Quiver Quantum Toroidal Algebra and Crystal Representations

Go Noshita

Department of Physics,
The University of Tokyo

MS seminar @Kavli, IPMU

- Based on arXiv: 2108.07104, 2109.02045 with A. Watanabe.
- See also arXiv: 2101.03953 with K. Harada, Y. Matsuo, and A. Watanabe for a related work.

Contents

Introduction: Affine Yangian (AY) $\mathfrak{gl}_1$ and corner vertex operator algebra

- AGT correspondence [Alday-Gaiotto-Tachikawa 2009]
  Nekrasov partition functions $\leftrightarrow$ conformal blocks of Liouville/Toda CFT
  4D supersymmetric gauge theory $\leftrightarrow$ algebra
- Virasoro and W_N algebra (free field realization)

$$\phi_i(z)\phi_j(w) \sim -\delta_{ij}\log(z-w),$$
$$R_i(z) = \alpha_0\partial + \partial\phi_i(z),$$
$$R_1R_2\cdots R_N = \sum_{k=0}^{N} U_k(z)(\alpha_0\partial)^{N-k}$$

- W_N algebra includes higher spin currents and is a generalization of Virasoro algebra. It is obtained by Miura transformation.

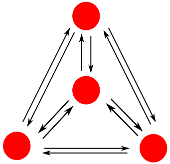
- Corner vertex operator algebra $Y_{L,M,N}$ [Gaiotto-Rapčák 2017]
  - Algebra appearing in brane junctions.
  - Understood as a pit reduction of the plane partition representation of affine Yangian $\mathfrak{gl}_1$. (AY picture).[Prochazak-Rapčák 2017]
    The central charge is specialized to
    $$-\psi_0\sigma = Lh_1 + Mh_2 + Nh_3, \quad \sigma = h_1h_2h_3.$$
  - Generalized Miura transformation with fractional power gives free field realizations. ($W_{1+\infty}$ picture)[Prochazka-Rapčák 2018]
    $$R^c =: (\alpha_0\partial + \frac{h_3}{h_c}J^{(c)})^{\frac{h_c}{h_3}} :, \quad J^{(c)}(z)J^{(c)}(w) = -\frac{h_c}{\sigma}\frac{1}{(z-w)^2}, \quad c = 1,2,3,$$
    $$R^{(c_1)}R^{(c_2)}\cdots R^{(c_{L+M+N})} = \sum_{s=0}^{\infty} U_s(z)\partial^{\frac{Lh_1+Mh_2+Nh_3}{h_3}-s}$$
  - $Y_{L,M,N}$ algebra is a generalization of the W_N algebra and can be understood as truncations of both AY $\mathfrak{gl}_1$ and $W_{1+\infty}$:
    $$\text{Virasoro} \subset W_N \subset Y_{L,M,N} \subset \text{AY } \mathfrak{gl}_1 / W_{1+\infty}$$
  - AY $\mathfrak{gl}_1 \simeq W_{1+\infty}$ Prochazak 2015, Gaberdiel et al. 2017

- Corner VOA should give new AGT dualities. Understanding such truncations should be useful. $\rightarrow$ We need new affine Yangians and new W-algebras.
- Quiver Yangian [Li-Yamazaki 2020, Galakhov-Yamazaki 2020, Galakhov-Li-Yamazaki 2021]
  - Generalizations of affine Yangian $\mathfrak{gl}_1$:

$$\text{Affine Yangian } \mathfrak{gl}_1 \rightarrow \text{Quiver Yangian}$$

$$\text{plane partition} \rightarrow \text{3D BPS crystal}$$ [Ooguri-Yamazaki]

  - Algebra is defined from the quiver associated with the toric Calabi-Yau 3-fold [Hanany et al.].



- Studying the W-algebra picture is also important.

Introduction: Deformed $\mathcal{W}$ algebras and 5D AGT

- There is a 5D lift-up of the correspondence (5D AGT).
- On the algebra side, quantum algebras appear (q-Virasoro, q-W_N, q-$Y_{L,M,N}$...)
- q-Virasoro [Shiraishi et al. 1995]: $q, t, p = q/t$

$$T(z) = \sum_{n\in\mathbb{Z}} T_z z^{-n},$$

$$f(w/z)T(z)T(w) - f(z/w)T(w)T(w) = -\frac{(1-q)(1-t^{-1})}{1-p}\left[\delta\left(\frac{pw}{z}\right) - \delta\left(\frac{p^{-1}w}{z}\right)\right],$$

$$f(z) = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\frac{(1-q^n)(1-t^{-n})}{1+p^n}z^n\right)$$

- In the limit $t = q^\beta, \quad q = e^h \to 1$

$$T(z) = 2 + \beta\left(z^2L(z) + \frac{(1-\beta)^2}{4\beta}\right)h^2 + \cdots,$$

where $L(z)$ is the Virasoro algebra.

- q-Virasoro and q-W_N are understood as truncations of quantum toroidal $\mathfrak{gl}_1$.
- Both the affine Yangian picture and W algebra picture are understood in a unified way by the quantum toroidal $\mathfrak{gl}_1$. We have two central charges $(C, C^{\perp})$,

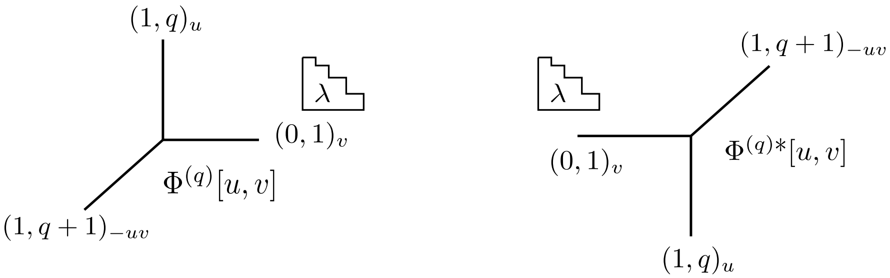
$$(C, C^{\perp}) = (1, q_1^{L/2} q_2^{M/2} q_3^{N/2}) \to \text{pit reduction picture}$$

$$(C, C^{\perp}) = (q_1^{L/2} q_2^{M/2} q_3^{N/2}, 1) \to \text{free field realizations}$$ [FHSSY, Harada et al., Bershtein et al., Kojima]

- The deformed algebra captures the algebraic structure in a rather symmetric way than the degenerate case.
  $\to$ basic motivation of considering quantum algebras
- Studying truncations as representation theory of quantum toroidal algebras is useful.
- Finding the trigonometric deformation of QY should give new perspectives.
  $\to$ quiver quantum toroidal algebra [GN-Watanabe 2021,Galakhov-Li-Yamazaki 2021]

Introduction: Intertwiner formalism

- One application of quantum toroidal algebras is the intertwiner formalism [Awata-Feigin-Shiraishi 2011].
- Using two basic representations "horizontal" and "vertical" representations, we can construct algebraic objects called "intertwiners". Composition of them gives Nekrasov partition functions.

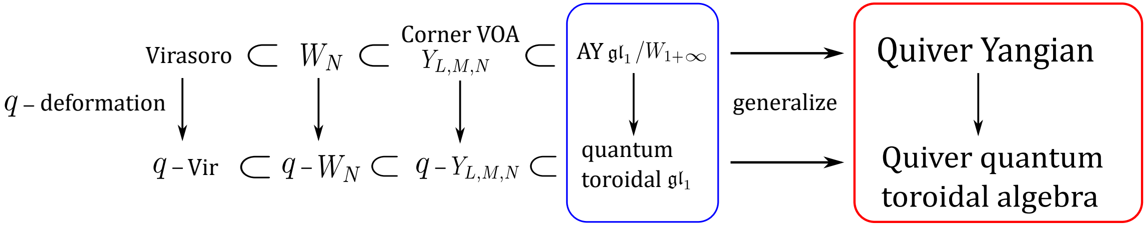


- If we change the vertical reps to other reps like Young diagrams with colored boxes and the horizontal reps to suitable vertex operator reps, we obtain other partition functions.

