The Ratios Conjecture and negative moments of L-functions

Alexandra Florea
UC Irvine

Kavli IPMU (Uni. Tokyo)
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$\zeta(s)$ and $L(s,\chi)$

For $\Re s > 1$,

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_p \left(1 - p^{-s}\right)^{-1}.$$

- It has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$.
- It satisfies a functional equation $\zeta(s) \leftrightarrow \zeta(1-s)$
- Trivial zeros at $s = -2m, m \geq 1$.
- (RH) The non-trivial zeros of $\zeta(s)$ lie on the line $\Re(s) = 1/2$.

$\zeta(s)$ and $L(s,\chi)$

A Dirichlet character is a completely multiplicative function

$$\chi : (\mathbb{Z}/d\mathbb{Z})^* \to \mathbb{C}^*$$

extended to $\mathbb{Z}$ by periodicity, with the property that

$$\chi(a) = 0 \text{ whenever } (a,d) \neq 1.$$

χ is primitive with conductor d if d is the smallest modulus for which χ is a character modulo d.
For $\Re(s) > 1$ and χ a character $\pmod d$,

$$L(s,\chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s} = \prod_{p\nmid d} \left(1 - \chi(p)p^{-s}\right)^{-1}.$$

- It has an analytic continuation to $\mathbb{C}$
- It satisfies a functional equation $L(s,\chi) \leftrightarrow L(1-s,\overline{\chi})$
- (GRH) The non-trivial zeros of $L(s,\chi)$ lie on the line $\Re(s) = 1/2$.

Relationship with RMT

Conjecture (Montgomery)

The pair correlation of zeros of $\zeta(s)$ is the same as the pair correlation of matrices from the GUE ensemble.

- Katz and Sarnak looked at low-lying zeros in families of L–functions over function fields and showed that they follow the laws governed by the corresponding scaling limit of the symmetry group of the family.
- Can model L–functions in families by random matrix ensembles

$$\begin{aligned} \zeta(s) &\Longleftrightarrow \text{ unitary ensemble} \\ L(s,\chi_d), \chi_d^2 = \chi_0 &\Longleftrightarrow \text{ symplectic ensemble} \\ L(s, E \otimes \chi_d), \chi_d^2 = \chi_0 &\Longleftrightarrow \text{ orthogonal ensemble} \end{aligned}$$

Moments of L–functions

- (RH) The non-trivial zeros of $\zeta(s)$ lie on the line $\Re(s) = 1/2$.
- (Lindelöf hypothesis) $|\zeta(1/2 + it)| = O(t^{\epsilon}), \forall \epsilon > 0$.
- Hardy and Littlewood (1916): moments of $\zeta(s)$

$$I_k(T) = \int_0^T |\zeta(1/2 + it)|^{2k} \, dt$$

$$\text{Lindelöf hypothesis} \iff I_k(T) \ll T^{1+\epsilon}, k = 1, 2, \ldots.$$

Conjecture (Keating, Snaith; Conrey, Farmer, Keating, Rubinstein, Snaith)

$$\int_0^T |\zeta(1/2 + it)|^{2k} \, dt \sim M_k T (\log T)^{k^2}.$$

- $k = 1$: Hardy, Littlewood (1916)
- $k = 2$: Ingham (1932), Heath-Brown (1979)

Moments of L–functions

For a prime p, let

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a = \square \bmod p, (a,p) = 1 \\ -1 & \text{if } a \neq \square \bmod p, (a,p) = 1 \\ 0 & \text{if } p | a. \end{cases}$$

Extend multiplicatively. Let

$$\chi_d(n) = \left(\frac{d}{n}\right).$$

Conjecture (Keating, Snaith; Conrey, Farmer, Keating, Rubinstein, Snaith)

$$\sum_{0<d\leq X}^{*} L\left(\tfrac{1}{2}, \chi_d\right)^k \sim C_k X (\log X)^{\frac{k(k+1)}{2}}.$$

Moments of L–functions

Conjecture (Keating, Snaith; Conrey, Farmer, Keating, Rubinstein, Snaith)

$$\sum_{0<d\leq X}^{*} L\left(\tfrac{1}{2}, \chi_d\right)^k = XP_k(\log X) + O(X^{1-\delta}),$$

where $\deg(P_k) = k(k+1)/2$, $\delta > 0$.

