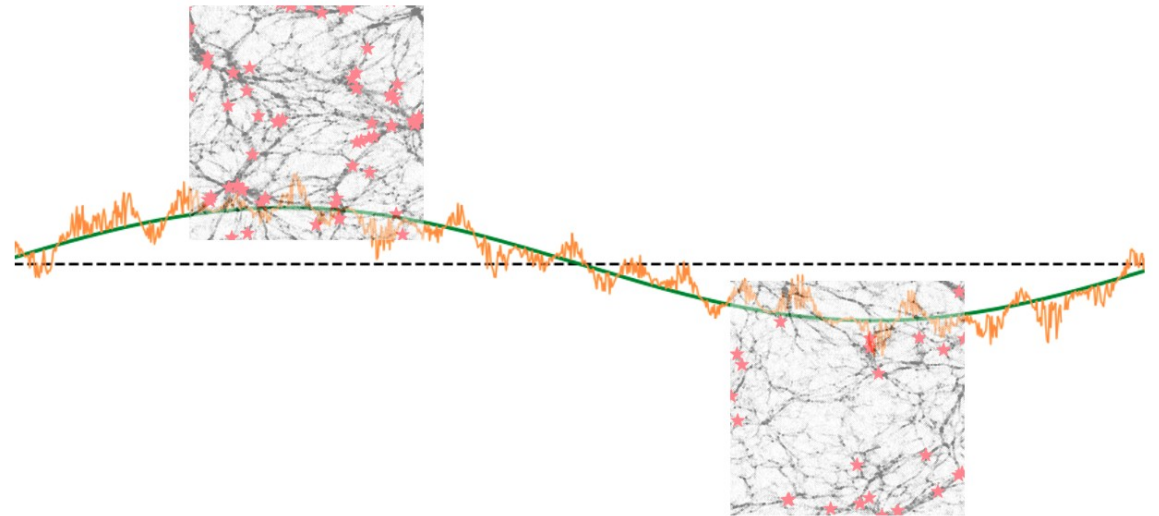


# Connecting the **visible** Universe to the **dark**

Recent advances on galaxy bias predictions

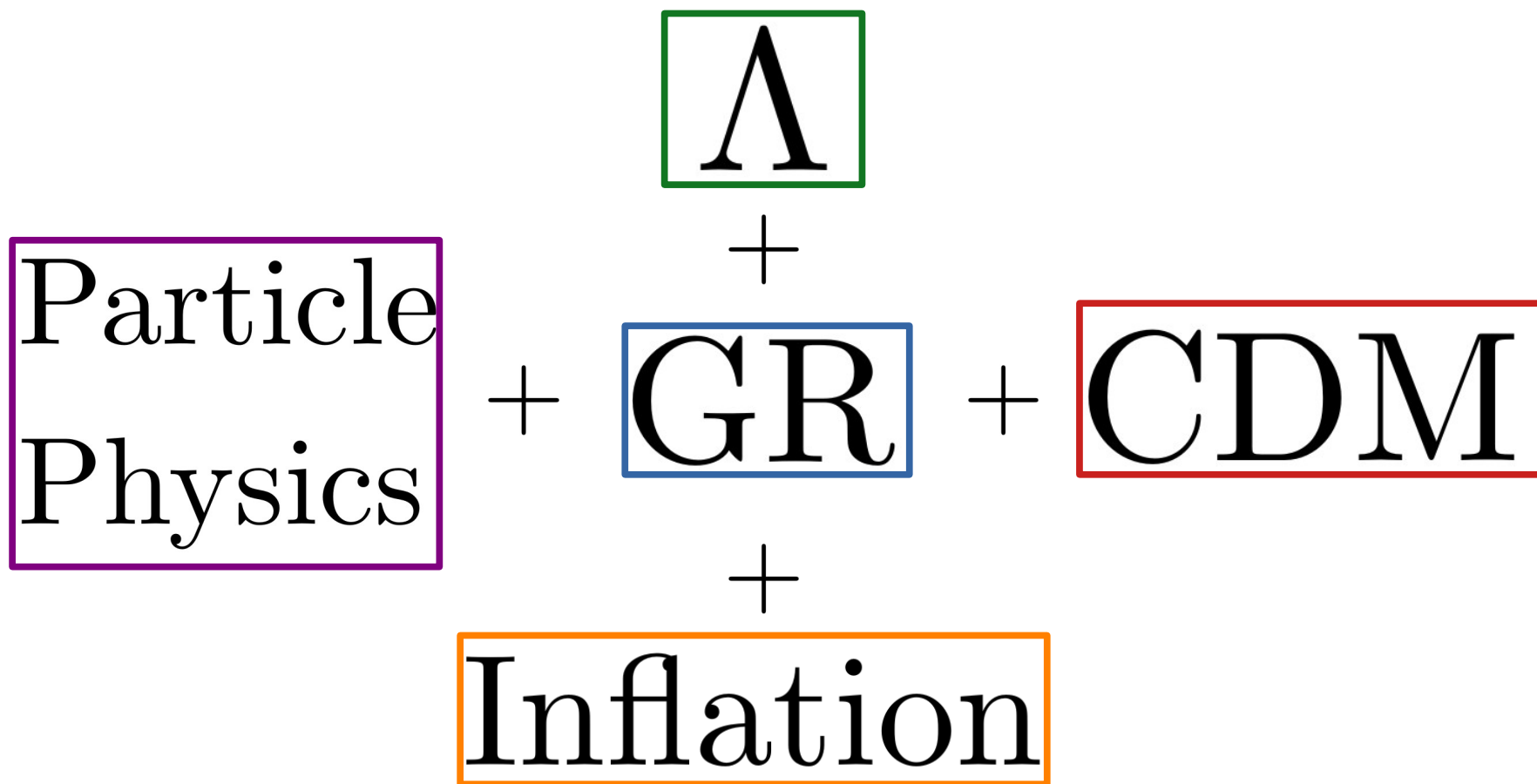
**Alex Barreira**

**ORIGINS Cluster**  
**LMU - Munich**



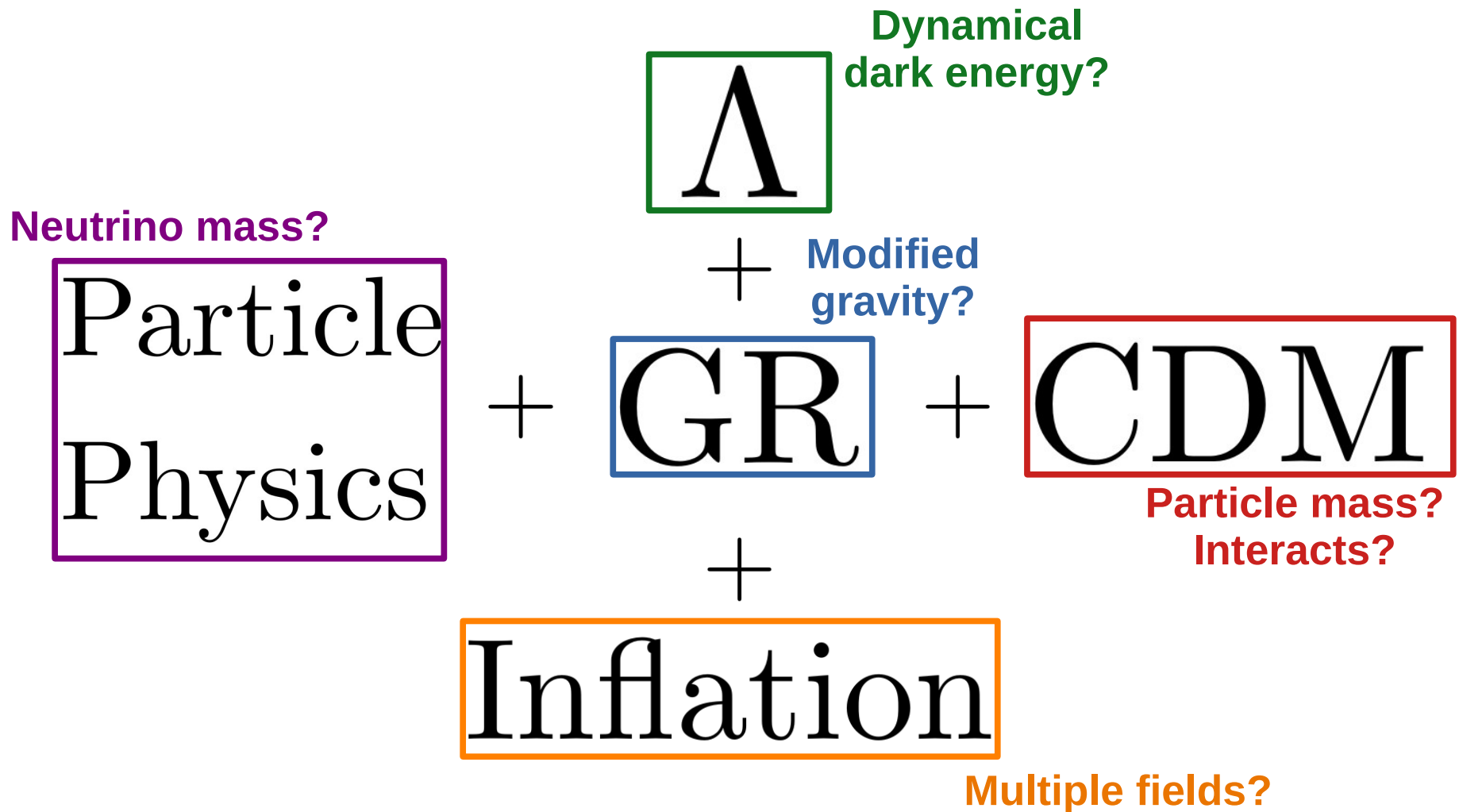
# Standard cosmological model: $\Lambda$ CDM

The  $\Lambda$ CDM model fits *almost* all of the data *almost* perfectly !



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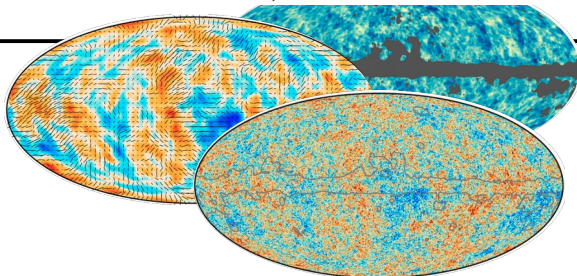
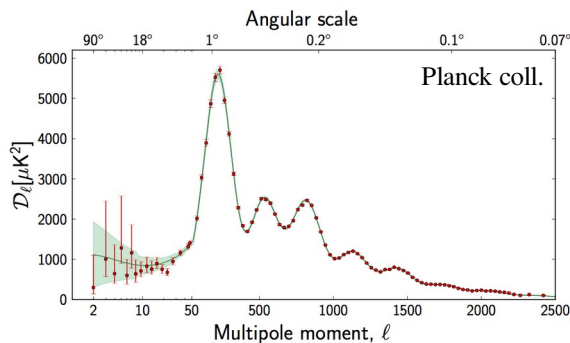
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# Lots of **observational** data