Algebra	Geometry	Gauge theory	Ref.
QT $\mathfrak{gl}_1$	$\mathbb{C}^3$	5D $\mathcal{N}=1$ on $\mathbb{C}^2 \times S^1$	[Awata-Feigin-Shiraishi 2011]
QT $\mathfrak{gl}_n$	$\mathbb{C}^2/\mathbb{Z}_n \times \mathbb{C}$, $\omega^n = 1$ $(x,y) \mapsto (\omega x, \omega^{-1} y)$	5D $\mathcal{N}=1$ on $\mathbb{C}^2/\mathbb{Z}_n \times S^1$ (x,y) plane	[Awata et al. 2017]
(ν_1,ν_2)-QT $\mathfrak{gl}_n$	$\mathbb{C}^3/\mathbb{Z}_n$, $\omega^n = 1$ $x_i \mapsto \omega^{\nu_i} x_i$ $\nu_1+\nu_2+\nu_3=0$	5D $\mathcal{N}=1$ on $\mathbb{C}^2/\mathbb{Z}_n \times S^1$ (x_1,x_2) plane	[Bourgine-Jeong 2019]
QQTA	Toric CY 3-fold	??	? ? ?

- Studying both reps entering the intertwiner is important.
  $\to$ We study two-dimensional crystal reps (vertical reps) here.
  $\to$ vertex operator reps (future work)

Goal



- Studying the quiver Yangian/quiver quantum toroidal algebra and their truncations should give generalizations of (q)-corner VOA.
  $\to$ Focus on one, two-dimensional crystal representations.

Affine Yangian (AY) $\mathfrak{gl}_1$

- Generators are

$$e(u) = \sum_{j=0}^{\infty} \frac{e_j}{u^{j+1}}, \quad f(u) = \sum_{j=0}^{\infty} \frac{f_j}{u^{j+1}}, \quad \psi(u) = 1 + \sigma \sum_{j=0}^{\infty} \frac{\psi_j}{u^{j+1}}$$

- Depends on parameters h_1, h_2, h_3

$$h_1 + h_2 + h_3 = 0, \quad \psi(u)\psi(v) \sim \psi(v)\psi(u),$$
$$e(u)e(v) \sim \varphi(u-v)e(v)e(u),$$
$$\psi(u)e(v) \sim \varphi(u-v)e(v)\psi(u), \text{ etc.}$$
$$\varphi(u) = \frac{(u+h_1)(u+h_2)(u+h_3)}{(u-h_1)(u-h_2)(u-h_3)}$$

- Symmetric under $h_1 \leftrightarrow h_2 \leftrightarrow h_3$.
- $\mathrm{AY}\mathfrak{gl}_1 \simeq W_{1+\infty}$

Quantum toroidal $\mathfrak{gl}_1$ [Ding-Iohara 1996, Miki 2007, Feigin et al.]

- Trigonometric deformation of AY $\mathfrak{gl}_1$.
- Generators are called Drinfeld currents:

$$E(z) = \sum_{m\in\mathbb{Z}} E_m z^{-m}, \quad F(z) = \sum_{m\in\mathbb{Z}} F_m z^{-m}, \; K^{\pm}(z) = (C^{\perp})^{\mp 1} \exp\left(\sum_{r>0} \pm H_{\pm r} z^{\mp r}\right)$$

- Defining relations are

$$q_1 q_2 q_3 = 1, \quad K^{-}(z)K^{+}(w) = \frac{\varphi(z, Cw)}{\varphi(Cz, w)} K^{+}(w)K^{-}(z)$$

$$E(z)E(w) = \varphi(z, w)E(w)E(z),$$

$$K^{\pm}(C^{(1\mp 1)/2}z)E(w) = \varphi(z, w)E(w)K^{\pm}(C^{(1\mp 1)/2}z), \text{etc.}$$

$$\varphi(z, w) = \prod_{i=1}^{3} \frac{(q_i^{1/2} z - q_i^{-1/2} w)}{(q_i^{-1/2} z - q_i^{1/2} w)}$$

Properties of quantum toroidal $\mathfrak{gl}_1$

- In the degenerate limit $q_i = e^{h_i} \to 1 + h_i$, it becomes AY $\mathfrak{gl}_1$.
- Coproduct structure

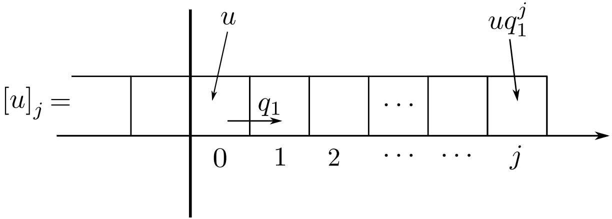
$$\begin{aligned}
\Delta E(z) &= E(z) \otimes 1 + K^-(C_1 z) \otimes E(C_1 z), \\
\Delta F(z) &= F(C_2 z) \otimes K^+(C_2 z) + 1 \otimes F(z), \\
\Delta K^+(z) &= K^+(z) \otimes K^+(C_1^{-1} z), \\
\Delta K^-(z) &= K^-(C_2^{-1} z) \otimes K^-(z), \text{ etc.}
\end{aligned}$$

- Triality: $q_1 \leftrightarrow q_2 \leftrightarrow q_3$
- Central elements are $C, C^\perp \to$ values of them determine representations
  1. Vertical representations $C = 1$: vector $(C^\perp = 1)$, Fock $(C^\perp = q_i^{1/2})$, MacMahon $(C^\perp = K)$
  2. Horizontal representations $C \neq 1$: Fock $(C, C^\perp) = (q_i^{1/2}, 1)$

Representations of quantum toroidal $\mathfrak{gl}_1$: Vector representation

- Vector representation

$$K^{\pm}(z)[u]_j = \left[\Psi_{[u]_j}(z)\right]_{\pm}[u]_j,$$
$$E(z)[u]_j = \mathcal{E}\delta\left(uq_1^{j+1}/z\right)[u]_{j+1},$$
$$F(z)[u]_{j+1} = \mathcal{F}\delta\left(uq_1^{j+1}/z\right)[u]_j$$

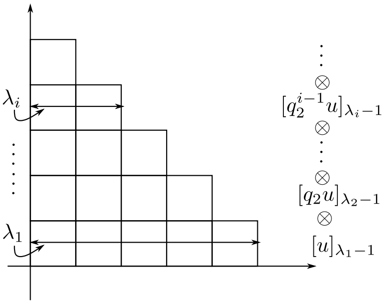


- $E(z)$ adds a box, $F(z)$ removes a box, and $K^{\pm}(z)$ acts diagonally.

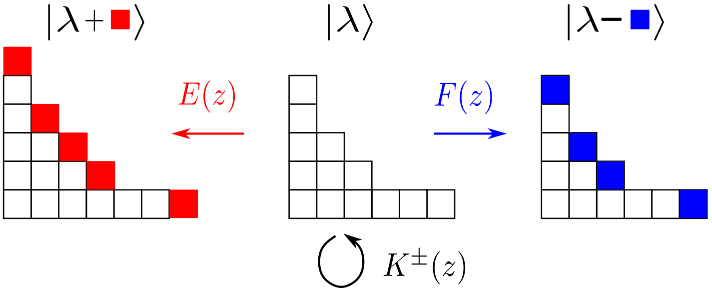
Representations of quantum toroidal $\mathfrak{gl}_1$: Fock representation

- Fock representation is obtained by taking tensor products of vector representations.

$$|\lambda\rangle = \otimes_{j=1}^{\infty}[uq_2^{j-1}]_{\lambda_j-1}$$

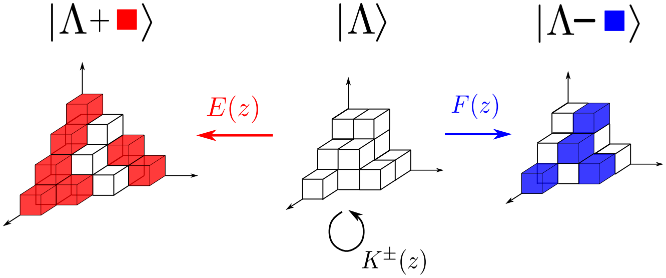


- $E(z)$ adds a box, $F(z)$ removes a box, and $K^{\pm}(z)$ acts diagonally.