- $k = 1, 2$: Jutila (1981)
- $k = 2, 3$: Soundararajan (2000)
- $k = 3$: Diaconu, Goldfeld, Hoffstein (2003)
- $k = 4$: Shen (2019) on GRH, using ideas of Soundararajan and Young
- Upper bounds of the right order of magnitude, on GRH: Soundararajan, Harper
- Lower bounds of the right order of magnitude: Soundararajan, Rudnick

The Ratios Conjecture for $\zeta(s)$

Conjecture (Farmer, 1993)

For $s = 1/2 + it$ and complex numbers $\alpha, \beta, \gamma, \delta$ of size $c/\log T$, such that $\Re\alpha, \Re\beta, \Re\gamma, \Re\delta > 0$ we have

$$\frac{1}{T}\int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)}\,dt \sim 1 + (1 - T^{-\alpha-\beta})\frac{(\alpha-\gamma)(\beta-\delta)}{(\alpha+\beta)(\gamma+\delta)}.$$

- The conjecture implies many interesting results about zeros of $\zeta(s)$, such as the pair correlation conjecture of Montgomery.
- By adapting the "recipe" used by Conrey, Farmer, Keating, Rubinstein and Snaith to conjecture asymptotic formulas for moments, one can make the following conjectures.

Conjecture (Conrey, Farmer, Zirnbauer, 2007)

$$\int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)}\,dt$$

$$= \int_0^T \Big(\frac{\zeta(1+\alpha+\beta)\zeta(1+\gamma+\delta)}{\zeta(1+\alpha+\delta)\zeta(1+\beta+\gamma)} A(\alpha,\beta,\gamma,\delta)$$

$$+ \left(\frac{t}{2\pi}\right)^{-\alpha-\beta} \frac{\zeta(1-\alpha-\beta)\zeta(1+\gamma+\delta)}{\zeta(1-\beta+\delta)\zeta(1-\alpha+\gamma)} A(-\beta,-\alpha,\gamma,\delta)\Big)\,dt + O\left(T^{1/2+\epsilon}\right),$$

where

$$A(\alpha,\beta,\gamma,\delta) = \prod_p \frac{\left(1-\frac{1}{p^{1+\gamma+\delta}}\right)\left(1-\frac{1}{p^{1+\beta+\gamma}}-\frac{1}{p^{1+\alpha+\delta}}+\frac{1}{p^{1+\gamma+\delta}}\right)}{\left(1-\frac{1}{p^{1+\beta+\gamma}}\right)\left(1-\frac{1}{p^{1+\alpha+\delta}}\right)}$$

for $|\Re\alpha|, |\Re\beta| < 1/4$,

$$\frac{1}{\log T} \ll \Re\gamma, \Re\delta < 1/4, \ \Im\alpha, \Im\beta, \Im\gamma, \Im\delta \ll T^{1-\epsilon}.$$

The Ratios Conjecture for quadratic Dirichlet L–functions

Conjecture (Conrey, Farmer, Zirnbauer, 2007)

$$\sum_{d\le X}^{*} \frac{L(1/2+\alpha,\chi_d)}{L(1/2+\beta,\chi_d)} = \sum_{d\le X}^{*} \Big(\frac{\zeta(1+2\alpha)}{\zeta(1+\alpha+\beta)} A(\alpha,\beta) + \left(\frac{d}{\pi}\right)^{-\alpha} \frac{\Gamma(1/4-\alpha/2)}{\Gamma(1/4+\alpha/2)} \frac{\zeta(1-2\alpha)}{\zeta(1-\alpha+\beta)} A(-\alpha,\beta) \Big) + O\Big(X^{1/2+\epsilon}\Big),$$

where

$$A(\alpha,\beta) = \prod_p \Big(1-\frac{1}{p^{1+\alpha+\beta}}\Big)^{-1} \Big(1-\frac{1}{(p+1)p^{1+2\alpha}} - \frac{1}{(p+1)p^{\alpha+\beta}}\Big),$$

for $|\Re\alpha| < 1/4$,

$$\frac{1}{\log X} \ll \Re\beta < 1/4,\ \Im\beta \ll X^{1-\epsilon}.$$

- M. Cech (2021): restricted range of parameters, nonuniform

The Ratios Conjecture in Random Matrix Theory

One can compute ratios of characteristic polynomials in matrix ensembles:

- Conrey-Farmer-Zirnbauer
- Borodin-Strahov
- Conrey-Forrester-Snaith
- Bump-Gamburd
- Huckleberry-Puttmann-Zirnbauer

Theorem (Conrey-Farmer-Zirnbauer)

For $\Re\gamma_k > 0$, we have

$$\int_{Usp(2N)} \frac{\prod_{k=1}^{K} \Lambda_A(e^{-\alpha_k})}{\prod_{k=1}^{K} \Lambda_A(e^{-\gamma_k})}\, dA$$

$$= \sum_{\epsilon\in\{-1,1\}^K} e^{N\sum_{k=1}^{K}(\epsilon_k\alpha_k-\alpha_k)} \frac{\prod_{j\le k\le K} z(\epsilon_j\alpha_j+\epsilon_k\alpha_k)\prod_{q<r\le K} z(\gamma_q+\gamma_r)}{\prod_{k=1}^{K}\prod_{q=1}^{K} z(\epsilon_k\alpha_k+\gamma_q)},$$

where $z(x) = (1-e^{-x})^{-1}$.

Applications of the Ratios Conjecture

- Compute the one-level density of zeros in families of L–functions, for test functions whose Fourier transforms have any support.

Conjecture (Chowla's conjecture)

$L(1/2,\chi) \neq 0$ *for any* χ *a Dirichlet character.*

  - Soundararajan: $\geq 87.5\%$ of $L(1/2,\chi_d) \neq 0$
  - Ozluk-Snyder: $\geq 93.75\%$ of $L(1/2,\chi_d) \neq 0$ by computing the one-level density of zeros with support $(-2,2)$ (GRH)
  - The Ratios Conjecture $\Rightarrow$ 100% of $L(1/2,\chi_d) \neq 0$

- Compute the lower order terms for the pair correlation of the zeros of $\zeta(s)$, which were previously heuristically computed by Bogomolny and Keating.
- Compute mollified moments of $\zeta(s)$ or other L–functions
- Obtain conjectures for moments of $|\zeta'(\rho)|$
- Etc.

Negative moments of $\zeta(s)$

Conjecture (Gonek, 1989)

Let $k > 0$ be fixed. Uniformly for $\frac{1}{\log T} \le \delta \le 1$, we have

$$\int_1^T \left|\zeta\left(\frac{1}{2}+\delta+it\right)\right|^{-2k} dt \asymp T\left(\frac{1}{\delta}\right)^{k^2},$$

and uniformly for $0 < \delta \le \frac{1}{\log T}$, we have

$$\int_1^T \left|\zeta\left(\frac{1}{2}+\delta+it\right)\right|^{-2k} dt \asymp \begin{cases} T(\log T)^{k^2} & \text{if } k < 1/2 \\ \log(e/(\delta \log T))\, T(\log T)^{k^2} & \text{if } k = 1/2 \\ (\delta \log T)^{1-2k}\, T(\log T)^{k^2} & \text{if } k > 1/2. \end{cases}$$

- Random matrix theory computations (Berry-Keating; Forrester-Keating) suggest transition regimes when $k = (2n+1)/2$, for n a positive integer

Negative moments of $\zeta(s)$

- Gonek obtained lower bounds consistent with the conjecture for all $k > 0$ in the range $\frac{1}{\log T} \leq \delta \leq 1$ and for $k < 1/2$ in the range $0 < \delta \leq \frac{1}{\log T}$.
- Upper bounds??
- Work in progress with H. Bui: upper bounds when $\log(1/\delta) \ll \log\log T$.

Function fields background

Dictionary

$\mathbb{N}$	monic polynomials in $\mathbb{F}_q[t]$
$\lvert n \rvert$	$\lvert f \rvert := q^{\deg(f)}$
primes p	monic irreducible polynomials P

Define the zeta-function

$$\zeta_q(s) = \sum_{f \text{ monic}} \frac{1}{|f|^s} = \sum_{n=0}^{\infty} \frac{q^n}{q^{ns}} = \frac{1}{1-q^{1-s}}.$$

$$u = q^{-s} : \mathcal{Z}(u) = \sum_{f \text{ monic}} u^{\deg(f)} = \frac{1}{1-qu}.$$