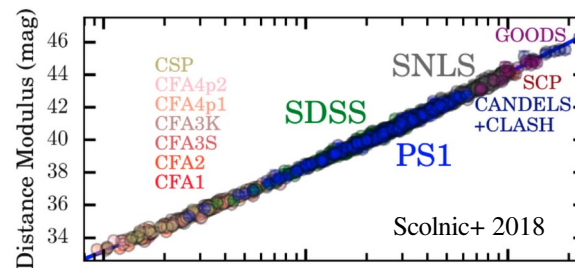
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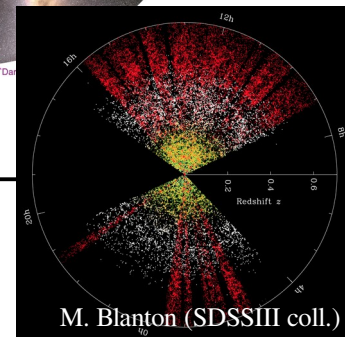
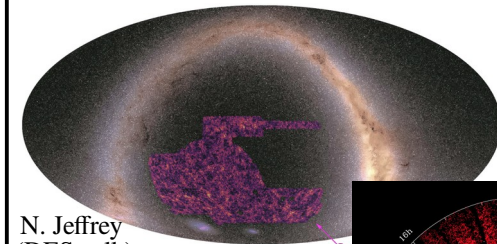
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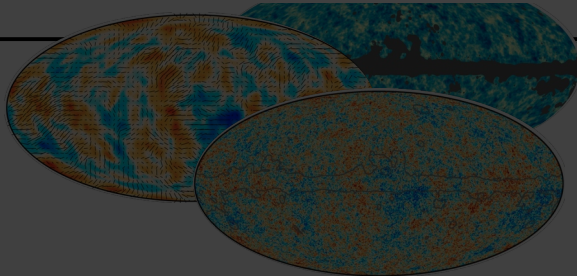
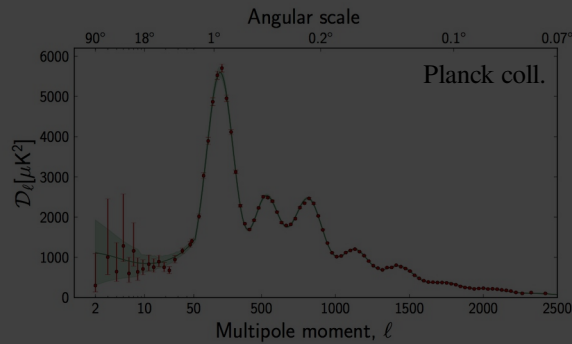
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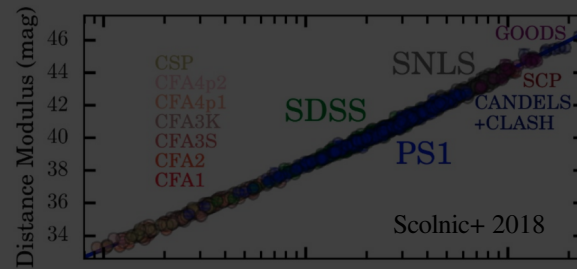
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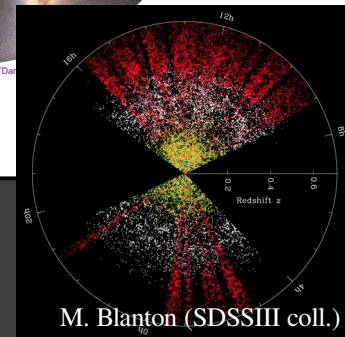
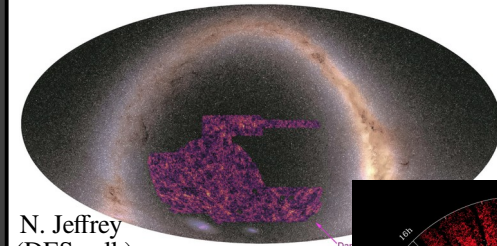
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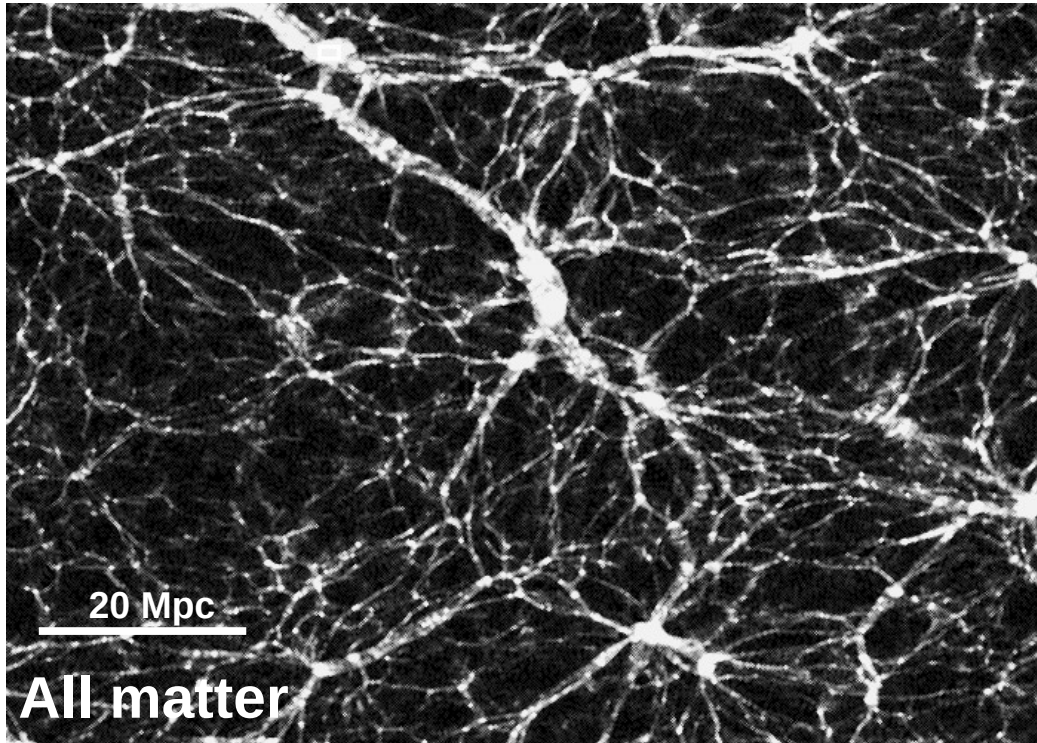


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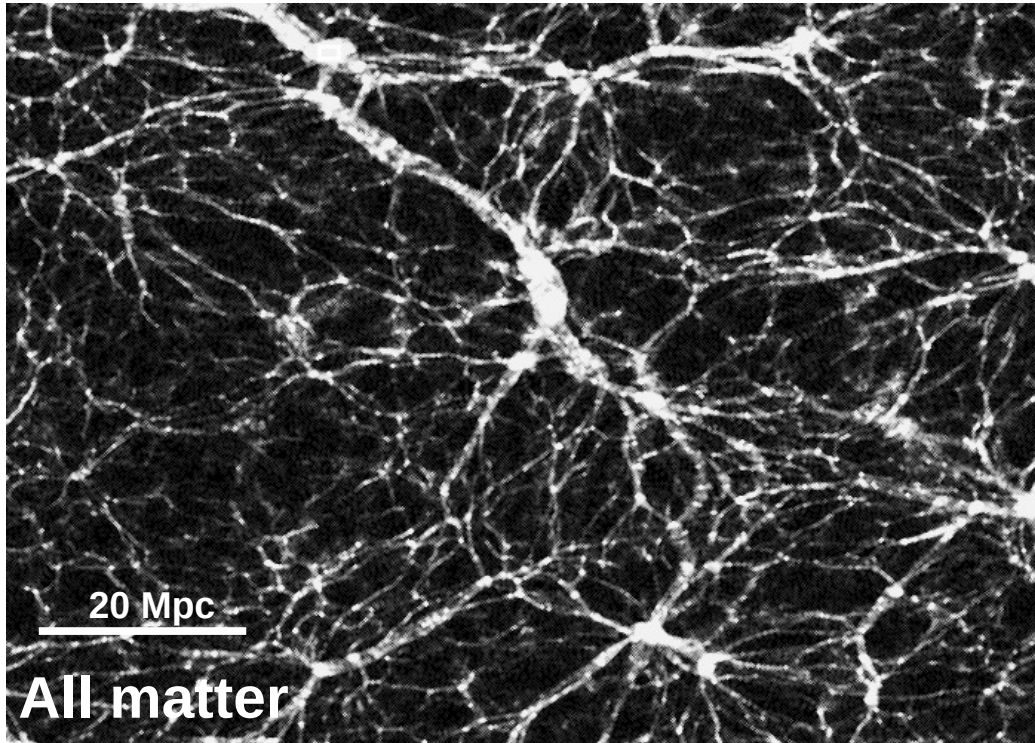
# Large-scale structure and cosmology



## Large-scale structure

- Forms under gravitational instability out of the primordial density fluctuations.
- Nonlinear, non-Gaussian field.
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# Large-scale structure and cosmology



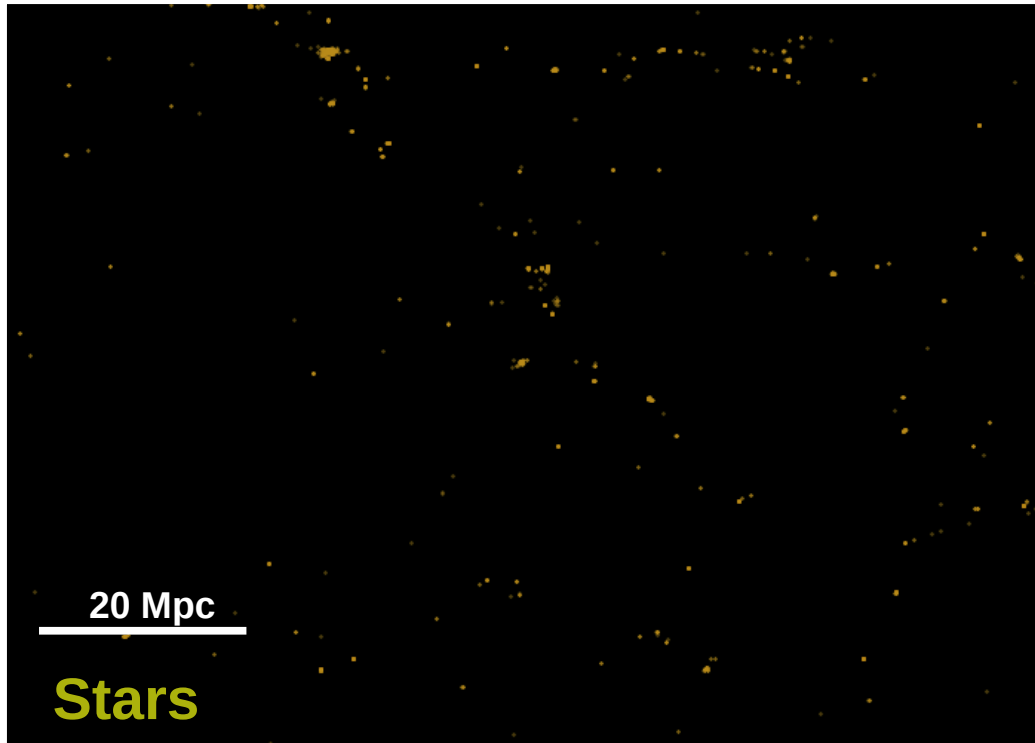
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$$\text{Total matter} = \underbrace{\text{dark matter}}_{\sim 85\%} + \underbrace{\text{baryonic matter}}_{\sim 15\%} \text{ (gas, galaxies, clusters)}$$

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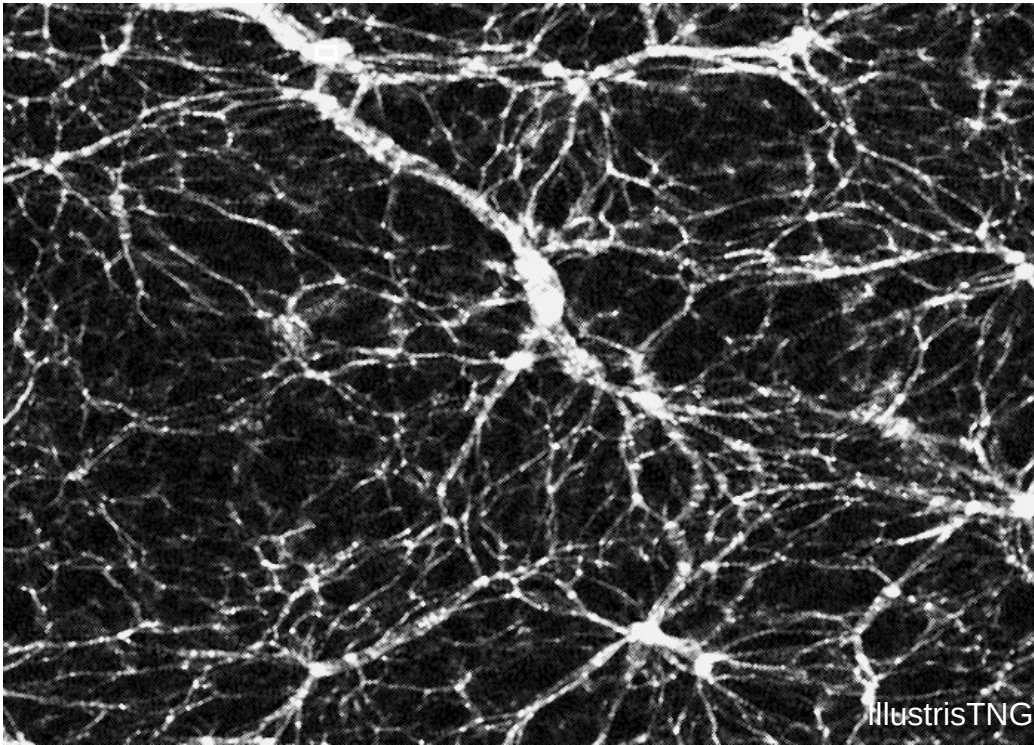
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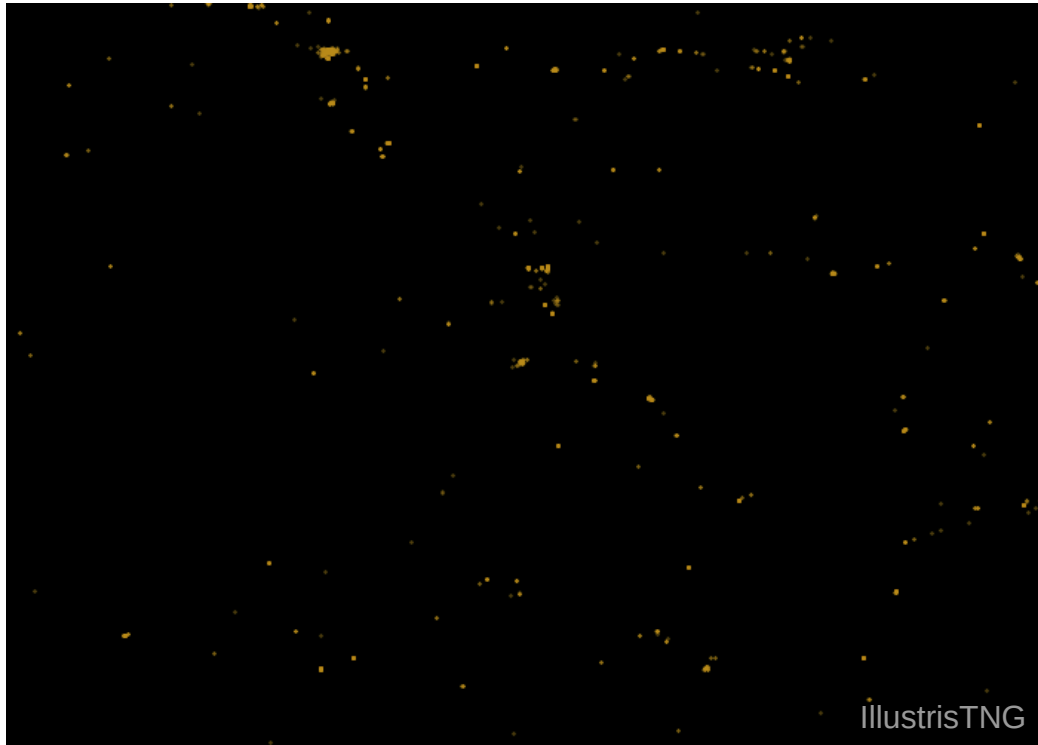