Representations of quantum toroidal $\mathfrak{gl}_1$: MacMahon representation

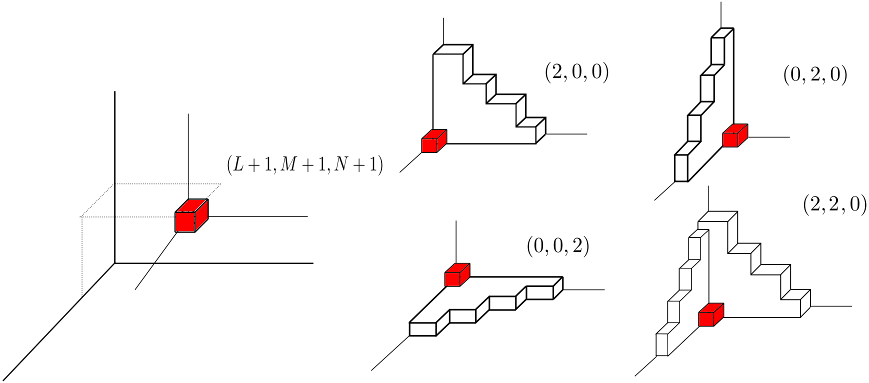
- Similar to the previous case, MacMahon representation is obtained by taking tensor products of Fock representations.
- $E(z)$ adds a box, $F(z)$ removes a box, and $K^{\pm}(z)$ acts diagonally.



- Actually, we can also consider the action of the algebra on plane partitions with nontrivial boundary conditions.
- Vertical representations captures the AY picture.

q-corner VOA from MacMahon representation

- Deformed corner VOA (q-$Y_{L,M,N}$) is understood as a pit reduction of the MacMahon representation. The central charge is $(C, C^{\perp}) = (1, q_1^{L/2} q_2^{M/2} q_3^{N/2})$.



Representations of quantum toroidal $\mathfrak{gl}_1$: Horizontal representation

- Horizontal representations are vertex operator representations: $\mathcal{F}_c(u)$, $(c = 1, 2, 3)$.
- The Drinfeld currents are vertex operators now:

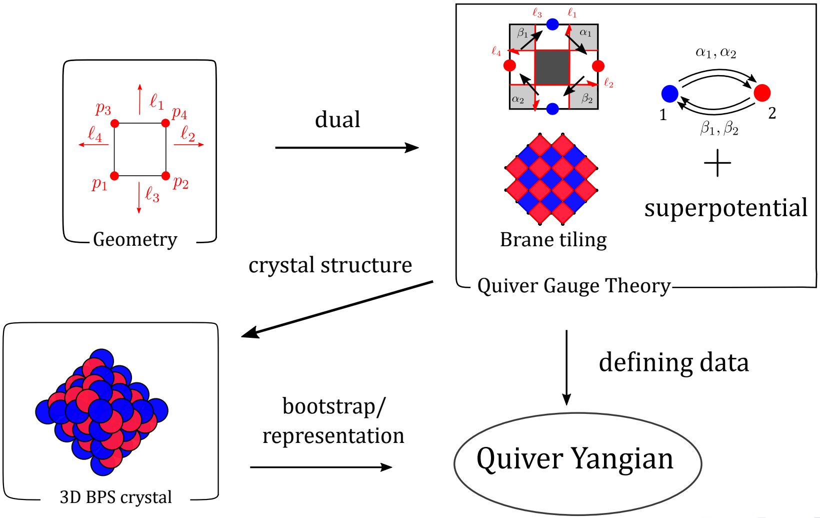
$$E(z) \to \eta_c(z), \quad F(z) \to \xi_c(z), \quad K^{\pm}(z) \to \varphi_c^{\pm}(z), \quad (c = 1, 2, 3)$$

$$\eta_c(z)\eta_c(w) = \frac{(1 - w/z)(1 - q_c^{-1}w/z)}{(1 - q_{c+1}w/z)(1 - q_{c-1}w/z)} : \eta_c(z)\eta_c(w) :,$$

$$\eta_c(z)\varphi_c^-(q_c^{1/2}w) = \frac{\prod_{i=1}^{3}(1 - q_i^{-1}w/z)}{\prod_{i=1}^{3}(1 - q_i w/z)} : \varphi_c^-(q_c^{1/2}w)\eta_c(z) :, \quad \text{etc.}$$

- Tensor products of these representations give free field realizations of deformed $\mathcal{W}$ algebras (q-Virasoro, q-W_N) or deformed corner vertex operator algebras (q-$Y_{L,M,N}$)[FHSSY 2010, Harada-Matsuo-GN-Watanabe 2021, Kojima 2019,2021].
- For the general q-$Y_{L,M,N}$, consider $\mathcal{F}_{\vec{c}}(\vec{u}) = \mathcal{F}_{c_1} \otimes \mathcal{F}_{c_2} \otimes \cdots \otimes \mathcal{F}_{c_n}$, where $\#\{i|c_i = 1\} = L$, $\#\{i|c_i = 2\} = M$, $\#\{i|c_i = 3\} = N$, $n = L + M + N$. The central charge is $(C, C^{\perp}) = (q_1^{L/2} q_2^{M/2} q_3^{N/2}, 1)$.

Quiver Yangian [Li-Yamazaki 2020, Galakhov-Yamazaki 2020, Galakhov-Li-Yamazaki 2021]



Definition of Quiver Yangian

- Quiver Yangian is defined from the quiver data associated with the geometry. It is generally a superalgebra.
- Quiver data: $Q = (Q_0, Q_1)$, W (vertices, arrows, and superpotential)
- Assign generators $e^{(a)}(u), f^{(a)}(u), \psi^{(a)}(u)$ for each vertex $a \in Q_0$

$$e^{(a)}(u) = \sum_{n=0}^{\infty} \frac{e_n^{(a)}}{u^{n+1}}, \quad \psi^{(a)}(u) = \sum_{n=-\infty}^{\infty} \frac{\psi_n^{(a)}}{u^{n+1}}, \quad f^{(a)}(u) = \sum_{n=0}^{\infty} \frac{f_n^{(a)}}{u^{n+1}}$$

- Statistics: If node $a \in Q_0$ has no self-loop, then $e^{(a)}(u), f^{(a)}(u)$ are fermionic. Otherwise, $e^{(a)}(u), f^{(a)}(u)$ are bosonic. The $\psi^{(a)}$'s are bosonic.
- Define bond factors as

$$\varphi^{a\Rightarrow b}(u) = (-1)^{\chi_{a\to b}} \frac{\prod_{I\in\{b\to a\}}(u+h_I)}{\prod_{I\in\{a\to b\}}(u-h_I)}, \quad \varphi^{a\Rightarrow b}(u)\varphi^{b\Rightarrow a}(-u) = 1$$

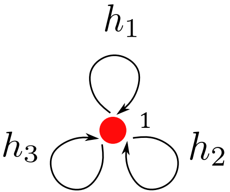
- Loop and vertex constraints are imposed on parameters h_I ($I \in Q_1$) (will not discuss).

- The defining relations are

$$\begin{aligned}\psi^{(a)}(u)\psi^{(b)}(v) &= \psi^{(b)}(v)\psi^{(a)}(u),\\ e^{(a)}(u)e^{(b)}(v) &\sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(u-v)e^{(b)}(v)e^{(a)}(u),\\ \psi^{(a)}(u)e^{(b)}(v) &\sim \varphi^{b\Rightarrow a}(u-v)e^{(b)}(v)\psi^{(a)}(u), \quad \text{etc.}\end{aligned}$$

- Generalization of AY $\mathfrak{gl}_1$:

$$\begin{aligned}\text{geometry} &\to \mathbb{C}^3,\\ \text{loop condition} &\to h_1+h_2+h_3=0,\\ \text{bond factor} &\to \varphi(u)=\frac{\prod_{i=1}^{3}(u+h_i)}{\prod_{i=1}^{3}(u-h_i)}.\end{aligned}$$



Quiver Quantum Toroidal Algebra (QQTA) [GN-Watanabe 2021, Galakhov-Li-Yamazaki 2021]

- QQTA is a trigonometric deformation of the quiver Yangian. Generally, it is a superalgebra.
- The defining data is the same $Q = (Q_0, Q_1)$, W.
- We focus on geometries when there are no compact 4-cycles, which gives non-chiral quivers.
- In this case, we assign parameters $q_I = e^{\epsilon h_I}$ to each edge $I \in Q_1$. They satisfy the loop and vertex constraints.
- The generators are

$$E_i(z) = \sum_{k\in\mathbb{Z}} E_{i,k} z^{-k}, \quad F_i(z) = \sum_{k\in\mathbb{Z}} F_{i,k} z^{-k}, \quad K_i^{\pm}(z) = \sum_{r\geq 0} K_{i,\pm r}^{\pm} z^{\mp r}, \quad (i \in Q_0).$$