Function fields background

- Let χ be a primitive character $\pmod{h}$. The L–function associated to χ is defined by

$$L(s,\chi) = \mathcal{L}(u,\chi) = \sum_{f \text{ monic}} \chi(f) u^{\deg(f)} = \prod_{P \nmid h} \left(1 - \chi(P) u^{\deg(P)}\right)^{-1}.$$

- $\mathcal{L}(u,\chi)$ satisfies the following:
  - It is a polynomial of degree $\leq \deg(h) - 1$.
  - It has a functional equation. If χ is an odd character, then

$$\mathcal{L}(u,\chi) = \omega(\chi)(\sqrt{q}u)^{\deg(h)-1} \mathcal{L}\left(\frac{1}{qu}, \overline{\chi}\right),$$

    where $\omega(\chi)$ is the root number.
  - All the nontrivial zeros lie on the circle of radius $|u| = \frac{1}{\sqrt{q}}$ (RH).

The Ratios Conjecture over function fields

$\mathcal{H}_n$ = squarefree polynomials of degree n in $\mathbb{F}_q[t]$.

Conjecture (Andrade, Keating, 2014)

$$\frac{1}{|\mathcal{H}_{2g+1}|}\sum_{D\in\mathcal{H}_{2g+1}}\frac{L(1/2+\alpha,\chi_D)}{L(1/2+\beta,\chi_D)}=\frac{\zeta_q(1+2\alpha)}{\zeta_q(1+\alpha+\beta)}A(\alpha,\beta)$$
$$+\,q^{-2g\alpha}\frac{\zeta_q(1-2\alpha)}{\zeta_q(1-\alpha+\beta)}A(-\alpha,\beta)+O(q^{-g+\epsilon g}),$$

where

$$A(\alpha,\beta)=\prod_P\left(1-\frac{1}{|P|^{1+\alpha+\beta}}\right)^{-1}\left(1-\frac{1}{(|P|+1)|P|^{1+2\alpha}}-\frac{1}{(|P|+1)P|^{\alpha+\beta}}\right),$$

and for

$$|\Re\alpha|<\frac{1}{4},\ \ \frac{1}{g}\ll\Re\beta<\frac{1}{4}.$$

Heuristic arguments

- Using the approximate functional equation:

$$L(1/2+\alpha,\chi_D) = \sum_{\deg(f)\le g} \frac{\chi_D(f)}{|f|^{1/2+\alpha}} + q^{-2g\alpha} \sum_{\deg(f)\le g-1} \frac{\chi_D(f)}{|f|^{1/2-\alpha}}.$$

- $$L(1/2+\beta,\chi_D)^{-1} = \sum_{h \text{ monic}} \frac{\mu(h)\chi_D(h)}{|h|^{1/2+\beta}}$$

- The first piece from the approximate functional equation gives

$$\frac{1}{|\mathcal{H}_{2g+1}|} \sum_{f,h} \frac{\mu(h)}{|h|^{1/2+\beta}|f|^{1/2+\alpha}} \sum_{D\in\mathcal{H}_{2g+1}} \chi_D(fh),$$

  and keep $fh = \square$. Rewrite $f \mapsto f^2 h$.

- $$\sum_{f,h} \frac{\mu(h)}{|h|^{1+\alpha+\beta}|f|^{1+2\alpha}} \prod_{P|fh} \left(1+\frac{1}{|P|}\right)^{-1} \Longrightarrow \frac{\zeta_q(1+2\alpha)}{\zeta_q(1+\alpha+\beta)} A(\alpha,\beta).$$

The Ratios Conjecture over function fields

Theorem (Bui, F., Keating, 2021)

- *For* $\frac{1}{g^{\frac{1}{2}-\epsilon}} \ll \Re\beta$, *and* $|\Re\alpha| < 1/2$, *we have*

$$\frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D\in\mathcal{H}_{2g+1}} \frac{L(1/2+\alpha,\chi_D)}{L(1/2+\beta,\chi_D)} = \frac{\zeta_q(1+2\alpha)}{\zeta_q(1+\alpha+\beta)} A(\alpha,\beta)$$
$$+ q^{-2g\alpha} \frac{\zeta_q(1-2\alpha)}{\zeta_q(1-\alpha+\beta)} A(-\alpha,\beta) + O\Big(q^{-g\Re\beta(3+2\alpha-\epsilon)}\Big).$$

- $k=2$*: RC for* $\frac{1}{g^{1/4-\epsilon}} \ll \Re\beta_1, \Re\beta_2$ *and* $|\Re\alpha_1|, |\Re\alpha_2| < \frac{1}{4}$.
- $k=3$*: RC for* $\frac{1}{g^{1/6-\epsilon}} \ll \Re\beta_1, \Re\beta_2, \Re\beta_3$ *and* $|\Re\alpha_1|, |\Re\alpha_2|, |\Re\alpha_3| < \frac{1}{16}$.