# Exit light enter night

Large scale structure

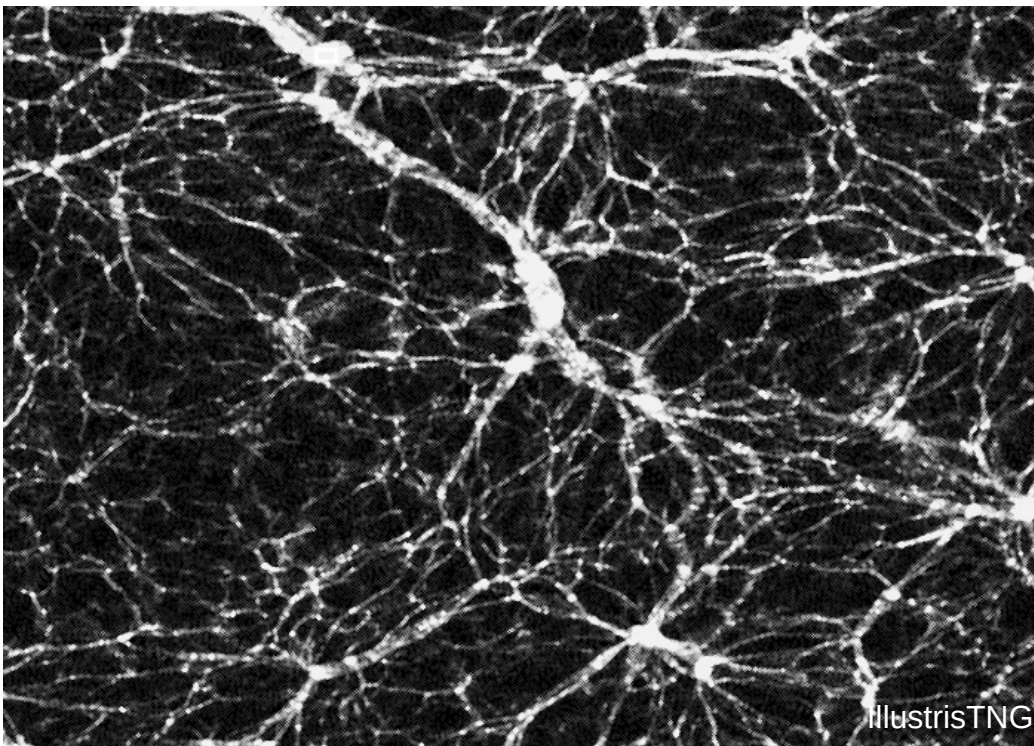


Stars

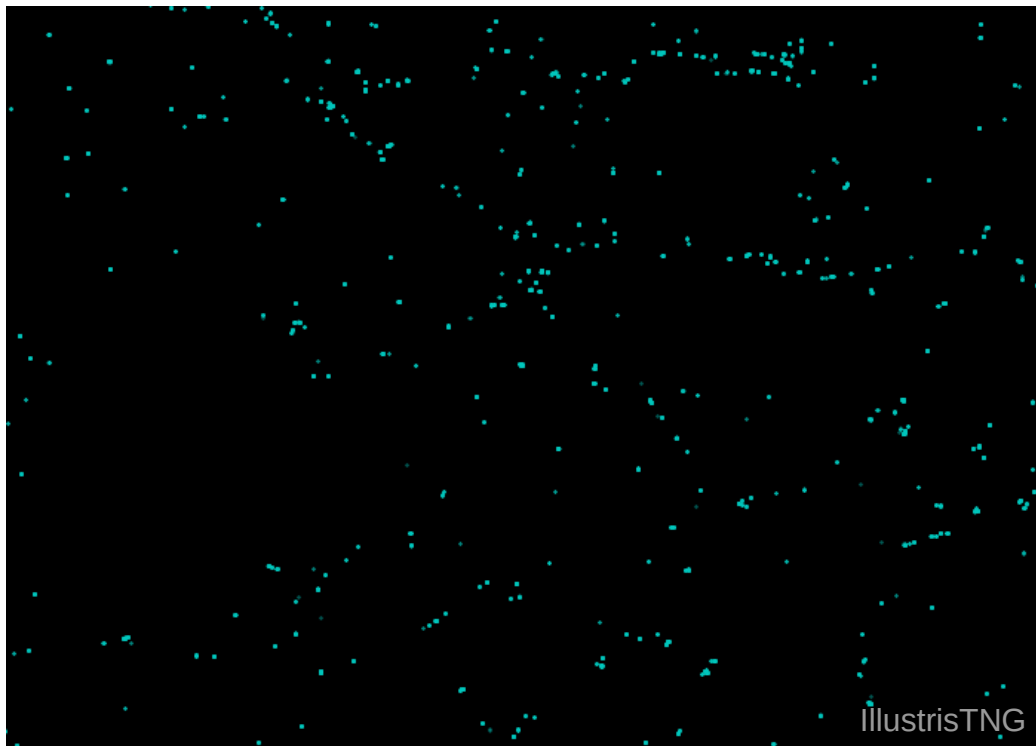


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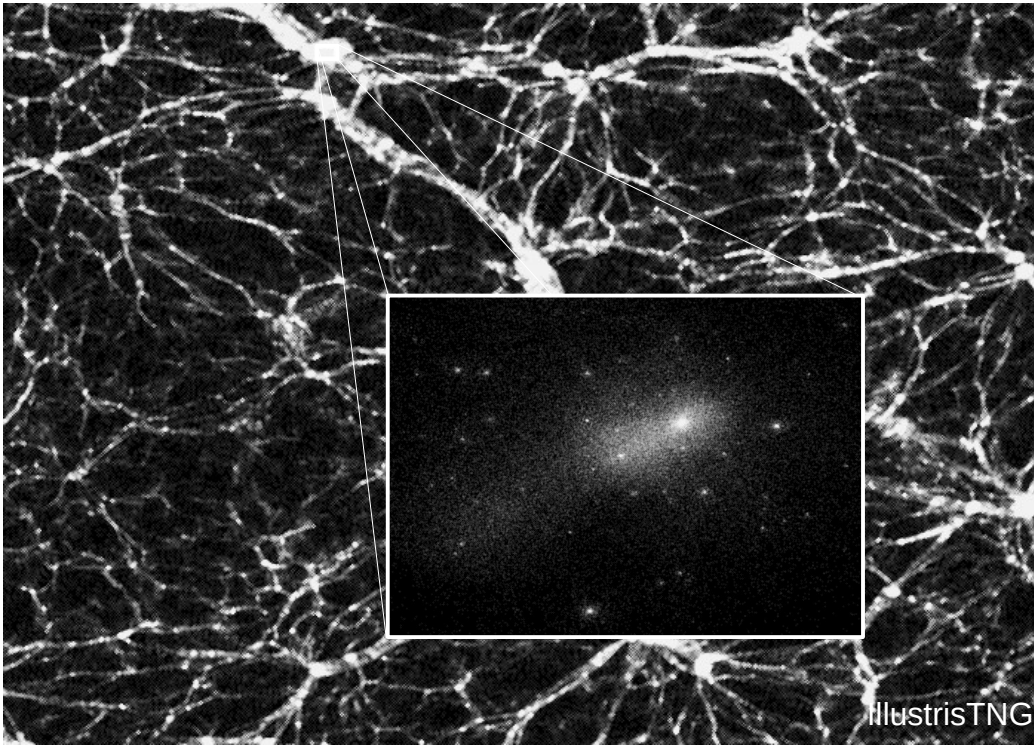


Neutral Hydrogen (21cm emission)

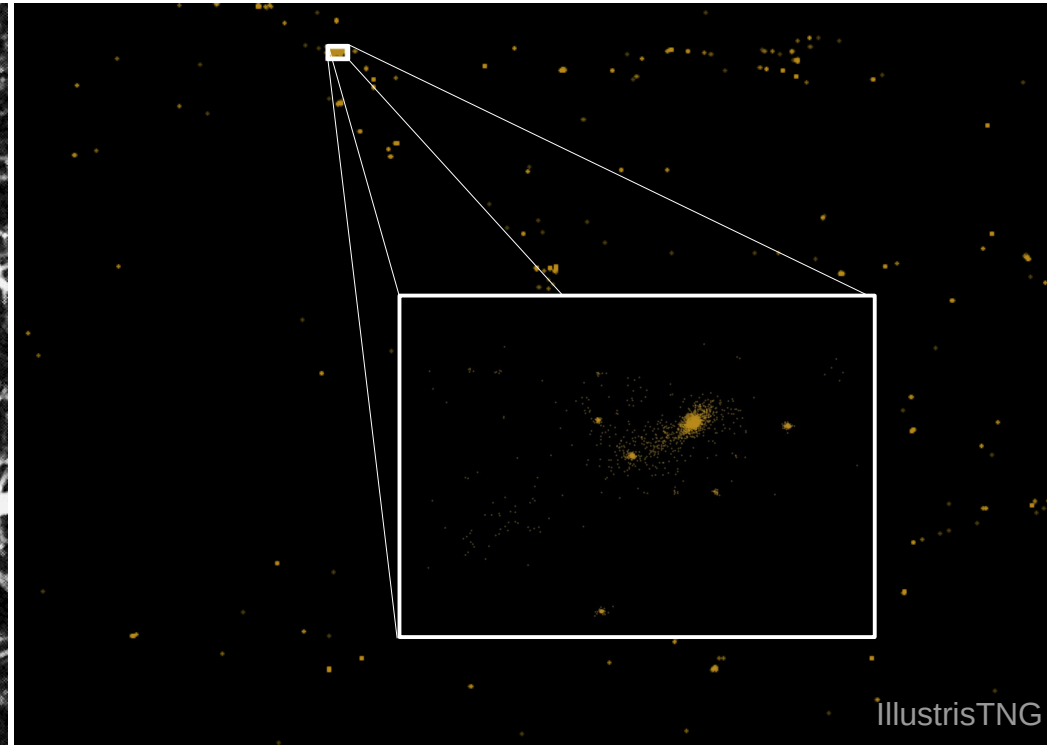


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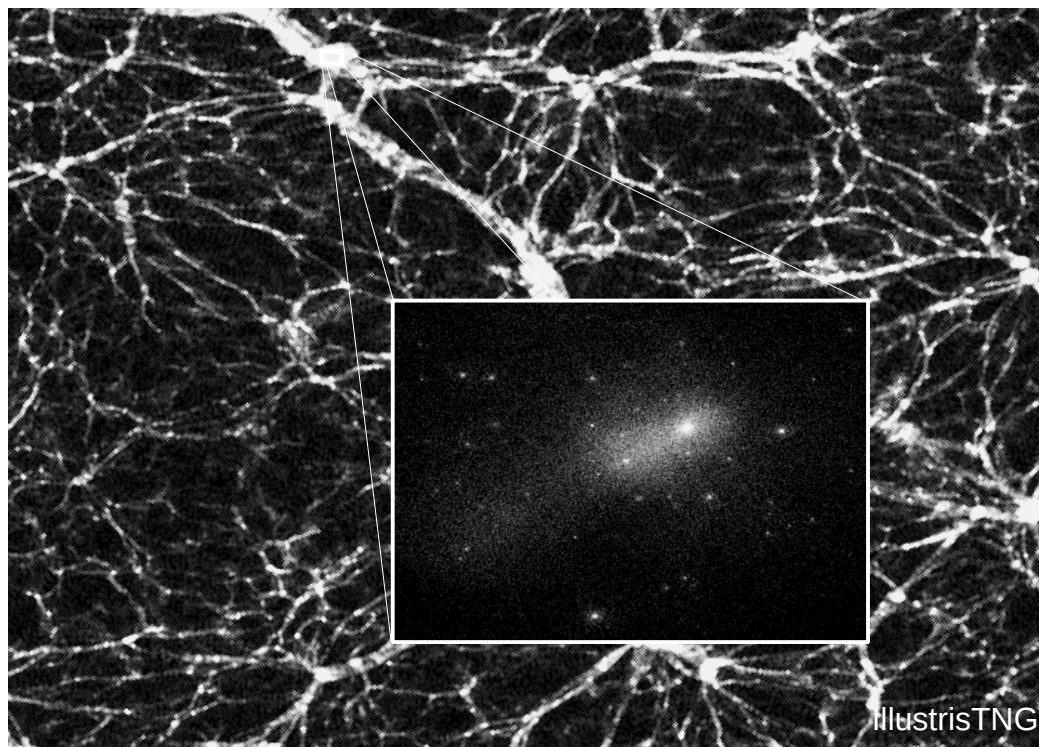


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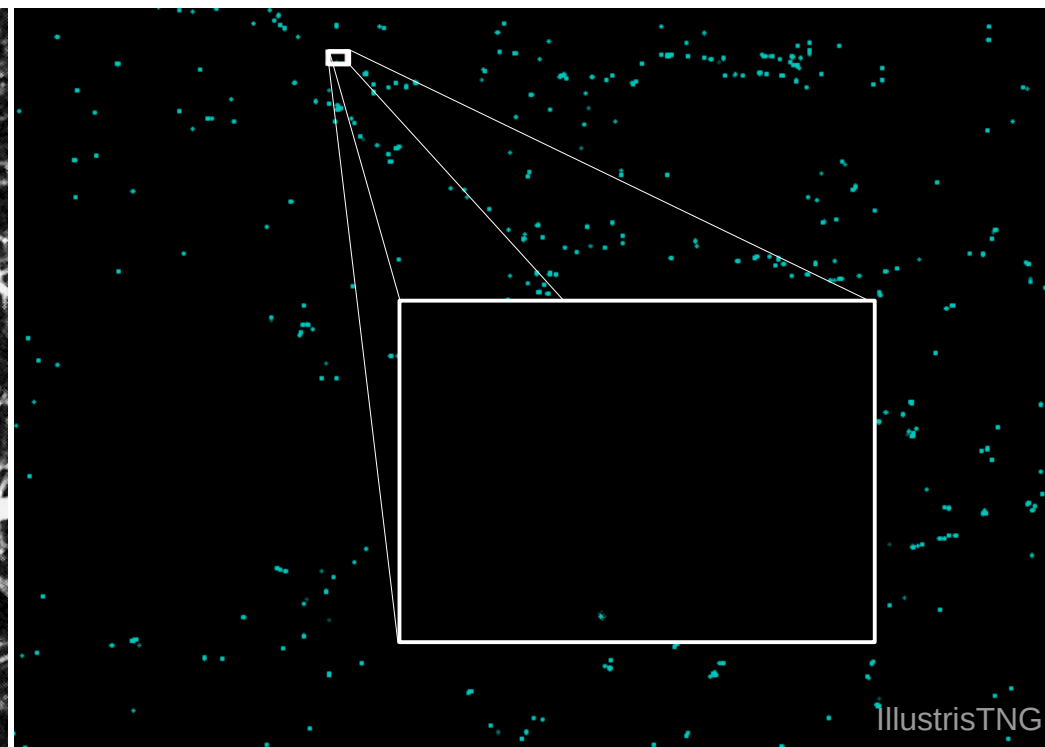