- The statistics are assign similarly. Currents $E_i(z), F_i(z)$ are fermionic when node $i \in Q_0$ does not have a self-loop. Otherwise, they are bosonic. $K_i^{\pm}(z)$ are always bosonic.
- Bond factors are

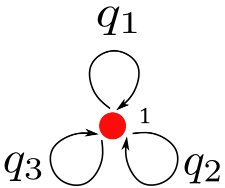
$$\varphi^{i\Rightarrow j}(z,w) = (-1)^{\chi_{i\to j}} \frac{\prod_{I\in\{j\to i\}} (q_I^{1/2} z - q_I^{-1/2} w)}{\prod_{I\in\{i\to j\}} (q_I^{-1/2} z - q_I^{1/2} w)}, \quad \varphi^{i\Rightarrow j}(z,w)\varphi^{j\Rightarrow i}(w,z) = 1.$$

- The defining relations are

$$K_i^-(z)K_j^+(w) = \frac{\varphi^{j\Rightarrow i}(z, Cw)}{\varphi^{j\Rightarrow i}(Cz, w)} K_j^+(w)K_i^-(z),$$
$$E_i(z)E_j(w) = (-1)^{|i||j|}\varphi^{j\Rightarrow i}(z, w)E_j(w)E_i(z),$$
$$K_i^\pm(C^{\frac{1\mp 1}{2}}z)E_j(w) = \varphi^{j\Rightarrow i}(z, w)E_j(w)K_i^\pm(C^{\frac{1\mp 1}{2}}z), \text{ etc.}$$

- Generalization of quantum toroidal $\mathfrak{gl}_1$:

$$\text{geometry} \to \mathbb{C}^3,$$
$$\text{loop condition} \to q_1q_2q_3 = 1,$$
$$\text{bond factor} \to \varphi(z, w) = \frac{\prod_{i=1}^3 (q_i^{1/2}z - q_i^{-1/2}w)}{\prod_{i=1}^3 (q_i^{-1/2}z - q_i^{1/2}w)}.$$



Properties

- Degenerate limit $\epsilon \to 0$:

$$q_I \to 1 + \epsilon h_I, \quad z \to 1 + \epsilon x, \quad w \to 1 + \epsilon y$$
$$\frac{\prod_{I\in\{j\to i\}}(q_I^{1/2} z - q_I^{-1/2} w)}{\prod_{I\in\{i\to j\}}(q_I^{-1/2} z - q_I^{1/2} w)} \to \frac{\prod_{I\in\{j\to i\}}(x - y + h_I)}{\prod_{I\in\{i\to j\}}(x - y + h_I)}$$

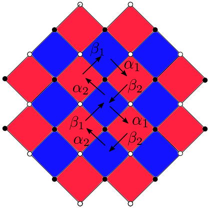
- It has a Hopf superalgebra structure. In particular, the coproduct is written as

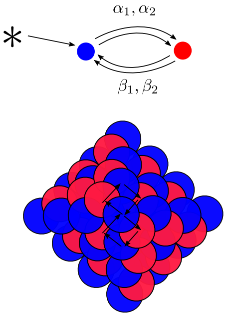
$$\begin{aligned}
\Delta E_i(z) &= E_i(z) \otimes 1 + K_i^-(C_1 z) \otimes E_i(C_1 z), \\
\Delta F_i(z) &= F_i(C_2 z) \otimes K_i^+(C_2 z) + 1 \otimes F_i(z), \\
\Delta K_i^+(z) &= K_i^+(z) \otimes K_i^+(C_1^{-1} z), \\
\Delta K_i^-(z) &= K_i^-(C_2^{-1} z) \otimes K_i^-(z)
\end{aligned}$$

- The coproduct gives various representations.

3D crystal representation

- As the quantum toroidal $\mathfrak{gl}_1$ acts on the plane partition, QQTA acts on some kind of 3D crystal. These 3D crystals are defined from the quiver data [Ooguri-Yamazaki 2008].
- From the quiver data, we can take copies of the periodic quiver and consider the universal covering (we also call this periodic quiver).
- Paths in the periodic quiver ↔ atoms in the crystal





- QQTA acts on these 3D crystals. $E_i(z)$ adds an atom of vertex i, $F_i(z)$ removes an atom of vertex i, and $K_i^{\pm}(z)$ acts diagonally.

$$K_i^{\pm}(z)\,|\Lambda\rangle = \left[\Psi_{\Lambda}^{(i)}(z,u)\right]_{\pm}|\Lambda\rangle\,,$$

$$E_i(z)\,|\Lambda\rangle = \sum_{\boxed{i}\in\mathrm{Add}(\Lambda)} E^{(i)}(\Lambda\to\Lambda+\boxed{i})\,\delta\left(\frac{z}{uq(\boxed{i})}\right)|\Lambda+\boxed{i}\rangle\,,$$

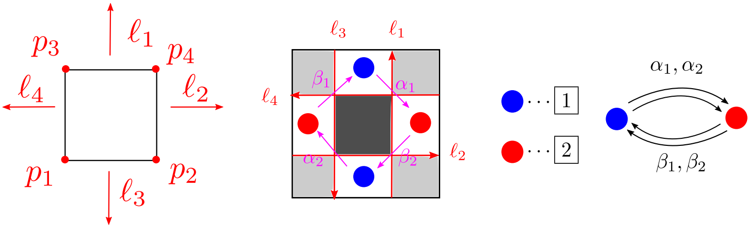
$$F_i(z)\,|\Lambda\rangle = \sum_{\boxed{\mathrm{i}}\in\mathrm{Rem}(\Lambda)} F^{(i)}(\Lambda\to\Lambda-\boxed{i})\,\delta\left(\frac{z}{uq(\boxed{i})}\right)|\Lambda-\boxed{i}\rangle$$

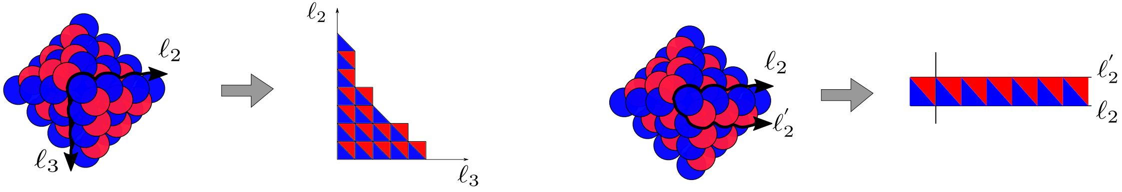
- It is a generalization of the quantum toroidal $\mathfrak{gl}_1$, where the 3D crystal is a plane partition. $\rightarrow$ MacMahon representation

Subcrystal

- Quantum toroidal $\mathfrak{gl}_1$ has not only MacMahon representation, but also vector and Fock representations.
- QQTA should also have such kind of representations.
- The generalizations of the Fock and vector representations are 2D and 1D crystal representations.
- The 2D crystals are surfaces of the 3D crystal [Nishinaka-Yamaguchi-Yoshida], while the 1D crystals are an extended version of the edges of the 2D crystals.
- The 2D crystals are associated with corner divisors of the toric diagram, while the 1D crystals are associated with the external legs of the toric diagram.

