}\right)}e^{-2\pi i\left(n\sigma-\frac{1}{\gamma(\gamma\sigma+\delta)}\right)}.$$

$$\frac{L}{\tilde{\ell}}=\frac{a-\frac{N}{a}}{a+\frac{N}{a}}$$

a is summed over divisors of N: for each $a \mid N$, there is $a' = \dfrac{N}{a} \mid N$

Only $N = 0$ survives. Sum over a, $\displaystyle\sum_{a=1}^{+\infty} a^0 \sim \zeta(0) = -\frac{1}{2}$ yields

$$\sum_{\gamma=1}^{+\infty} Kl(n,-1,\gamma)\frac{-24}{n^6}I_{12}\left(\frac{4\pi\sqrt{n}}{\gamma}\right)$$

Correct!

Conclusions

One can use the $Sp(2,\mathbb{Z})$ symmetries of $1/\Phi_{10}$ to perform the sum over residues with $n_2 \neq 0$ and obtain a finer-grained generalized Rademacher expansion where the continued fraction structure for the polar coefficients arises naturally.

Answer parametrized by two sets of $SL(2,\mathbb{Z})$ matrices, the S-duality being one of them. Physics input: polar states as bound states. Mathematics output: two $SL(2,\mathbb{Z})$ groups needed to characterize the physical degeneracies $d(m,n,\ell)$.

Only input $d(n)$ of $\eta^{-24}(\sigma)$: non-perturbative from perturbative.

Future directions

Can we use the microscopic structure as a guide for a macroscopic path integral computation? The work [Murthy, Reys '15] matches the expression of our first coefficient, $n_2 = 1$.

$$d(n,\ell,m) = \frac{1}{2^{12}} \int \frac{d^2\tau}{\tau_2^{13}} \left(m + E_2(\tau) + E_2(-\bar{\tau})\right) \left(\eta^{24}(\tau)\eta^{24}(-\bar{\tau})\right)^{-1} e^{\frac{\pi}{\tau_2}(n-\ell\tau_1+m\tau_1^2+m\tau_2^2)}, \tag{4.7}$$

How does one modify the Quantum Entropy Function proposal to get the full answer? What is the role of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$? And $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$? [Work in progress w/ Abhiram Kidambi & Valentin Reys]

$1/4$-BPS dyons in CHL orbifolds are also counted by Siegel modular forms with a quadratic pole at $v = 0$. Polar coefficients known. Generalized Rademacher expansion not known.

Thank you