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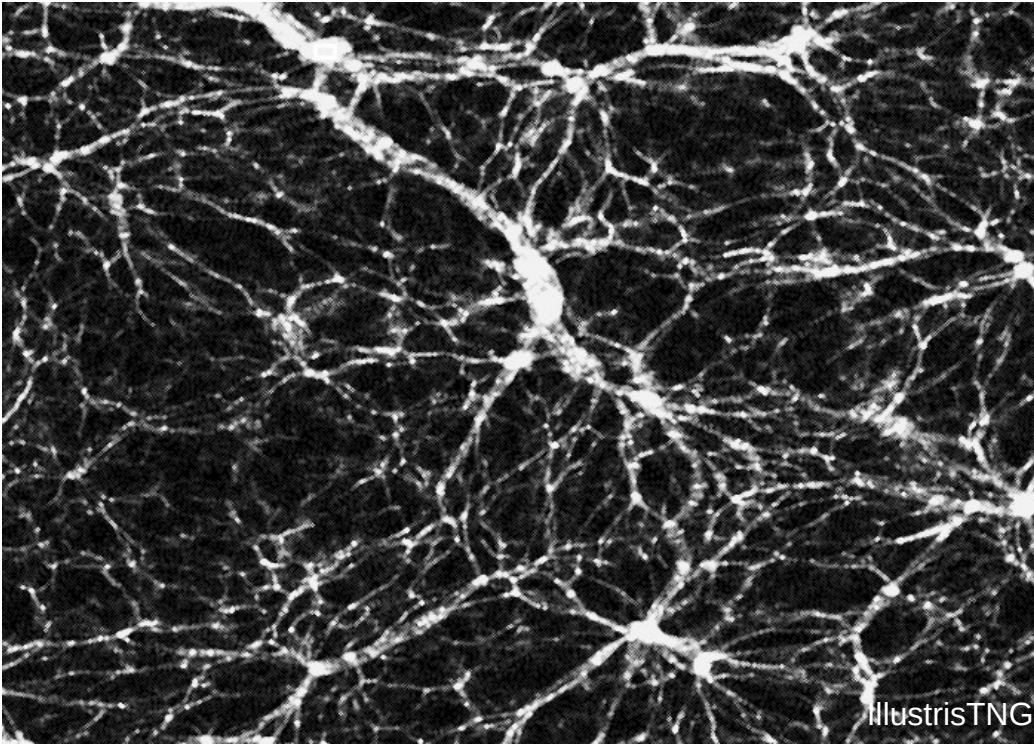


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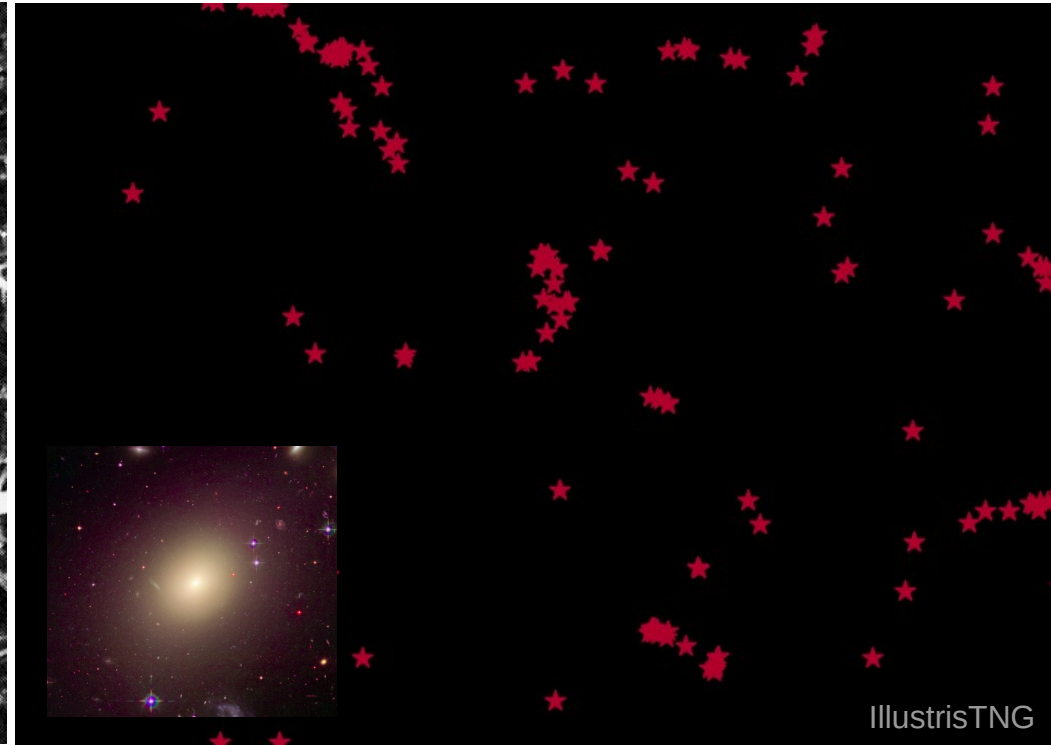


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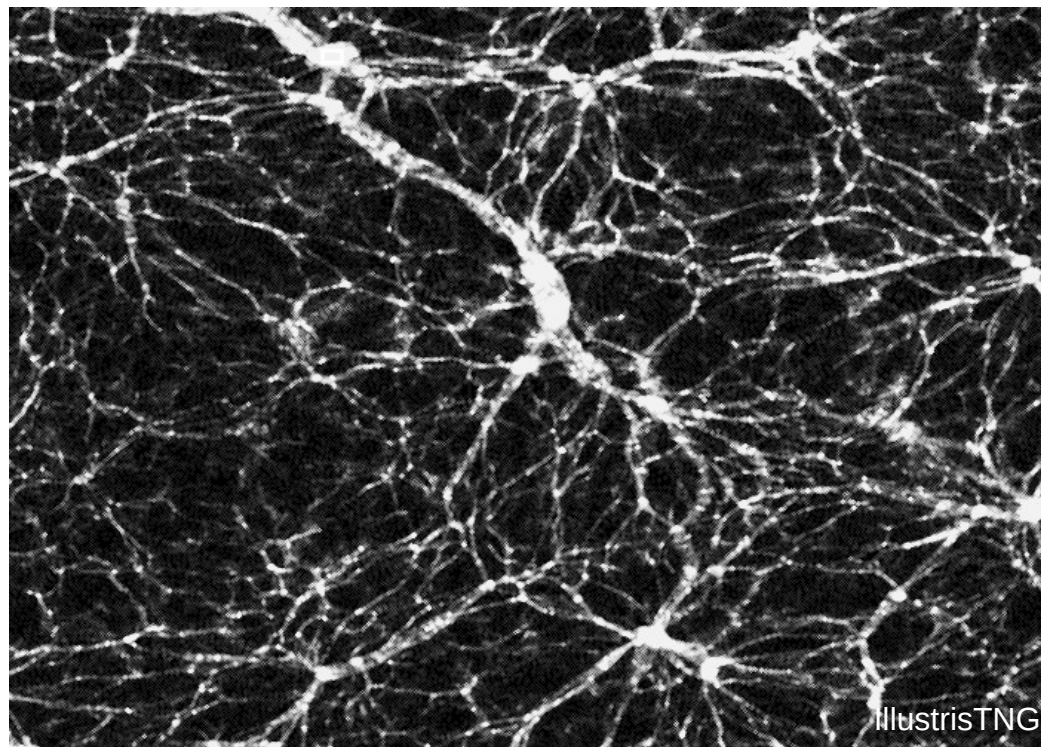
Quenched galaxies  $s\text{SFR} < 10^{-7} \text{yr}^{-1}$



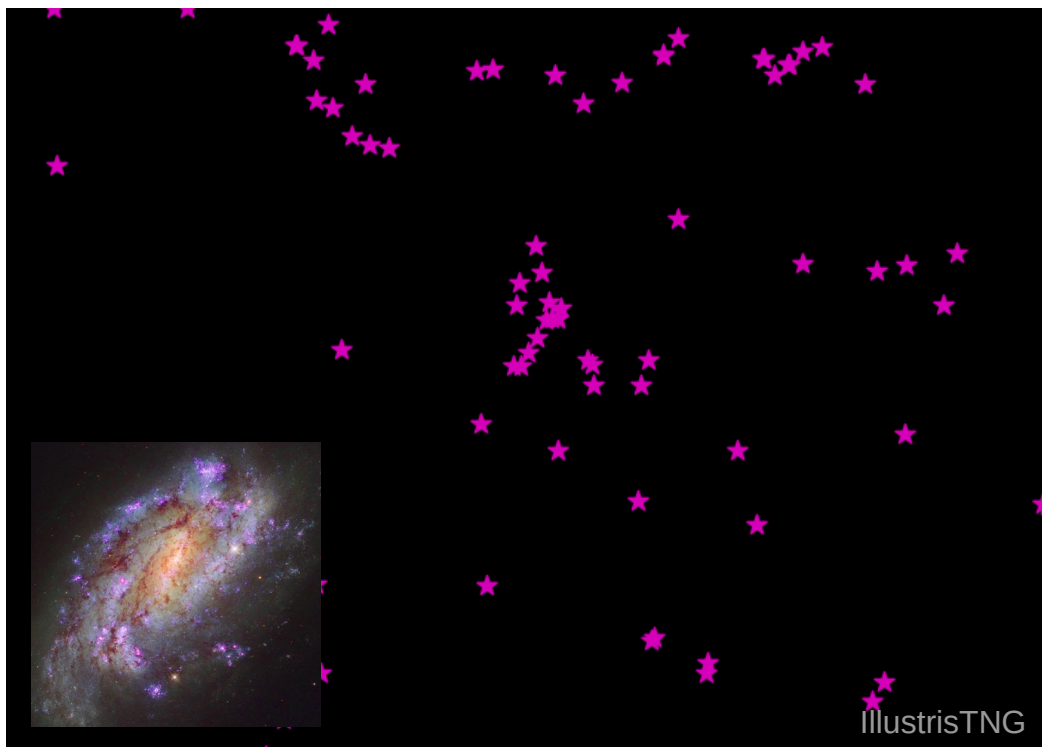
**Definition:** galaxy is any dark matter halo in the simulations that also contains stars.

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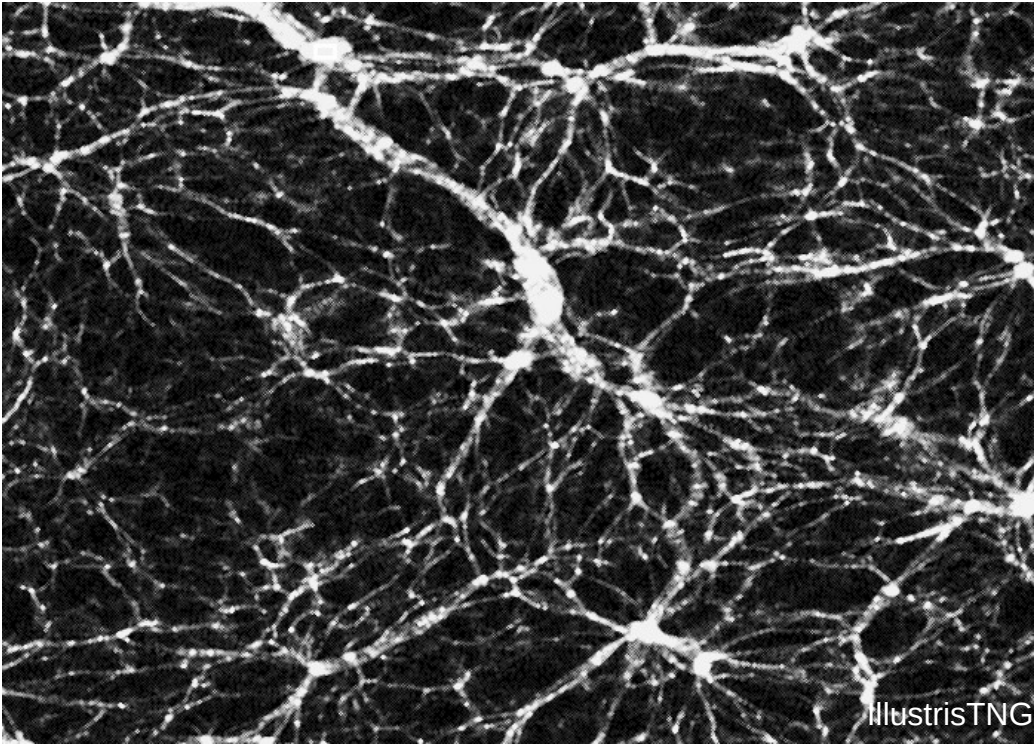
Star-forming gals.  $s\text{SFR} \geq 10^{-7} \text{yr}^{-1}$