- Conifold $\to$ quantum toroidal $\mathfrak{gl}_{1|1}$
  2D crystal associated with divisor p_2
  1D crystal associated with ℓ_2



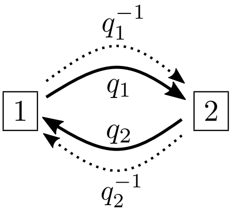


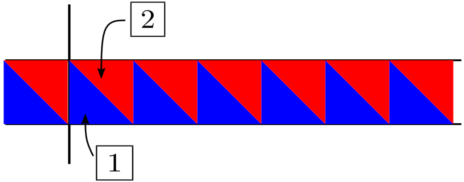
- Generally, a "shifted" version of the QQTA (shifted QQTA) acts on these crystals.
- The generators are modified as $K_i^{+}(z) \to z^{r_i} K_i^{+}(z)\,(i \in Q_0)$.
- $\mathbf{r} = (r_i)_{i\in Q_0}$ are shift parameters depending on the subcrystal considered.
- Denoting the shifted QQTA as $\ddot{\mathcal{U}}_Q^{\mathbf{r}}$, we also have a "shifted coproduct"

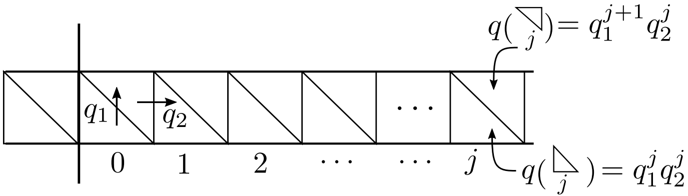
$$\Delta_{\mathbf{r},\mathbf{r}'} : \ddot{\mathcal{U}}_Q^{\mathbf{s}} \to \ddot{\mathcal{U}}_Q^{\mathbf{r}} \otimes \ddot{\mathcal{U}}_Q^{\mathbf{r}'}, \quad \mathbf{s} = \mathbf{r} + \mathbf{r}'.$$

- Using this shifted coproduct, we can take tensor products of the 1D crystal representations and obtain 2D crystal representations. This is a generalization of the story between vector and Fock representation of quantum toroidal $\mathfrak{gl}_1$.
- We expect we can obtain nontrivial three-dimensional subcrystals by taking tensor products of the 2D crystal representations.

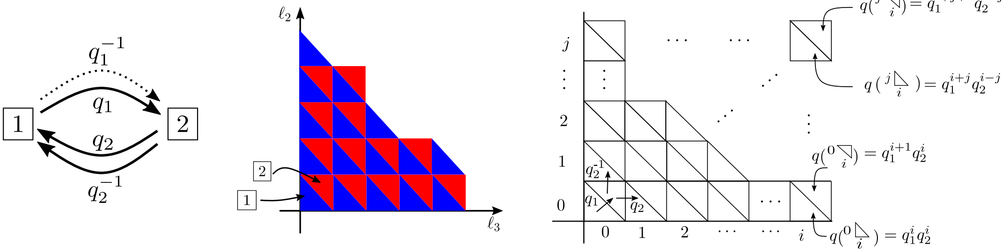
One-dimensional crystal representations







Two-dimensional crystal representations

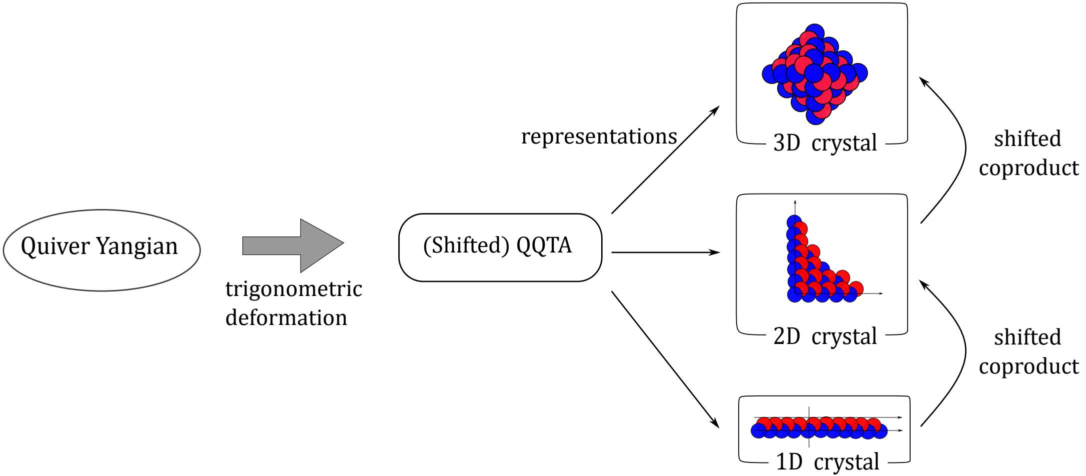


- Stacking the 1D crystal rep and using the shifted coproducts, we obtain the 2D crystal rep.
- The action of $K_s^{\pm}(z)$ on the vacuum $|\emptyset\rangle$ is

$$K_s^{\pm}(z)\,|\emptyset\rangle = \left[\frac{(q_1^{1/2}z - q_1^{-1/2}u)^{\delta_{s,2}}}{(z-u)^{\delta_{s,1}}}\right]_{\pm} |\emptyset\rangle\,.$$

- The shifts are $r_1 = -1$, $r_2 = 1$.

Summary


Quiver Yangian
trigonometric deformation
(Shifted) QQTA
representations
3D crystal
2D crystal
1D crystal
shifted coproduct
shifted coproduct

Future directions

- Deriving vertex operator representations ($C \neq 1$) of QQTA.
- Shifted intertwiner formalism.
- Quantum toroidal algebras associated with brane tilings including orientifolds.
- Deriving crystal representations of quantum toroidal algebras of DE-type orbifolds.
- Crystal representations and intertwiner formalism of general quantum N-toroidal algebras.
- Gluing of quantum toroidal algebras.
- $\cdots$

Appendix: Deformed $\mathcal{W}$ algebra and quantum toroidal $\mathfrak{gl}_1$

[FHSSY 2010, Bershtein-Feigin-Merzon 2015, Harada-Matsuo-GN-Watanabe 2021, Kojima 2019,2021]

- We have three horizontal representations denoted $\mathcal{F}_c(u)$ $(c = 1, 2, 3)$ [Bershtein-Feigin-Merzon 2015]:

$$[a_r, a_s] = r\frac{(q_c^{r/2} - q_c^{-r/2})^3}{-\kappa_r}\delta_{r+s,0}, \quad \kappa_r = \prod_{i=1}^{3}(q_i^{r/2} - q_i^{-r/2}),$$

$$E(z) \to \frac{1-q_c}{\kappa_1}\eta_c(z), \quad F(z) \to \frac{1-q_c^{-1}}{\kappa_1}\xi_c(z), \quad K^{\pm}(z) \to \varphi_c^{\pm}(z),$$

$$\eta_c(z) = u\exp\left(\sum_{r=1}^{\infty}\frac{q_c^{-r/2}\kappa_r}{r(q_c^{r/2} - q_c^{-r/2})^2}a_{-r}z^r\right)\exp\left(\sum_{r=1}^{\infty}\frac{\kappa_r}{r(q_c^{r/2} - q_c^{-r/2})^2}a_r z^{-r}\right),$$

$$\xi_c(z) = u^{-1}\exp\left(\sum_{r=1}^{\infty}\frac{-\kappa_r}{r(q_c^{r/2} - q_c^{-r/2})^2}a_{-r}z^r\right)\exp\left(\sum_{r=1}^{\infty}\frac{-q_c^{r/2}\kappa_r}{r(q_c^{r/2} - q_c^{-r/2})^2}a_r z^{-r}\right),$$

$$\varphi_c^{\pm}(z) = \exp\left(\sum_{r=1}^{\infty}\frac{-\kappa_r}{r(q_c^{r/2} - q_c^{-r/2})}a_{\pm r}z^{\mp r}\right)$$

- Deformed $\mathcal{W}_N$ algebra and deformed corner vertex operator algebra q-$Y_{L,M,N}$ are obtained by tensor products of the horizontal representations of quantum toroidal $\mathfrak{gl}_1$. [FHSSY 2010, Harada-Matsuo-GN-Watanabe 2021, Kojima 2019,2021]
- For the general q-$Y_{L,M,N}$, consider $\mathcal{F}_{\vec{c}}(\vec{u}) = \mathcal{F}_{c_1} \otimes \mathcal{F}_{c_2} \otimes \cdots \otimes \mathcal{F}_{c_n}$, where $\#\{i|c_i = 1\} = L$, $\#\{i|c_i = 2\} = M$, $\#\{i|c_i = 3\} = N$, $n = L + M + N$.
- Actually, there is an extra Heisenberg algebra and we need to remove it. We can define a current $t(z)$ that commutes with the Heisenberg H_r as

$$E(z) \to t(z) = \alpha(z)E(z)\beta(z), \quad H_r \to \frac{a_r}{q_c^{r/2} - q_c^{-r/2}}$$

$$\alpha(z) = \exp\left(\sum_{r=1}^{\infty} \frac{-\kappa_r}{r(1-q_c^r)} H_{-r} z^r\right), \quad \beta(z) = \exp\left(\sum_{r=1}^{\infty} \frac{-q_c^{-r/2}\kappa_r}{r(1-q_c^{-r})} H_r z^{-r}\right).$$