Upper bounds for negative moments of L–functions

The main ingredients in the proof are:

- Obtaining asymptotic formulas for twisted, shifted moments of L–functions
- Obtaining upper bounds for negative moments of L–functions; builds on work of Soundararajan, Harper, Radziwill-Soundararajan

Theorem (Bui, F., Keating, 2021)

Let k be a positive integer such that $k > 1/2$. Let $\frac{1}{g^{\frac{1}{2k}-\epsilon}} \ll \beta$. Then

$$\frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} \frac{1}{|L(1/2+\beta+it, \chi_D)|^k} \ll \left(\frac{1}{\beta}\right)^{k(k-1)/2} (\log g)^{k(k+1)/2}.$$

Theorem (F., 2021)

The bound above and the Ratios Conjecture hold for $\beta \gg \frac{\log g}{g}$.

Ideas of proof

$$\frac{1}{|\mathcal{H}_{2g+1}|}\sum_{D\in\mathcal{H}_{2g+1}}\frac{L(1/2+\alpha,\chi_D)}{L(1/2+\beta,\chi_D)}=\frac{1}{|\mathcal{H}_{2g+1}|}\sum_{D\in\mathcal{H}_{2g+1}}L(1/2+\alpha,\chi_D)$$
$$\times\sum_{h\text{ monic}}\frac{\mu(h)}{|h|^{1/2+\beta}}\chi_D(h)$$

For some parameter X to be chosen later, let

$$S_{\leq X}=\frac{1}{|\mathcal{H}_{2g+1}|}\sum_{D\in\mathcal{H}_{2g+1}}L(1/2+\alpha,\chi_D)\sum_{\deg(h)\leq X}\frac{\mu(h)}{|h|^{1/2+\beta}}\chi_D(h),$$

$$S_{>X}=\frac{1}{|\mathcal{H}_{2g+1}|}\sum_{D\in\mathcal{H}_{2g+1}}L(1/2+\alpha,\chi_D)\sum_{\deg(h)>X}\frac{\mu(h)}{|h|^{1/2+\beta}}\chi_D(h).$$

Ideas of proof

Using Perron's formula for the sum over h, we rewrite

$$S_{>X} = \frac{1}{|\mathcal{H}_{2g+1}|} \frac{1}{2\pi i} \oint \sum_{D \in \mathcal{H}_{2g+1}} \frac{L(1/2+\alpha, \chi_D)}{\mathcal{L}\Big(\frac{z}{q^{1/2+\beta}}, \chi_D\Big) z^X (z-1)} \frac{dz}{z},$$

where we are integrating over a circle $|z| > 1$. We pick $|z| = q^{\Re\beta/2}$, use Holder's inequality for the sum over D and then use upper bounds for negative moments of L–functions.

A key inequality

We have

$$\log|L(1/2+\beta+it,\chi_D)| \geq \frac{2g}{N+1}\log\left(1-q^{-(N+1)\beta}\right) + \Re\Bigg(\sum_{\deg(P)\leq N}\frac{\chi_D(P)}{|P|^{1/2+\beta+it}}\Bigg) + \Re\Bigg(\frac{1}{2}\sum_{\deg(P)\leq N/2}\frac{1}{|P|^{1+2\beta+2it}}\Bigg) + O(1),$$

so

$$\frac{1}{|L(1/2+\beta+it,\chi_D)|^k} \leq \sqrt{N}^k\left(\frac{1}{1-q^{-(N+1)\beta}}\right)^{\frac{2gk}{N+1}} \times \exp\Bigg(k\sum_{\deg(P)\leq N}\frac{\chi_D(P)a(P)}{|P|^{1/2+\beta}}\Bigg),$$

where $a(P) = -\cos(t\deg(P)\log q)$.
Think of $N \approx \frac{\log g}{\beta}$.