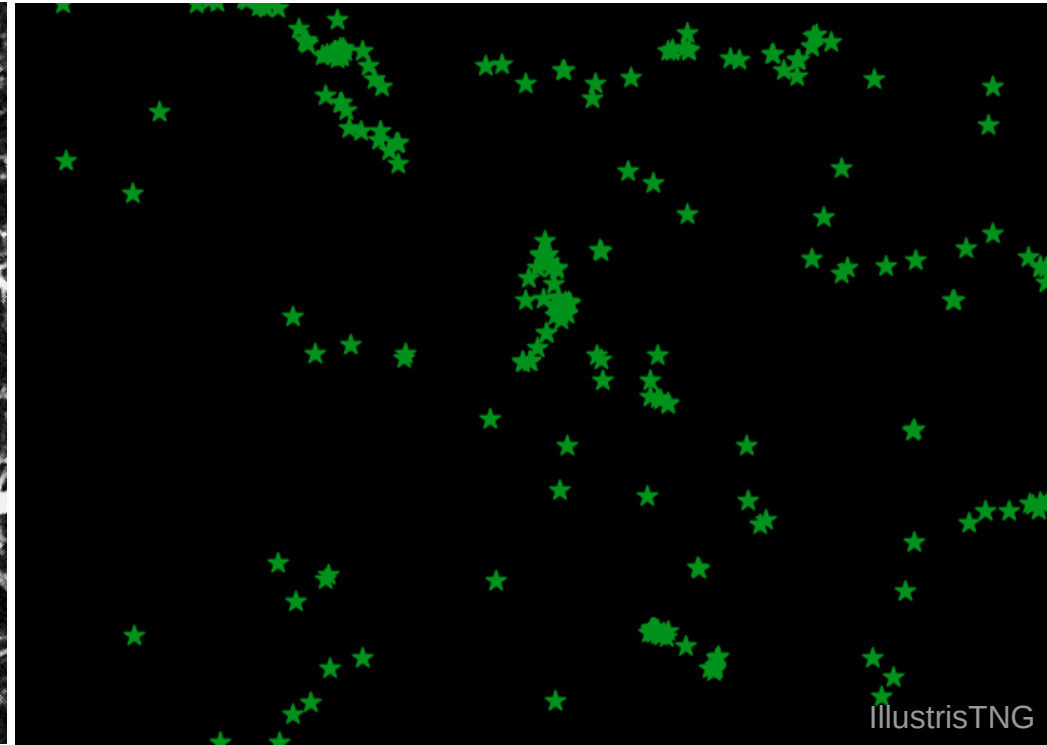
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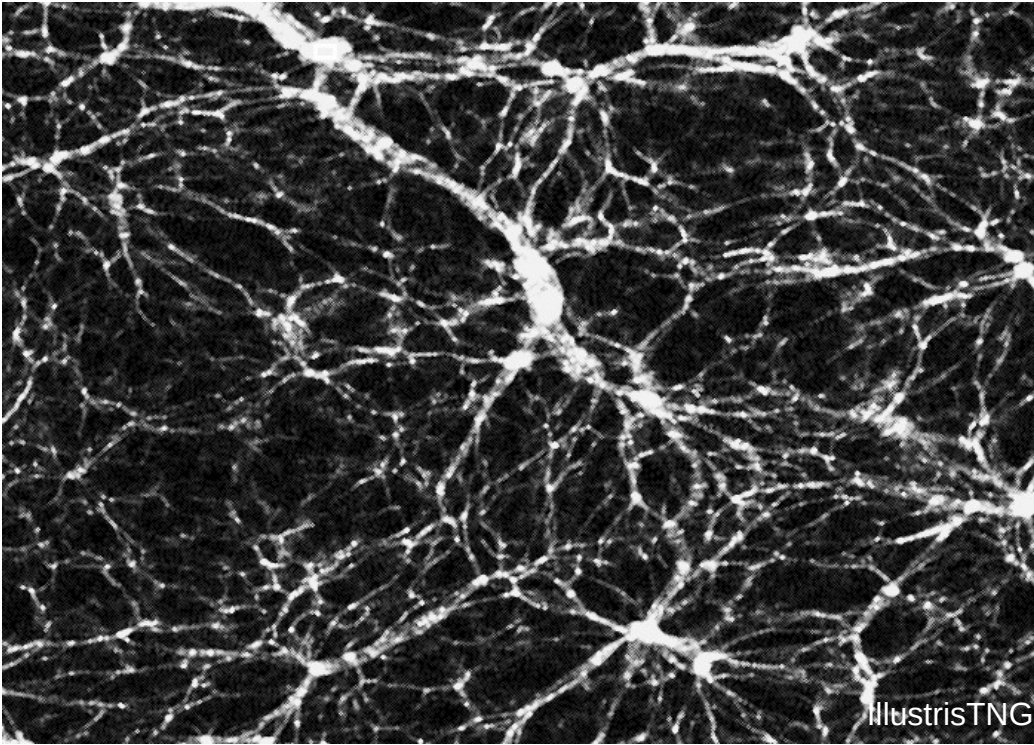
Low-mass galaxies  $M_* \geq 10^9 M_\odot$



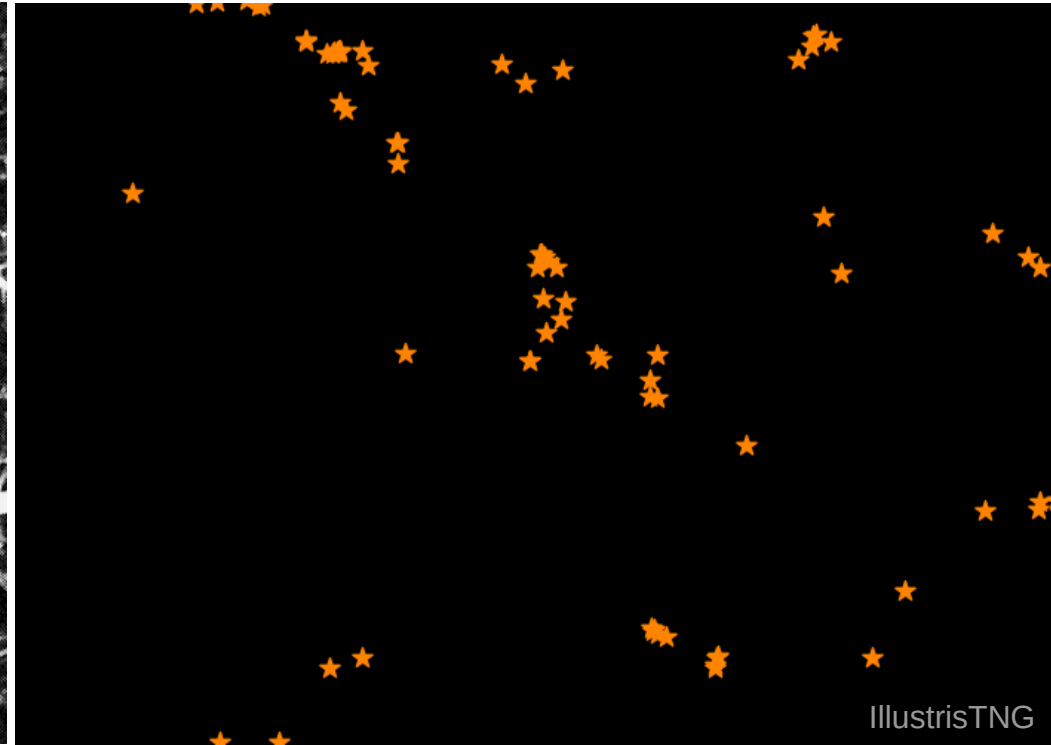
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# Exit light enter night

Large scale structure



Mid-mass galaxies  $M_* \geq 10^{10} M_\odot$



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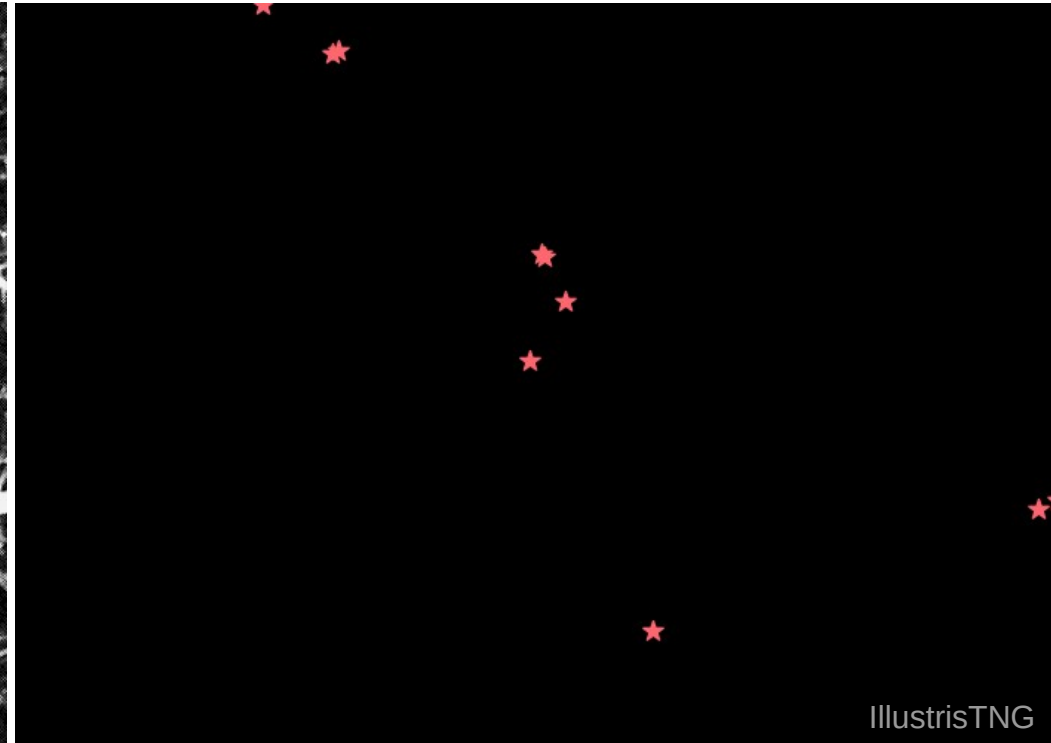


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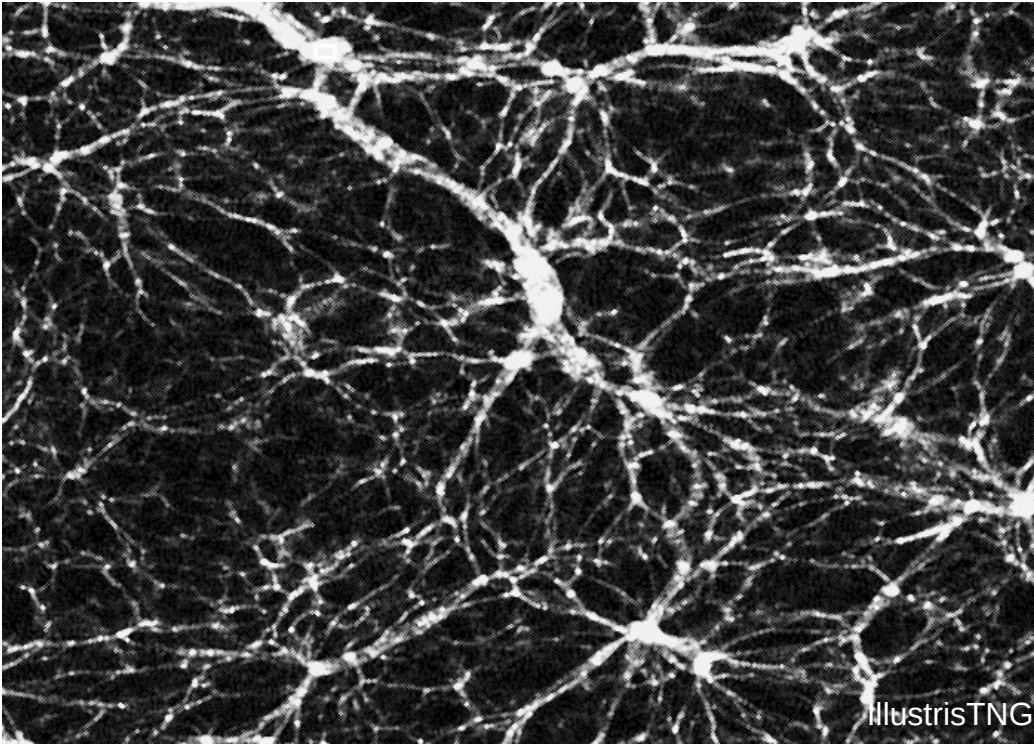
High-mass galaxies  $M_* \geq 10^{11} M_{\odot}$