- Choose the Drinfeld current $E(z)$ and apply the coproduct $\Delta^{(n-1)}$:

$$\Delta^{n-1}(E(z)) \to \sum_{i=1}^{n} y_i \Lambda_i(z), \quad y_i = \frac{q_{c_i}^{1/2} - q_{c_i}^{-1/2}}{q_3^{1/2} - q_3^{-1/2}}$$

$$\Lambda_i(z) = \underbrace{1 \otimes \cdots \otimes 1}_{i-1} \otimes \eta_{c_i}(q_{c_{i+1}}^{-1/2} \cdots q_{c_n}^{-1/2} z) \otimes \varphi^+_{c_{i+1}}(q_{c_{i+1}}^{-1/2} \cdots q_{c_n}^{-1/2} z) \otimes \cdots \otimes \varphi^+_{c_n}(q_{c_n}^{-1/2} z).$$

- After the decoupling process, we obtain

$$\Delta^{n-1}(t(z)) \to \sum_{i=1}^{n} y_i \tilde{\Lambda}_i(z).$$

- Define currents

$$T_m(z) = t(q_3^{-m+1}z)t(q_3^{-m+2}z)\cdots t(z)$$

and these are the currents of deformed corner VOA. Actually they satisfy the following quadratic relation:

$$f_{i,j}\left(\frac{q_3^{\frac{i-j}{2}}w}{z}\right)T_i(z)T_j(w) - f_{j,i}\left(\frac{q_3^{\frac{j-i}{2}}z}{w}\right)T_j(w)T_i(z) = \frac{(q_1^{\frac{1}{2}}-q_1^{-\frac{1}{2}})(q_2^{\frac{1}{2}}-q_2^{-\frac{1}{2}})}{q_3^{\frac{1}{2}}-q_3^{-\frac{1}{2}}}$$

$$\times\sum_{k=1}^{i}\prod_{l=1}^{k-1}\frac{(1-q_1q_3^{-l})(1-q_2q_3^{-l})}{(1-q_3^{-l-1})(1-q_3^{-l})}\left(\delta\left(\frac{q_3^k w}{z}\right)f_{i-k,j+k}(q_3^{\frac{i-j}{2}})T_{i-k}(q_3^{-k}z)T_{j+k}(q_3^k w)\right.$$

$$\left. -\,\delta\left(\frac{q_3^{i-j-k}w}{z}\right)f_{i-k,j+k}(q_3^{\frac{j-i}{2}})T_{i-k}(z)T_{j+k}(w)\right), \quad (i\leq j)$$

- The function $f_{i,j}(z)$ does not depend on the ordering of the tensor products.
  $\rightarrow$ relation with R-matrix (?)

$$f_{i,j}(z) = \exp\left[\sum_{m=1}^{\infty} \frac{1}{m}(q_3^{\frac{i}{2}m} - q_3^{-\frac{i}{2}m})(q_{\vec{c}}^{\frac{m}{2}} q_3^{-\frac{j}{2}m} - q_{\vec{c}}^{-\frac{m}{2}} q_3^{\frac{j}{2}m}) \frac{(q_1^{\frac{m}{2}} - q_1^{-\frac{m}{2}})(q_2^{\frac{m}{2}} - q_2^{-\frac{m}{2}})}{(q_{\vec{c}}^{\frac{m}{2}} - q_{\vec{c}}^{-\frac{m}{2}})(q_3^{\frac{m}{2}} - q_3^{-\frac{m}{2}})} z^m\right] \quad (i \leq j).$$

- Examples:
  1. q-Virasoro=$\mathcal{F}_3 \otimes \mathcal{F}_3$ [Shiraishi et al. 1995]
  2. q-W_N=$\underbrace{\mathcal{F}_3 \otimes \cdots \otimes \mathcal{F}_3}_{N}$ [Awata et al. 1996, Feigin-Frenkel 1996]
  3. $W_{q,t}(\mathfrak{sl}_{2,1})$=$\mathcal{F}_3 \otimes \mathcal{F}_3 \otimes \mathcal{F}_2$ [Ding-Feigin 1999, Kojima 2019]
- We can also use $F(z)$ instead of $E(z)$ to get the same algebra.
- By some manipulation, we can also define a generating function of the currents which is a q-deformation of the generalized Miura transformation defined in [Prochazka-Rapčák 2018].

- q-deformed Miura operators

$$R^{(c)}(z)=\sum_{n=0}^{\infty}:\prod_{j=1}^{n}\left(-\frac{q_c^{\frac{1}{2}}q_3^{-\frac{j-1}{2}}-q_c^{-\frac{1}{2}}q_3^{\frac{j-1}{2}}}{q_3^{\frac{j}{2}}-q_3^{-\frac{j}{2}}}\Lambda(q_3^{-j+1}z)\right):q_3^{-nD_z},$$

$$R_1^{(c_1)}R_2^{(c_2)}\cdots R_n^{(c_n)}=\sum_{m=0}^{\infty}(-1)^m T_m(z)q_3^{-mD_z},$$

$$D_z=z\frac{d}{dz}$$

Appendix: Affine Yangian $\mathfrak{gl}_1$

- Generators and defining relations:

$$
\begin{aligned}
&e_n, \quad f_n, \quad \psi_n \quad (n \in \mathbb{Z}_{\geq 0}),\\
&\sigma_2 = h_1 h_2 + h_1 h_3 + h_2 h_3, \quad \sigma_3 = h_1 h_2 h_3,\\
&0 = [\psi_m, \psi_n],\\
&0 = [e_{m+3}, e_n] - 3[e_{m+2}, e_{n+1}] + 3[e_{m+1}, e_{n+2}] - [e_m, e_{n+3}]\\
&\quad + \sigma_2 [e_{m+1}, e_n] - \sigma_2 [e_m, e_{n+1}] - \sigma_3 \{e_m, e_n\},\\
&0 = [f_{m+3}, f_n] - 3[f_{m+2}, f_{n+1}] + 3[f_{m+1}, f_{n+2}] - [f_m, f_{n+3}]\\
&\quad + \sigma_2 [f_{m+1}, f_n] - \sigma_2 [f_m, f_{n+1}] + \sigma_3 \{f_m, f_n\},\\
&0 = [e_m, f_n] - \psi_{m+n},\\
&0 = [\psi_{m+3}, e_n] - 3[\psi_{m+2}, e_{n+1}] + 3[\psi_{m+1}, e_{n+2}] - [\psi_m, e_{n+3}]\\
&\quad + \sigma_2 [\psi_{m+1}, e_n] - \sigma_2 [\psi_m, e_{n+1}] - \sigma_3 \{\psi_m, e_n\},\\
&0 = [\psi_{m+3}, f_n] - 3[\psi_{m+2}, f_{n+1}] + 3[\psi_{m+1}, f_{n+2}] - [\psi_m, f_{n+3}]\\
&\quad + \sigma_2 [\psi_{m+1}, f_n] - \sigma_2 [\psi_m, f_{n+1}] + \sigma_3 \{\psi_m, f_n\},
\end{aligned}
$$

- Boundary conditions:

$$[\psi_0, e_m] = 0, \quad [\psi_1, e_m] = 0, \quad [\psi_2, e_m] = 2e_m,$$
$$[\psi_0, f_m] = 0, \quad [\psi_1, f_m] = 0, \quad [\psi_2, f_m] = -2f_m$$