The upper bound

- Pointwise bound

$$\frac{1}{|L(1/2+\beta+it,\chi_D)|} \leq \left(\frac{1}{1-g^{-2\beta}}\right)^{\frac{(1+\epsilon)g}{\log_q g}}.$$

- Use the inequality

$$e^t \leq (1+e^{-\ell/2})\sum_{s\leq \ell}\frac{t^s}{s!},$$

for $t \leq \ell/e^2$ and ℓ even.

- Split the primes into K intervals, where $K \sim c\log(1/\beta)$:

$$I_0 = (0, N_0],\ I_1 = (N_0, N_1], \ldots,\ I_K = (N_{K-1}, N_K].$$

The upper bound

- If the contribution from primes in I_0 is "big" (call this set of discriminants $\mathcal{T}_0$) i.e: if

$$\Big| \sum_{\deg(P) \in I_0} \frac{\chi_D(P) a(P)}{|P|^{1/2+\beta}} \Big| > \frac{\ell_0}{ke^2},$$

  we want to exploit the fact that $|\mathcal{T}_0|$ is small.

$$\sum_{D \in \mathcal{T}_0} \frac{1}{|L(1/2+\beta+it, \chi_D)|^k} \leq \sum_{D \in \mathcal{H}_{2g+1}} \frac{1}{|L(1/2+\beta+it, \chi_D)|^k} \times \Big(\frac{ke^2}{\ell_0} \sum_{\deg(P) \in I_0} \frac{\chi_D(P) a(P)}{|P|^{1/2+\beta}} \Big)^{s_0},$$

  for some even parameter s_0 for which $s_0 N_0 \leq g$.

The upper bound

- $$\sum_{D\in\mathcal{H}_{2g+1}} \frac{1}{|L(1/2+\beta+it,\chi_D)|^k} \Big(\frac{ke^2}{\ell_0} \sum_{\deg(P)\in I_0} \frac{\chi_D(P)a(P)}{|P|^{1/2+\beta}}\Big)^{s_0}$$
$$\leq \Big(\sum_{D\in\mathcal{H}_{2g+1}} \frac{1}{|L(1/2+\beta+it,\chi_D)|^{2k}}\Big)^{1/2} \frac{k^{s_0}e^{2s_0}}{\ell_0^{s_0}}$$
$$\times \Big(\sum_{D\in\mathcal{H}_{2g+1}} \Big(\sum_{\deg(P)\in I_0} \frac{\chi_D(P)a(P)}{|P|^{1/2+\beta}}\Big)^{2s_0}\Big)^{1/2}$$
- For the first term, use the pointwise bound for the L–function; for the second term, we compute the moments and keep the diagonal pieces. Choose $\ell_0 = s_0$.

The upper bound

- $$\sum_{D\in\mathcal{M}_{2g+1}}\Big(\sum_{\deg(P)\in I_0}\frac{\chi_D(P)a(P)}{|P|^{1/2+\beta}}\Big)^{2s_0}$$
$$= (2s_0)!\sum_{D\in\mathcal{M}_{2g+1}}\sum_{\substack{\Omega(f)=2s_0\\ P|f\Rightarrow\deg(P)\in I_0}}\frac{\chi_D(f)a(f)\nu(f)}{|f|^{1/2+\beta}},$$
  where ν is the multiplicative function given by $\nu(P^a)=1/a!$.
- Only keep $f=\square$. Get
$$\ll q^{2g}\frac{(2s_0)!}{s_0!}\Big(\sum_{\deg(P)\in I_0}\frac{1}{|P|}\Big)^{s_0}\ll q^{2g}\frac{(2s_0)!}{s_0!}(\log N_0)^{s_0}.$$

The upper bound

- $$\sum_{D\in\mathcal{T}_0}\frac{1}{|L(1/2+\beta+it,\chi_D)|^k} \ll q^{2g}\left(\frac{1}{1-g^{-2\beta}}\right)^{\frac{(1+\epsilon)g}{\log_q g}}$$
$$\times \exp\left(-\frac{s_0}{2}\log s_0\right)\exp(cs_0\log\log N_0) = o(q^{2g}),$$

for

$$s_0 = 2\Big[\frac{(2+3\epsilon)g\log q}{2(\log g)^2}\log\left(\frac{1}{1-g^{-2\beta}}\right)\Big],\ \ N_0=[g/s_0],$$
$$N_j = rN_{j-1},\ \ s_j=[ag/N_j].$$