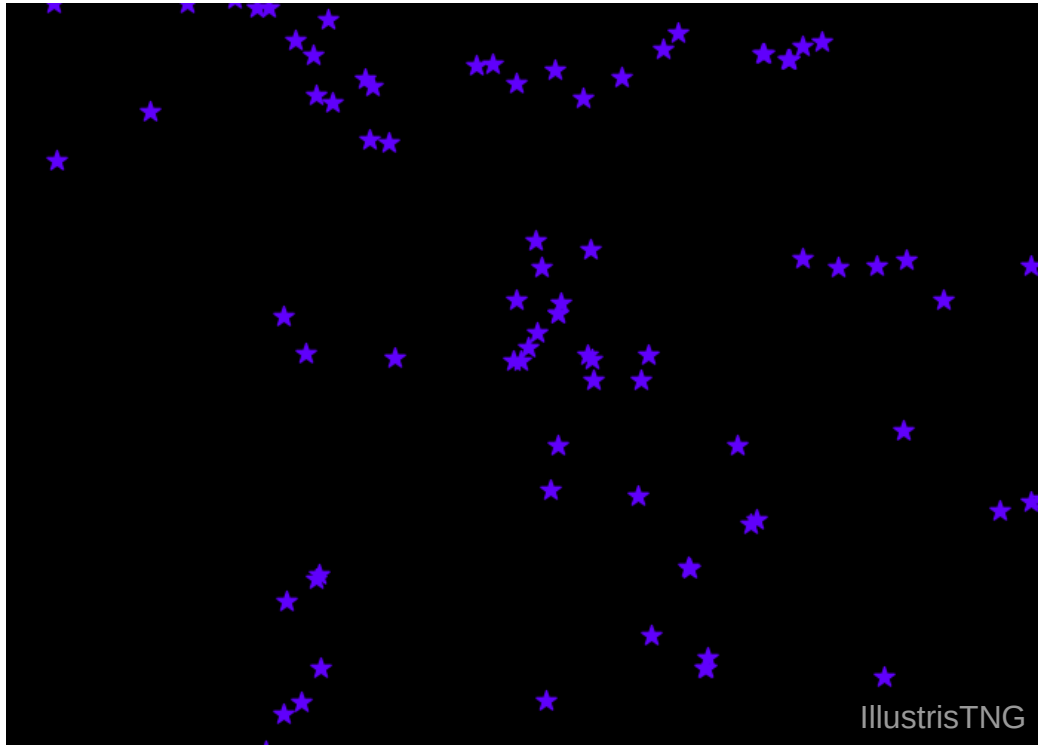
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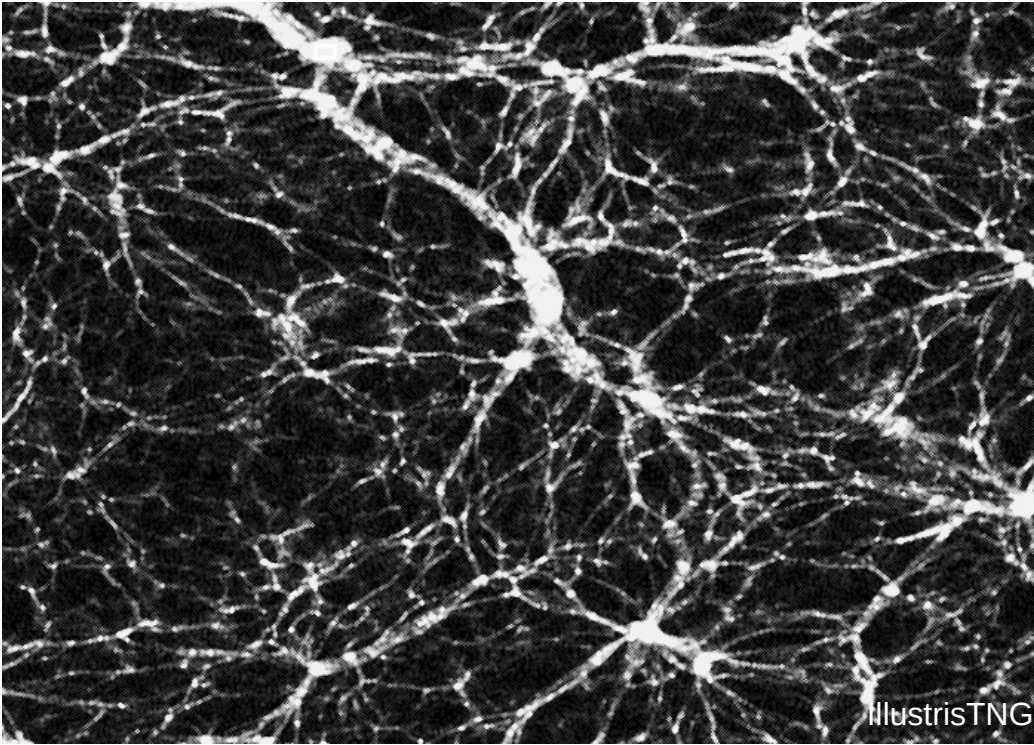
Blue galaxies  $(g - r) < 0.5$



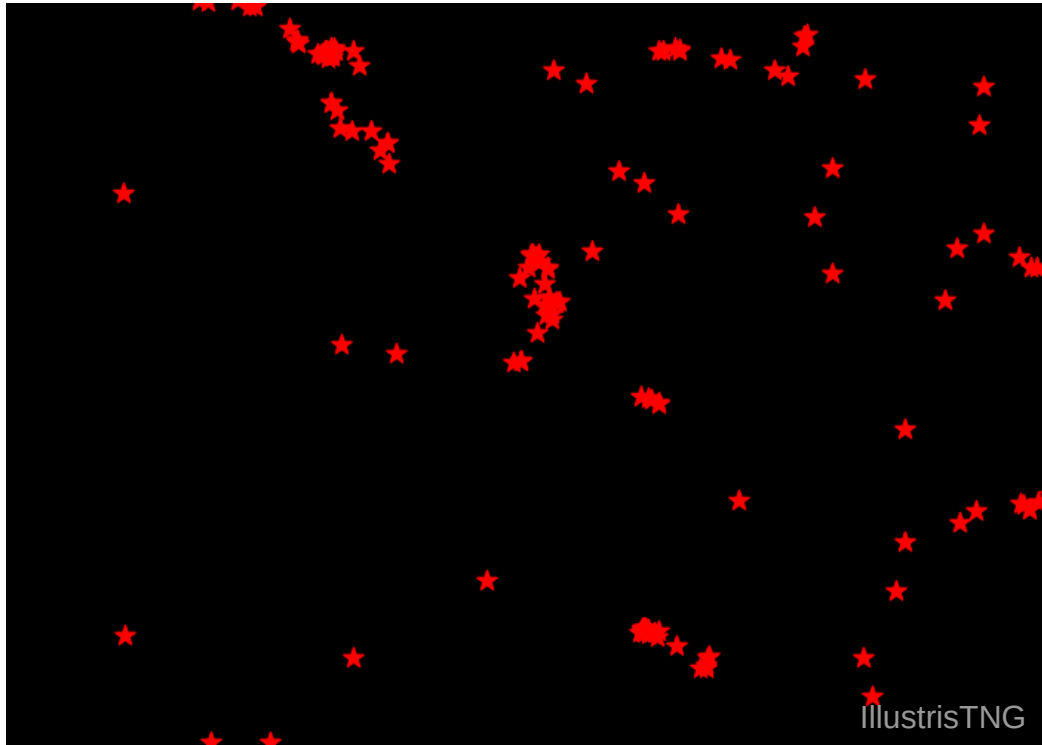
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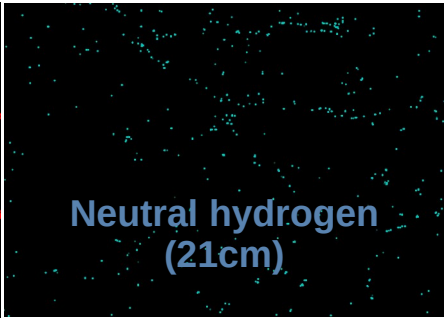
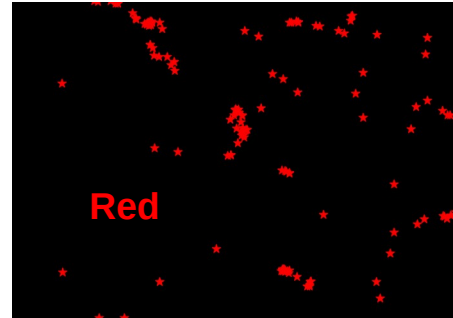
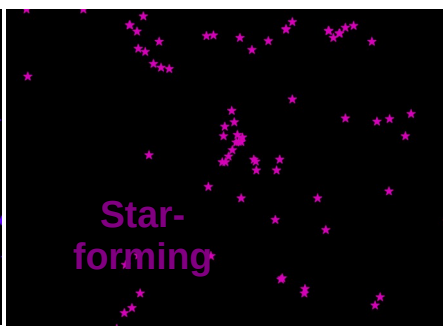
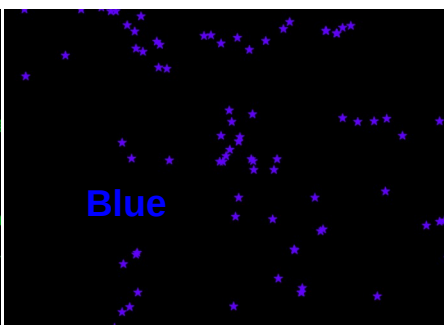
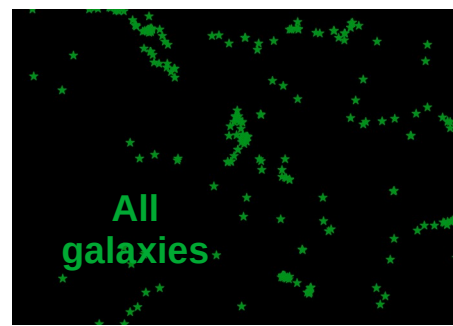
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Red galaxies  $(g - r) \geq 0.5$



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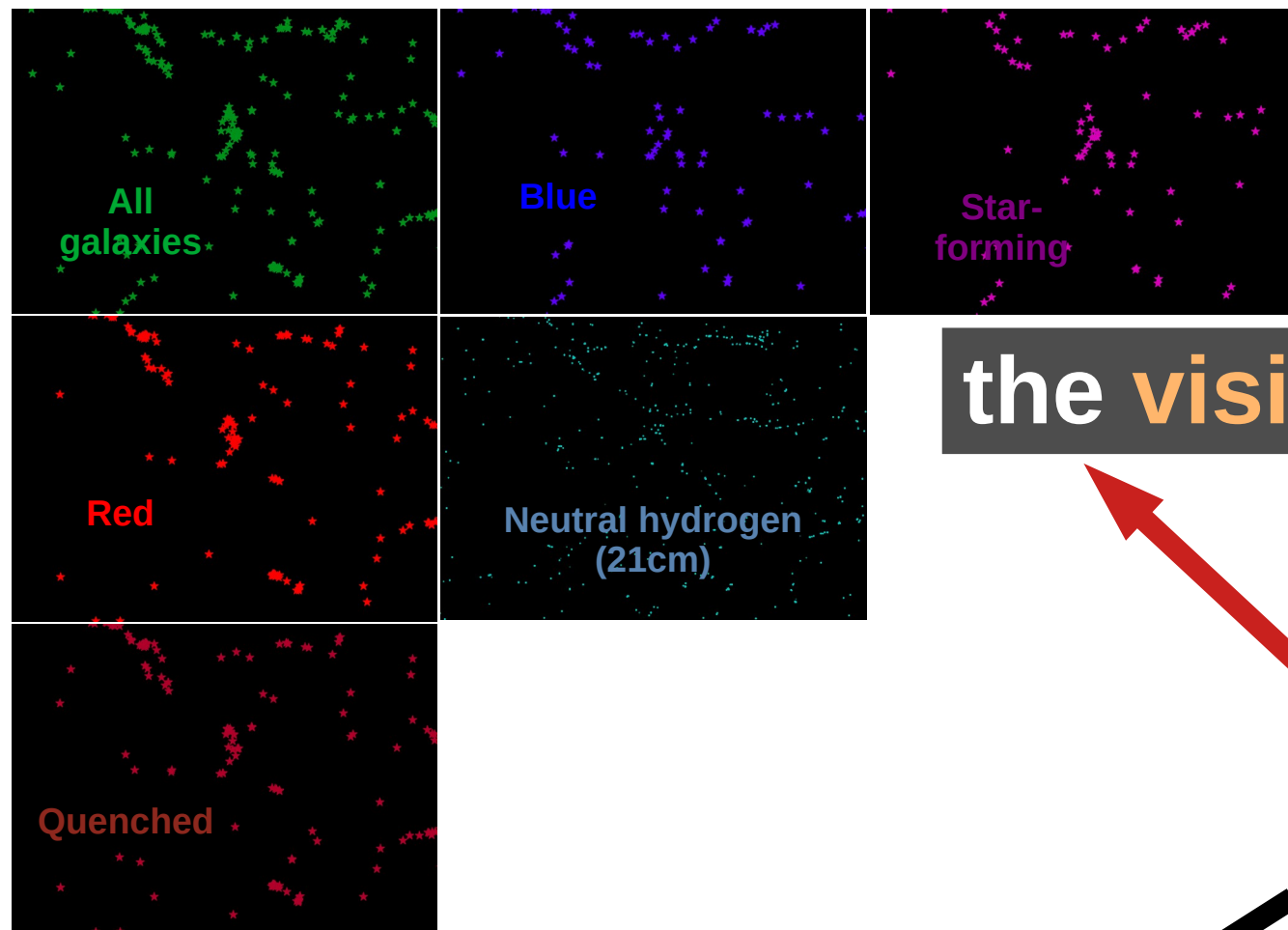


the visible



the dark





the **visible**

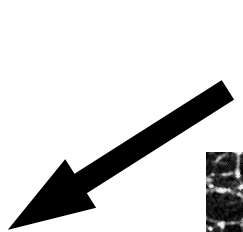
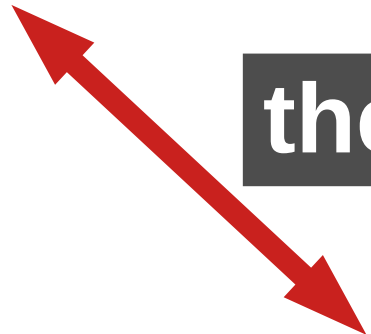
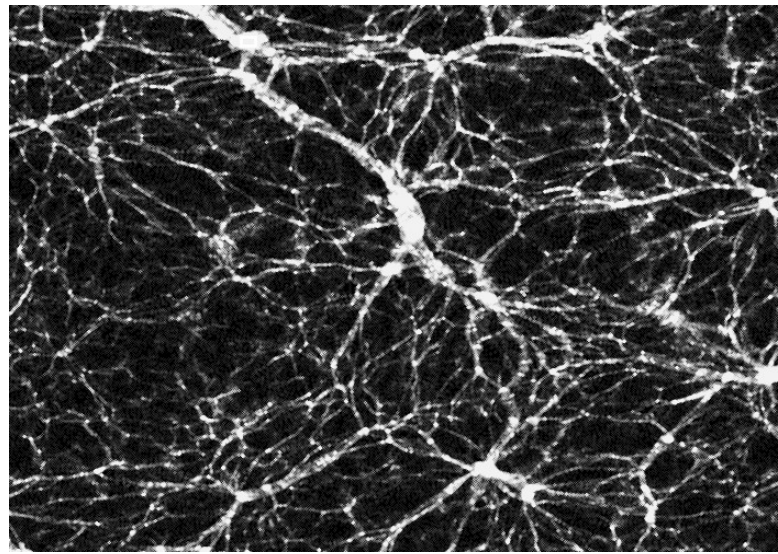
the **connection**

a.k.a. **galaxy bias**

the **dark**

our **cosmology**

- Gravity      Inflation
- Particle Physics
- Dark Matter
- Dark Energy



# Outline

- What is galaxy bias?
- Recent results from galaxy formation simulations.
- Consequences for tests of inflation using galaxy data.

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How does galaxy formation depend on its **dark** environment ?