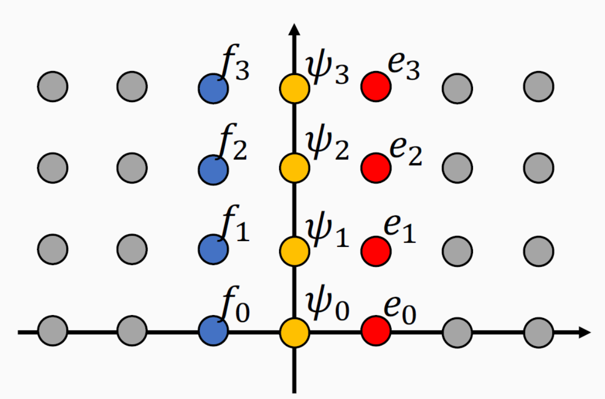
- Serre relations

$$\mathrm{Sym}_{(m_1,m_2,m_3)}[e_{m_1}, [e_{m_2}, e_{m_3+1}]] = 0,$$
$$\mathrm{Sym}_{(m_1,m_2,m_3)}[f_{m_1}, [f_{m_2}, f_{m_3+1}]] = 0,$$

  where Sym is the complete symmetrization over all indicated indices.
- We can rewrite the modes in a compact form:

$$e(u) = \sum_{j=0}^{\infty} \frac{e_j}{u^{j+1}}, \quad f(u) = \sum_{j=0}^{\infty} \frac{f_j}{u^{j+1}}, \quad \psi(u) = 1 + \sigma_3 \sum_{j=0}^{\infty} \frac{\psi_j}{u^{j+1}}$$

- Modes of affine Yangian $\mathfrak{gl}_1$:



- In the currents form, the defining relations are

$$
\begin{aligned}
\psi(u)\psi(v) &\sim \psi(v)\psi(u), \\
e(u)e(v) &\sim \varphi(u-v)e(v)e(u), \\
f(u)f(v) &\sim \varphi(u-v)^{-1}f(v)f(u), \\
\psi(u)e(v) &\sim \varphi(u-v)e(v)\psi(u), \\
\psi(u)f(v) &\sim \varphi(u-v)^{-1}f(v)\psi(u), \\
[e(u), f(v)] &\ -\frac{1}{\sigma_3}\frac{\psi(u)-\psi(v)}{u-v}, \\
\varphi(u) &= \frac{(u+h_1)(u+h_2)(u+h_3)}{(u-h_1)(u-h_2)(u-h_3)}.
\end{aligned}
$$

- The function $\varphi(u)$ is called "structure function" and it obeys

$$\varphi(u)\varphi(-u) = 1.$$

- MacMahon representation:

$$h(\square) = h_1 x_1(\square) + h_2 x_2(\square) + h_3 x_3(\square),$$

$$\psi(z)\,|\Lambda\rangle = [\Psi_\Lambda(z-u)]_+\,|\Lambda\rangle\,,$$

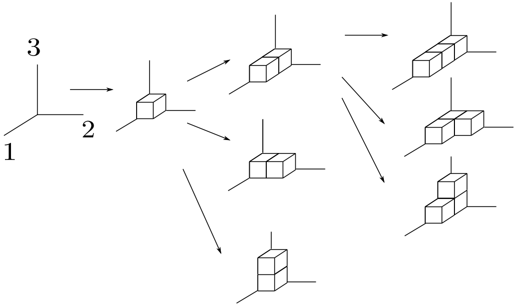
$$e(z)\,|\Lambda\rangle = \sum_{\square\in \mathrm{Add}(\Lambda)} \left[\frac{E(\Lambda\to\Lambda+\square)}{z-u-h(\square)}\right]_+ |\Lambda+\square\rangle\,,$$

$$f(z)\,|\Lambda\rangle = \sum_{\square\in \mathrm{Rem}(\Lambda)} \left[\frac{F(\Lambda\to\Lambda-\square)}{z-u-h(\square)}\right]_+ |\Lambda-\square\rangle\,,$$

$$\Psi_\Lambda(z) = \psi_0(z)\prod_{\square\in\Lambda}\varphi(z-h(\square)), \quad \psi_0(z) = 1+\sigma_3\frac{\psi_0}{z},$$

$$E(\Lambda\to\Lambda+\square) = \sqrt{-\frac{1}{\sigma_3}\operatorname*{Res}_{z\to u+h(\square)}\Psi_\Lambda(z)},$$

$$F(\Lambda\to\Lambda-\square) = \sqrt{\frac{1}{\sigma_3}\operatorname*{Res}_{z\to u+h(\square)}\Psi_\Lambda(z)}.$$



Appendix: Quiver Yangian

- Defining relations:

$$
\begin{aligned}
\psi^{(a)}(z)\psi^{(b)}(w) &= \psi^{(b)}(w)\psi^{(a)}(z),\\
\psi^{(a)}(z)e^{(b)}(w) &\simeq \varphi^{b\Rightarrow a}(z-w)e^{(b)}(w)\psi^{(a)}(z),\\
e^{(a)}(z)e^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(z-w)e^{(b)}(w)e^{(a)}(z),\\
\psi^{(a)}(z)f^{(b)}(w) &\simeq \varphi^{b\Rightarrow a}(z-w)^{-1}f^{(b)}(w)\psi^{(a)}(z),\\
f^{(a)}(z)f^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(z-w)^{-1}f^{(b)}(w)f^{(a)}(z),\\
[e^{(a)}(z),f^{(b)}(w)\} &\sim -\delta_{a,b}\frac{\psi^{(a)}(z)-\psi^{(b)}(w)}{z-w},
\end{aligned}
$$

for $a, b \in Q_0$. In the above equations, $\simeq$ means the equality up to $z^n w^{m\geq 0}$ terms and $\sim$ means the equality up to $z^{n\geq 0}w^m$ and $z^n w^{m\geq 0}$ terms.

$$
[e^{(a)}(z),f^{(b)}(w)\} = e^{(a)}(z)f^{(b)}(w) + f^{(b)}(w)e^{(a)}(z), \quad |a|=|b|=1
$$

$$
[e^{(a)}(z),f^{(b)}(w)\} = e^{(a)}(z)f^{(b)}(w) - f^{(b)}(w)e^{(a)}(z), \quad \text{otherwise.}
$$

Appendix: Symmetries and conditions imposed on QY

- Spectral shift:

$$e(z) \to e(z+u), \quad f(z) \to f(z+u), \quad \psi(z) \to \psi(z+u)$$

- Gauge-symmetry shift:

$$h_I \to h'_I = h_I + \epsilon_a \mathrm{sign}_a(I),$$

$$\mathrm{sign}_a(I) = \begin{cases} +1 & (s(I) = a, \quad t(I) \neq a), \\ -1 & (s(I) \neq a, \quad t(I) = a), \\ 0 & (\text{otherwise}) \end{cases}$$

- Vertex condition:

$$\sum_{I \in Q_1(a)} h_I = 0, \quad a \in Q_0,$$

where $Q_1(a)$ is the subset of Q_1 where the vertex $a \in Q_0$ is contained either in the start point of the endpoint.

- Loop condition: $\sum_{I \in L} h_I = 0, \quad$ arbitrary loop L

Appendix: Quantum Toroidal $\mathfrak{gl}_1$

- Generators and defining relations:

$$E(z)=\sum_{m\in\mathbb{Z}}E_m z^{-m},\quad F(z)=\sum_{m\in\mathbb{Z}}F_m z^{-m},\quad K^{\pm}(z)=(C^{\perp})^{\mp 1}\exp\left(\sum_{r>0}\pm H_{\pm r}z^{\mp r}\right),$$

$$C, C^{\perp}, D, D^{\perp},$$

$$DE(z)=E(qz)D,\quad DF(z)=F(qz)D,\quad DK^{\pm}(z)=K^{\pm}(qz)D,$$

$$D^{\perp}E(z)=qE(z)D^{\perp},\quad D^{\perp}F(z)=q^{-1}F(z)D^{\perp},\quad \left[D^{\perp},K^{\pm}(z)\right]=0\,,$$

$$E(z)E(w)=g(z/w)E(w)E(z),\quad F(z)F(w)=g(z/w)^{-1}F(w)F(z),$$

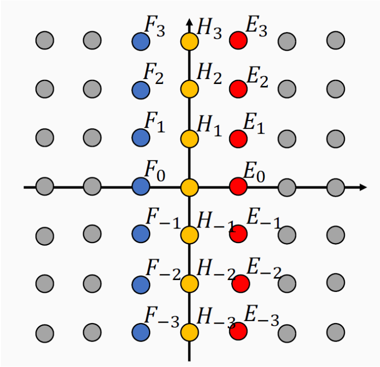
$$K^{\pm}(z)K^{\pm}(w)=K^{\pm}(w)K^{\pm}(z),\quad K^{-}(z)K^{+}(w)=\frac{g(C^{-1}z/w)}{g(Cz/w)}K^{+}(w)K^{-}(z),$$