The upper bound

- If the contribution from primes in I_0 is "small", move to the next interval and proceed as before.
- . . .
- The contribution from the term for which the sums over each interval are small will be bounded by

$$\ll \exp\left(\frac{2gk}{N_K}\log\left(\frac{1}{1-q^{-N_K\beta}}\right)\right)N_K^{\frac{k}{2}}\prod_{r=0}^{K}\sum_{s\leq\ell_r}\frac{1}{s!}\Big(\sum_{\deg(P)\in I_r}\frac{k\chi_D(P)a(P)}{|P|^{1/2+\beta}}\Big)^s$$

$$\ll \exp\left(\frac{2gk}{N_K}\log\left(\frac{1}{1-q^{-N_K\beta}}\right)\right)N_K^{\frac{k}{2}}$$

$$\times\prod_{r=0}^{K}\prod_{\substack{P|f_r\Rightarrow\deg(P)\in I_r\\ \Omega(f_r)\leq\ell_r}}\frac{k^{\Omega(f_r)}\chi_D(f_r)a(f_r)\nu(f_r)}{|f_r|^{1/2+\beta}},$$

where ν is the multiplicative function defined by $\nu(P^a)=1/a!$.

The upper bound

- Keeping the $f_r = \square$, this will be

$$\ll \exp\left(\frac{2gk}{N_K}\log\left(\frac{1}{1-q^{-N_K\beta}}\right)\right)N_K^{k/2}\prod_{\deg(P)\le N_K}\left(1-\frac{k^2}{2|P|}\right)^{-1}$$

$$\ll \exp\left(\frac{2gk}{N_K}\log\left(\frac{1}{1-q^{-N_K\beta}}\right)\right)N_K^{k^2/2+k/2}.$$

- With the choice $N_K \approx \frac{\log g}{\beta}$, the bound follows.

The asymptotic formula for $S_{\leq X}$

Recall

$$S_{\leq X} = \frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} L(1/2+\alpha, \chi_D) \sum_{\deg(h) \leq X} \frac{\mu(h)}{|h|^{1/2+\beta}} \chi_D(h),$$

so we need asymptotic formulas for

$$\sum_{D \in \mathcal{H}_{2g+1}} L(1/2+\alpha, \chi_D) \chi_D(h)$$

for h of small enough degree.

- $\alpha = 0, h = 1$: Andrade-Keating; F.
- $\alpha = 0$: Bui-F.

Sketch of proof

Use standard techniques such as

- Approximate functional equation
- Poisson summation
- Upper bounds for positive moments of L–functions, etc.

- Approximate functional equation

$$L\left(\tfrac{1}{2}, \chi_D\right) = \sum_{f \in \mathcal{M}_{\leq g}} \frac{\chi_D(f)}{\sqrt{|f|}} + \sum_{f \in \mathcal{M}_{\leq g-1}} \frac{\chi_D(f)}{\sqrt{|f|}}.$$

- $$\sum_{D \in \mathcal{H}_{2g+1}} \chi_D(f) = \sum_{C \mid f^\infty} \sum_{h \in \mathcal{M}_{2g+1-2d(C)}} \chi_f(h) - q \sum_{C \mid f^\infty} \sum_{h \in \mathcal{M}_{2g-1-2d(C)}} \chi_f(h)$$

Poisson summation formula

- For $a \in \mathbb{F}_q((\frac{1}{x}))$, $a = \displaystyle\sum_{i=-\infty}^{\infty} a_i \left(\frac{1}{x}\right)^i$, define
$$e(a) = e^{\frac{2\pi i a_1}{q}}.$$
- Generalized Gauss sums $G(V, \chi_f) = \displaystyle\sum_{u \pmod f} \chi_f(u) e\left(\frac{uV}{f}\right)$.
- If $\deg(f) = n$ is even, then
$$\sum_{h \in \mathcal{M}_m} \chi_f(h) = \frac{q^m}{|f|}\Bigg[G(0, \chi_f) + (q-1) \sum_{V \in \mathcal{M}_{\leq n-m-2}} G(V, \chi_f) - \sum_{V \in \mathcal{M}_{n-m-1}} G(V, \chi_f)\Bigg]$$