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$$n_g(\boldsymbol{x}, z) = \bar{n}_g(z) \left[ 1 + \sum_{\mathcal{O}} b_{\mathcal{O}}(z) \mathcal{O}(\boldsymbol{x}, z) \right] + \epsilon(\boldsymbol{x}, z)$$

Local galaxy number density

Global galaxy number density

Large-scale perturbations (density, tidal, potential, etc.)

Bias parameters

Stochasticity (shot-noise)

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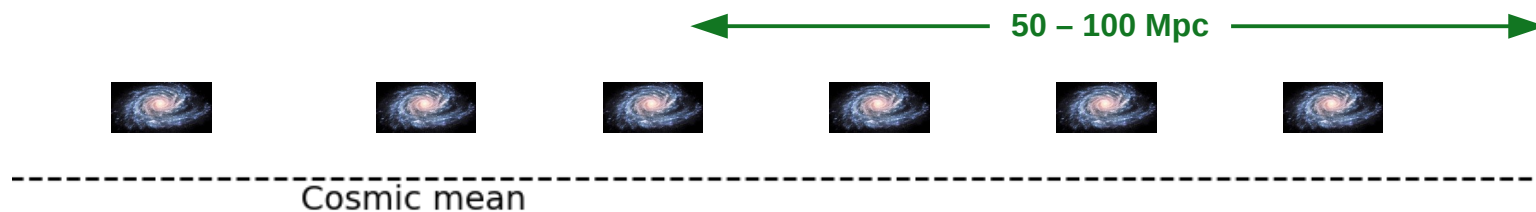
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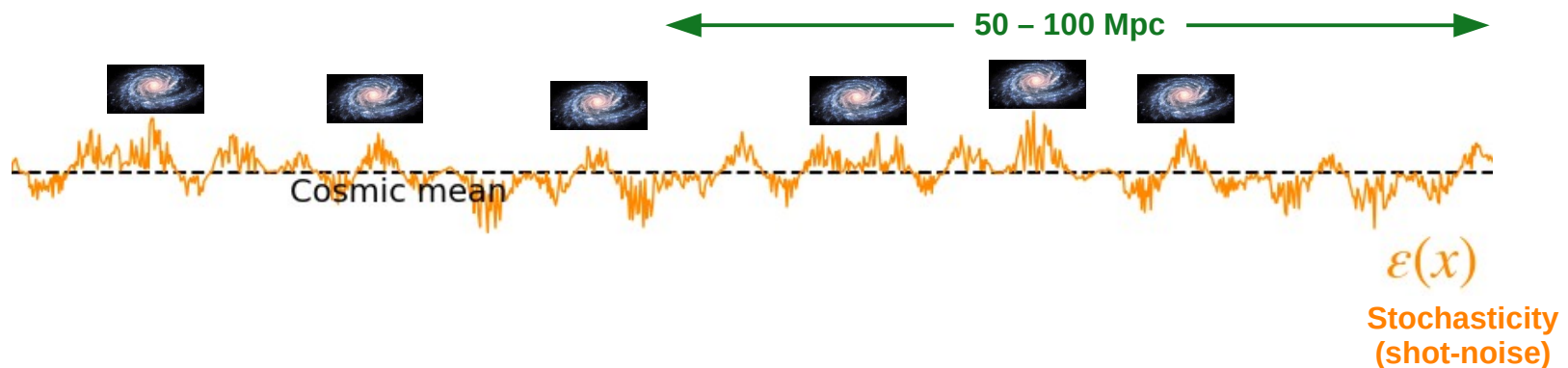
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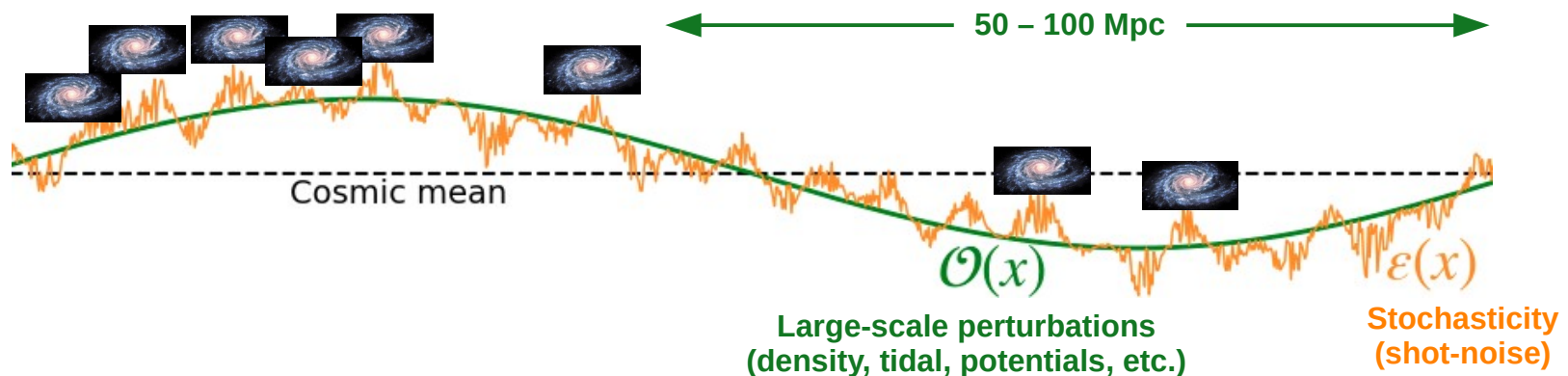
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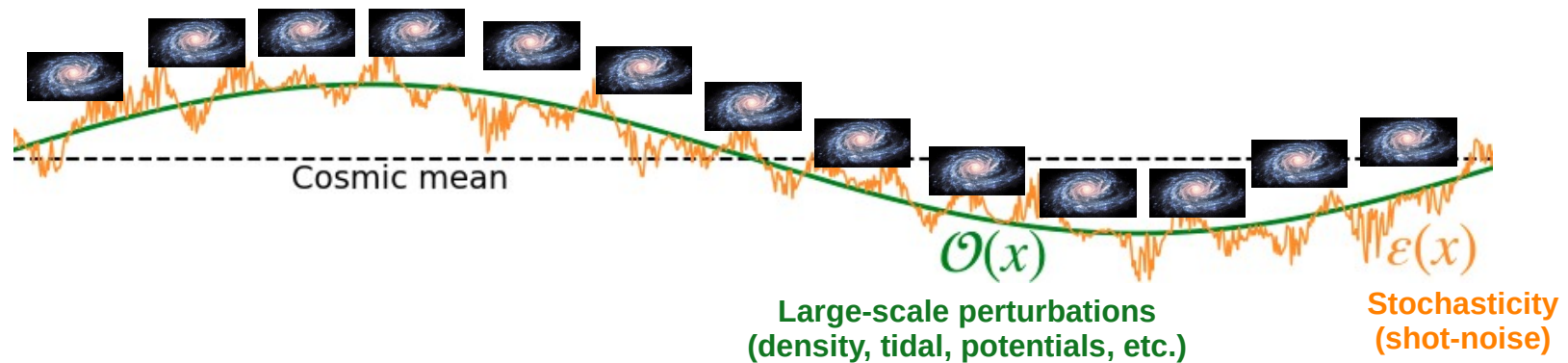


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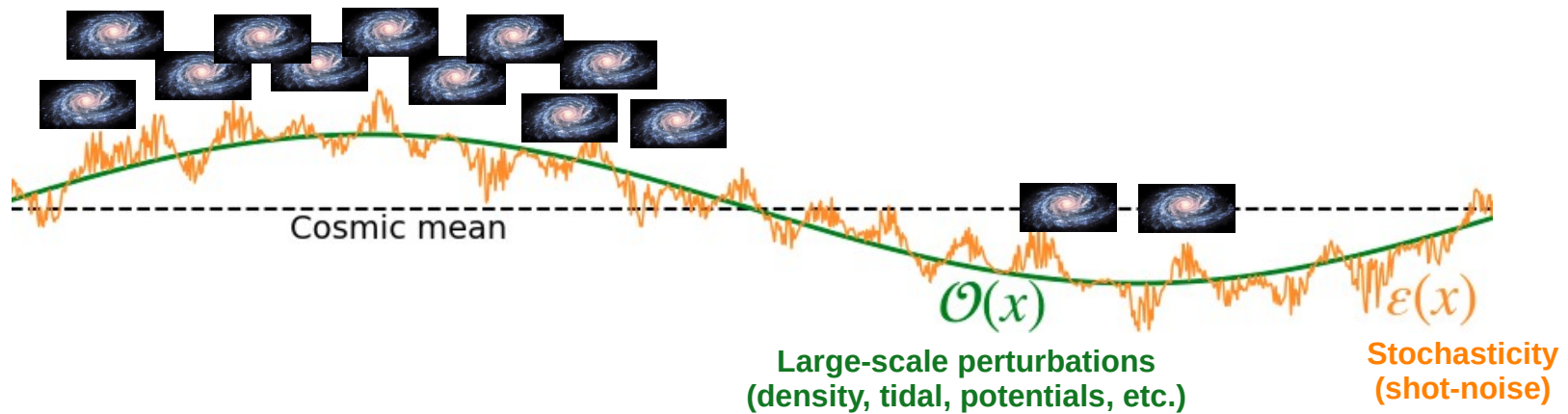


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$$b_{\mathcal{O}} > 0$$

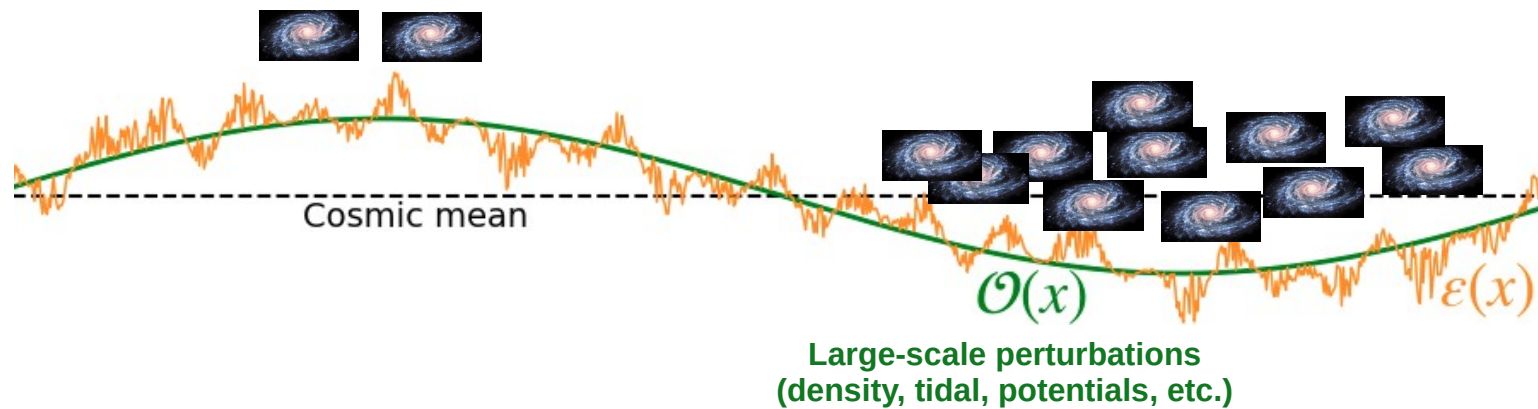


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# Two reasons to study galaxy bias

$$\text{Galaxy data} = \text{Cosmology} \times \text{Galaxy bias}$$

(dark energy, gravity, inflation, etc)

(galaxy-environment connection)

# Two reasons to study galaxy bias

Robust constraints on cosmology



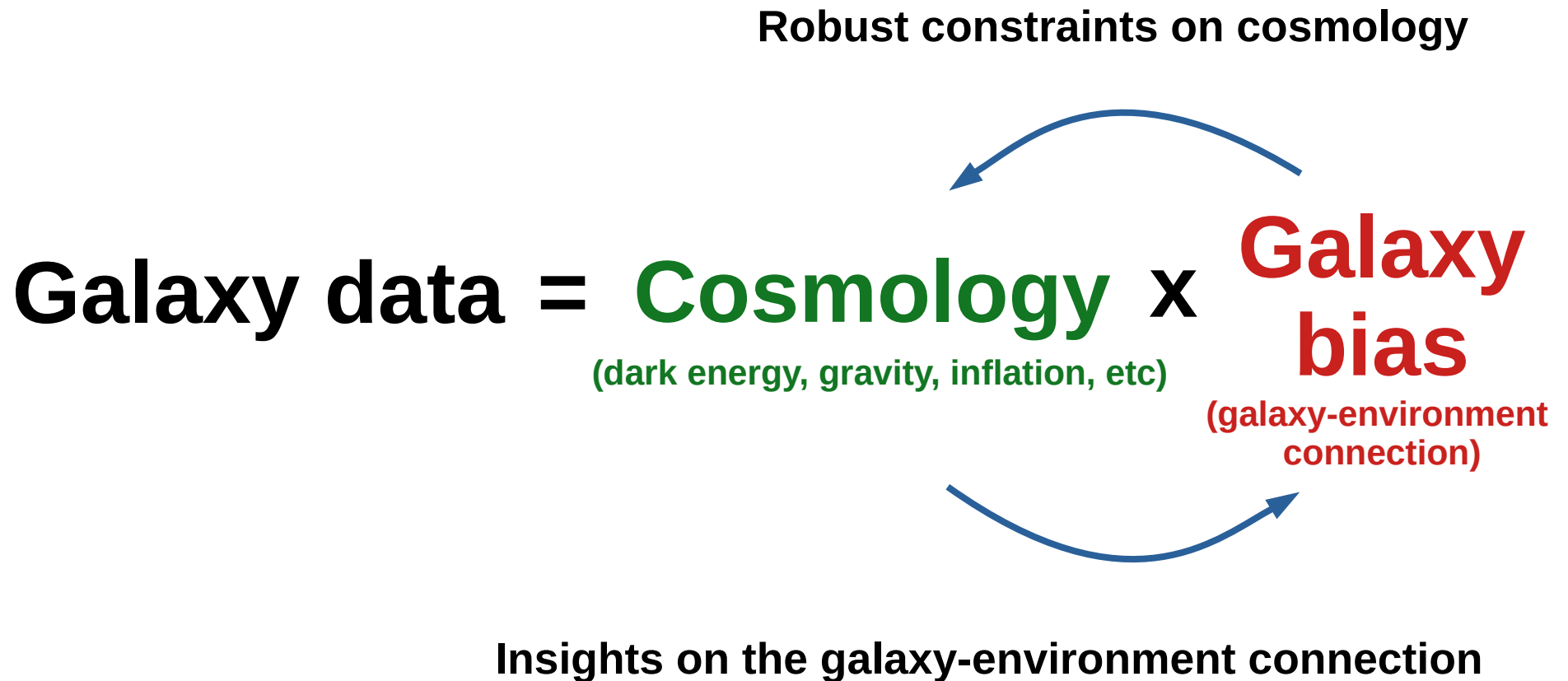
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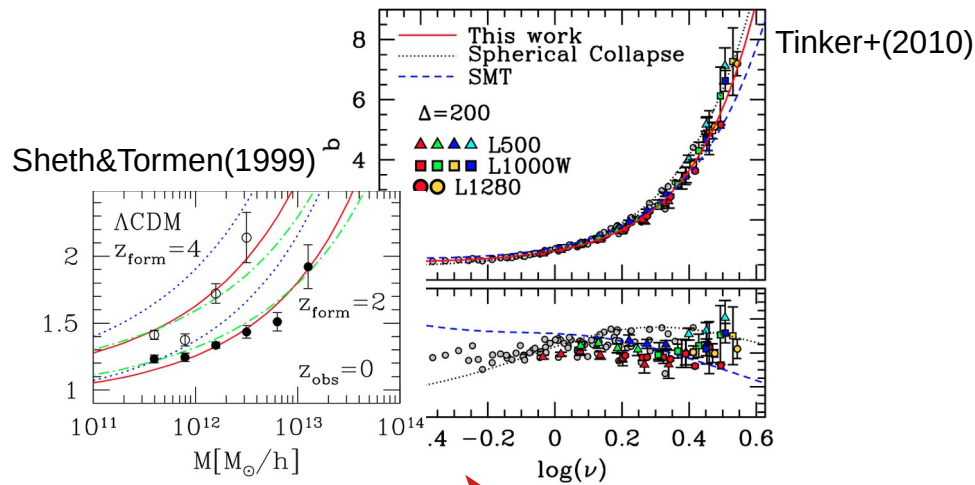
# A long history of halo bias studies