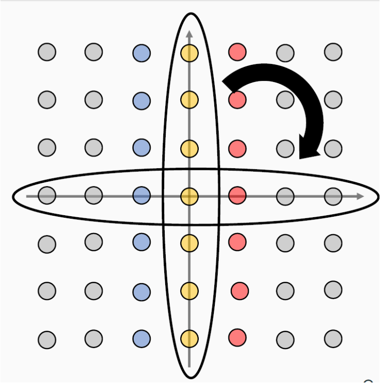
$$K^{\pm}(C^{(1\mp 1)/2}z)E(w)=g(z/w)E(w)K^{\pm}(C^{(1\mp 1)/2}z),$$

$$K^{\pm}(C^{(1\pm 1)/2}z)F(w)=g(z/w)^{-1}F(w)K^{\pm}(C^{(1\pm 1)/2}z)\,,$$

$$[E(z),F(w)]=\tilde{g}(\delta(\frac{Cw}{z})K^{+}(z)-\delta(\frac{Cz}{w})K^{-}(w)).$$

- Modes and Miki automorphism [Miki 2007]





Appendix: Quiver Quantum Toroidal Algebra

- Defining relations:

$$
\begin{aligned}
K_i K_i^{-1} &= K_i^{-1} K_i = 1, \\
C^{-1} C &= C C^{-1} = 1, \\
K_i^{\pm}(z) K_j^{\pm}(w) &= K_j^{\pm}(w) K_i^{\pm}(z), \\
K_i^{-}(z) K_j^{+}(w) &= \frac{\varphi^{j \Rightarrow i}(z, Cw)}{\varphi^{j \Rightarrow i}(Cz, w)} K_j^{+}(w) K_i^{-}(z), \\
K_i^{\pm}(C^{\frac{1 \mp 1}{2}} z) E_j(w) &= \varphi^{j \Rightarrow i}(z, w) E_j(w) K_i^{\pm}(C^{\frac{1 \mp 1}{2}} z), \\
K_i^{\pm}(C^{\frac{1 \pm 1}{2}} z) F_j(w) &= \varphi^{j \Rightarrow i}(z, w)^{-1} F_j(w) K_i^{\pm}(C^{\frac{1 \pm 1}{2}} z), \\
[E_i(z), F_j(w)] &= \delta_{i,j} \left(\delta\left(\frac{Cw}{z}\right) K_i^{+}(z) - \delta\left(\frac{Cz}{w}\right) K_i^{-}(w) \right), \\
E_i(z) E_j(w) &= (-1)^{|i||j|} \varphi^{j \Rightarrow i}(z, w) E_j(w) E_i(z), \\
F_i(z) F_j(w) &= (-1)^{|i||j|} \varphi^{j \Rightarrow i}(z, w)^{-1} F_j(w) F_i(z),
\end{aligned}
$$

Symmetries and conditions imposed on QQTA

- Spectral shift:

$$E_i(z) \to E_i(az), \quad F_i(z) \to F_i(az), \quad K_i^{\pm}(z) \to K_i^{\pm}(az)$$

- Gauge-symmetry shift:

$$q_I \to q_I' = q_I p_i^{\mathrm{sign}_i(I)}$$

- Vertex condition:

$$\prod_{I \in Q_1(i)} q_I^{\mathrm{sign}_i(I)} = 1$$

- Loop condition:

$$\prod_{I \in L} q_I = 1.$$

Appendix: (ν_1,ν_2) deformed quantum toroidal $\mathfrak{gl}_n$ [Bourgine-Jeong 1906.01625]

- Currents: $\omega,\ \omega' \in \mathbb{Z}_n$

$$x^{\pm}_{\omega}(z) = \sum_{k\in\mathbb{Z}} z^{-k} x^{\pm}_{\omega,k}, \quad \psi^{\pm}_{\omega}(z) = \psi^{\pm}_{\omega,0} z^{\mp a^{\pm}_{\omega,0}} \exp\left(\pm\sum_{k>0} z^{\mp k} a_{\omega,\pm k}\right)$$

- Defining relations: $(\nu_1,\nu_2)\in\mathbb{Z}_n,\ \nu_1+\nu_2+\nu_3=0$

$$x^{\pm}_{\omega}(z)x^{\pm}_{\omega'}(w) = g_{\omega\omega'}(z/w)^{\pm1}x^{\pm}_{\omega'}(w)x^{\pm}_{\omega}(z), \quad \psi^{\pm}_{\omega}(z)x^{\pm}_{\omega'}(w) = g_{\omega\omega'}(z/w)^{\pm1}x^{\pm}_{\omega'}(w)\psi^{+}_{\omega}(z),$$

$$\psi^{-}_{\omega}(z)x^{+}_{\omega'}(w) = g_{\omega-\nu_3 c}(q_3^{-c}z/w)x^{+}_{\omega'}(w)\psi^{-}_{\omega}(z), \quad \psi^{-}_{\omega}(z)x^{-}_{\omega'}(w) = g_{\omega\omega'}(z/w)^{-1}x^{-}_{\omega'}(w)\psi^{-}_{\omega}(z),$$

$$\psi^{+}_{\omega}(z)\psi^{-}_{\omega'}(w) = \frac{g_{\omega\omega'-\nu_3 c}(q_3^{c}z/w)}{g_{\omega\omega'}(z/w)}\psi^{-}_{\omega'}(w)\psi^{+}_{\omega}(z), \quad [\psi^{\pm}_{\omega}(z),\psi^{\pm}_{\omega'}(w)] = 0,$$

$$[x^{+}(z),x^{-}_{\omega'}(w)] = \Omega[\delta_{\omega,\omega'}\delta\left(z/w\right)\psi^{+}_{\omega}(z) - \delta_{\omega,\omega'-\nu_3 c}\delta\left(q_3^{c}z/w\right)\psi^{-}_{\omega+\nu_3 c}(q_3^{c}z)].$$

- Structure functions:

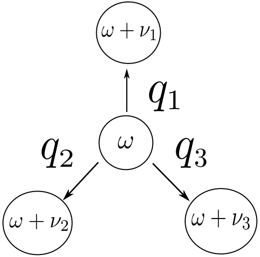
$$g_{\omega\omega'}(z) = f_{\omega\omega'}(z^{-1}) \prod_{i=1,2,3} \frac{(1-q_i z)^{\delta_{\omega,\omega'-\nu_i}}}{(1-q_i^{-1} z)^{\delta_{\omega,\omega'+\nu_i}}},$$

$$f_{\omega\omega'}(z) = F_{\omega\omega'} z^{\beta_{\omega\omega'}},$$

$$\beta_{\omega\omega'} = \delta_{\omega\omega'} + \delta_{\omega\omega'+\nu_1+\nu_2} - \delta_{\omega\omega'+\nu_1} - \delta_{\omega\omega'+\nu_2},$$

$$F_{\omega\omega'} = (-1)^{\delta_{\omega\omega'}} (-q_3)^{-\delta_{\omega,\omega'-\nu_3}} (-q_1)^{-\delta_{\omega\omega'+\nu_1}} (-q_2)^{-\delta_{\omega\omega'+\nu_2}}$$

- Quiver



- Coproduct:

$$
\begin{aligned}
\Delta(x^{+}_{\omega}(z)) &= x^{+}_{\omega}(z)\otimes 1 + \psi^{-}_{\omega+\nu_3 c_{(1)}}(q_3^{c_{(1)}}z)\otimes x^{+}_{\omega}(z),\\
\Delta(x^{-}_{\omega}(z)) &= x^{-}_{\omega}(z)\otimes \psi^{+}_{\omega-\nu_3 c(1)}(q_3^{-c_{(1)}}z) + 1\otimes x^{-}_{\omega-\nu_3 c(1)}(q_3^{-c_{(1)}}z)\\
\Delta(\psi^{+}_{\omega}(z)) &= \psi^{+}_{\omega}(z)\otimes \psi^{+}_{\omega-\nu_3 c(1)}(q^{-c(1)}z),\\
\Delta(\psi^{-}_{\omega}(z)) &= \psi^{-}_{\omega-\nu_3 c_{(2)} z}(q_3^{-c_{(2)}})\otimes \psi^{-}_{\omega-\nu_3 c_{(1)}}(q_3^{-c_{(1)}}z)
\end{aligned}
$$