Setup

$$\sum_{D\in\mathcal{H}_{2g+1}} L\left(\tfrac{1}{2},\chi_D\right) = S_g + S_{g-1} + O(q^{\frac{g}{2}(1+\epsilon)}),$$

where

$$S_g = q^{2g+1} \sum_{\substack{f\in\mathcal{M}_{\leq g} \\ \deg(f)\text{ even}}} \frac{1}{|f|} \sum_{C|f^\infty} \frac{1}{|C|^2} \Bigg[G(0,\chi_f) +$$

$$(q-1) \sum_{V\in\mathcal{M}_{\leq \deg(f)-2g-3+2d(C)}} \frac{G(V,\chi_f)}{\sqrt{|f|}} - \sum_{V\in\mathcal{M}_{\deg(f)-2g-2+2d(C)}} \frac{G(V,\chi_f)}{\sqrt{|f|}} \Bigg]$$

- $G(0,\chi_f) \neq 0$ iff $f = \square$, in which case $G(0,\chi_f) = \phi(f)$.

Main term from $V = 0$

$$M_{g-1} = q^{2g+1} \sum_{\substack{f \in \mathcal{M}_{\leq g-1} \\ \deg(f) \text{ even}}} \frac{1}{|f|} \sum_{C | f^\infty} \frac{1}{|C|^2} G(0, \chi_f)$$

$$= \frac{q^{2g+1}}{\zeta_q(2)} \sum_{l \in \mathcal{M}_{\leq [\frac{g-1}{2}]}} \frac{1}{|l|} \left(\prod_{P | l} \frac{|P|}{|P|+1} \right)$$

$$= \frac{q^{2g+1}}{\zeta_q(2)} \frac{1}{2\pi i} \oint_{|u|<\frac{1}{q}} \mathcal{C}(u) \frac{(qu)^{-[\frac{g-1}{2}]}}{(1-qu)^2} \frac{du}{u},$$

$$\mathcal{C}(u) = \prod_P \left(1 - \frac{u^{d(P)}}{|P|+1} \right).$$

$V \neq 0$

- $$\sum_f \frac{G(V,\chi_f)}{\sqrt{|f|}} w^{d(f)} = \mathcal{L}(w,\chi_V) \prod_P \mathcal{R}_P(w).$$

  $\Rrightarrow$ Focus on $V = \square = l^2$.

$$S_g(V=\square) = q^{2g+1} \sum_{\substack{f \in \mathcal{M}_{\leq g} \\ \deg(f) \text{ even}}} \frac{1}{|f|} \sum_{C | f^\infty} \frac{1}{|C|^2} \Bigg[(q-1) \cdot \sum_{l \in \mathcal{M}_{\leq \frac{\deg(f)}{2} - g - 2 + d(C)}} \frac{G(l^2,\chi_f)}{\sqrt{|f|}} - \sum_{l \in \mathcal{M}_{\frac{\deg(f)}{2} - g - 1 + d(C)}} \frac{G(l^2,\chi_f)}{\sqrt{|f|}} \Bigg]$$

- For the term $S_g(V=\square)$, since $C|f^\infty$, we roughly have $\deg(C) \leq \deg(f)$. Combining this with the condition on the degree of l implies that $\deg(f) \geq 2g/3$. Thus, heuristically we have a sum of the form $\sum_{\frac{2g}{3} \leq \deg(f) \leq g} 1$.

$$M_{g-1} = \frac{q^{2g+1}}{\zeta_q(2)} \oint_{|u|<\frac{1}{q}} \mathcal{C}(u) \frac{(qu)^{-[\frac{g-1}{2}]}}{(1-qu)^2} \frac{du}{u}$$

$$S_g(V=\square) = q^{\frac{2g+1}{3}} P_1(g) - \frac{q^{2g+1}}{\zeta_q(2)} \oint_{|u|=\sqrt{q}} \mathcal{C}(u) \frac{(qu)^{-[\frac{g-1}{2}]}}{(1-qu)^2} \frac{du}{u}$$

$$M_{g-1} + S_g(V=\square) = -\mathrm{Res}(u=\frac{1}{q}) + q^{\frac{2g+1}{3}} P_1(g).$$

- By matching up the terms, we get rid of the error of size $q^{g(1+\epsilon)}$.
- The term corresponding to $V \neq \square$ is bounded by $q^{\frac{g}{2}(1+\epsilon)}$.

Thank you!