Density contrast

$$\delta_g(\mathbf{x}, z) \stackrel{\text{det.}}{=} b_1(z)\delta_m(\mathbf{x}, z) + b_\phi(z)f_{\text{NL}}\phi(\mathbf{x})$$
$$+ \frac{b_2(z)}{2}\delta_m(\mathbf{x}, z)^2 + b_{K^2}(z)K_{ij}(\mathbf{x}, z)^2 + b_{\phi\delta}(z)f_{\text{NL}}\phi(\mathbf{x})\delta_m(\mathbf{x}, z)$$

**P.S.** No such thing as scale-dependent bias: the bias parameters are only a function of  $z$ .

# A long history of halo bias studies



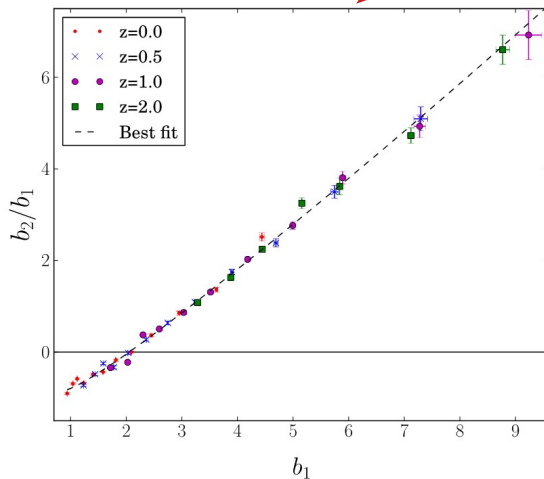
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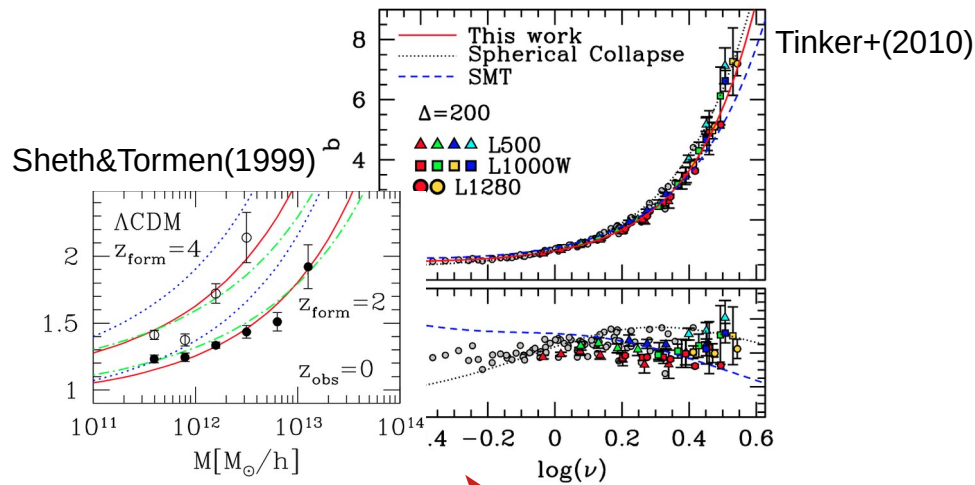
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Lazeyras+(2016)



# A long history of halo bias studies



Density contrast

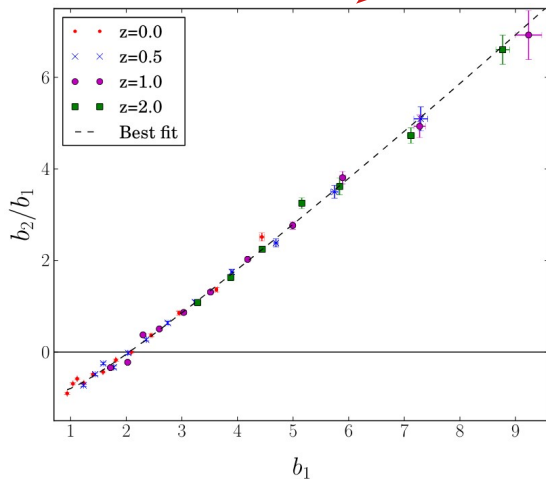
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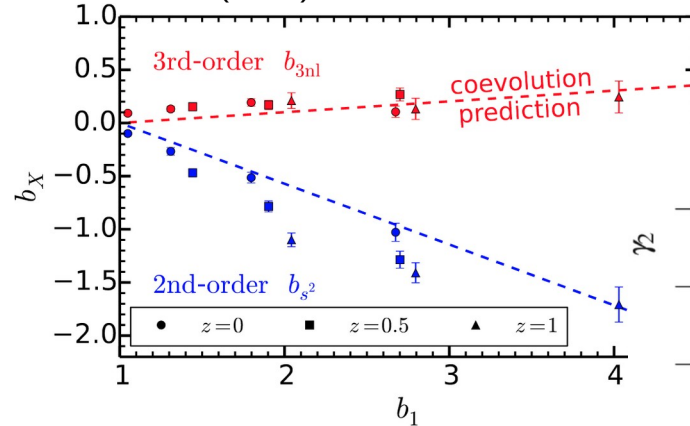
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Tidal field

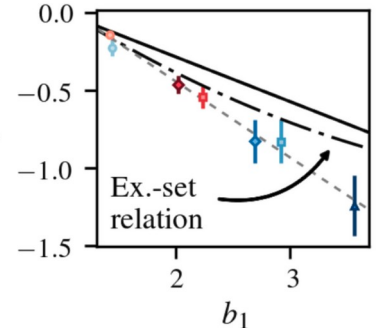
Lazeyras+(2016)



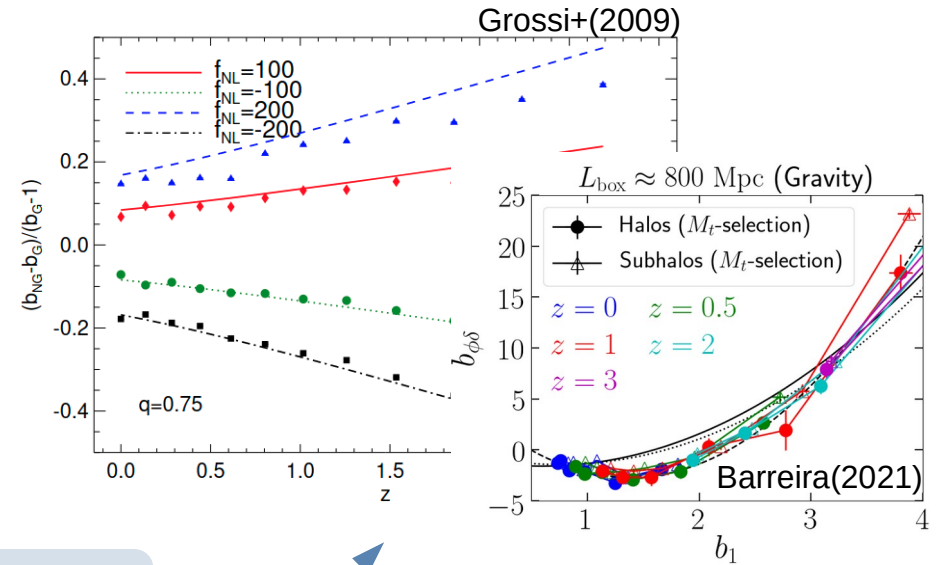
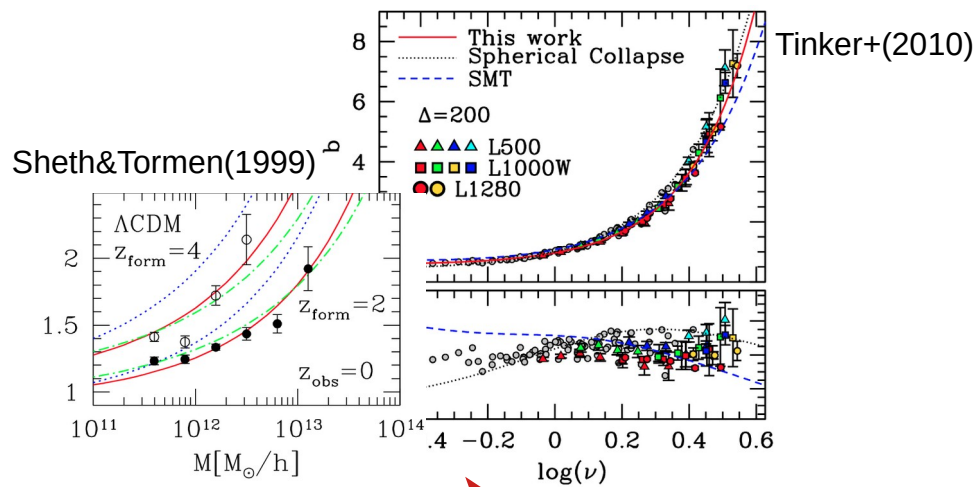
Saito+(2014)



Eggemeier+(2021)



# A long history of halo bias studies



Density contrast

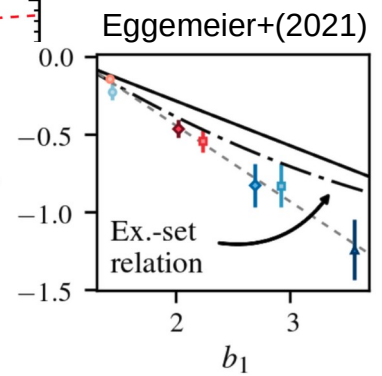
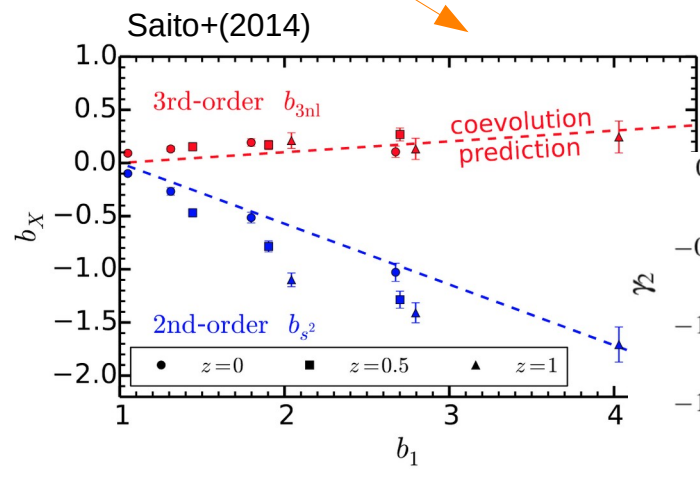
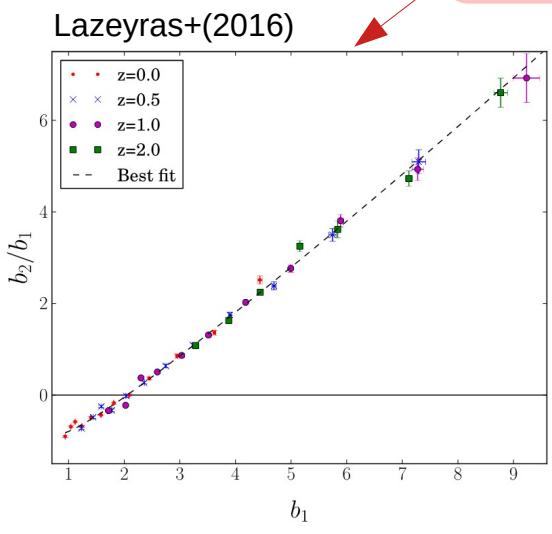
$$\delta_g(\mathbf{x}, z) \stackrel{\text{det.}}{=} b_1(z)\delta_m(\mathbf{x}, z) + b_\phi(z)f_{\text{NL}}\phi(\mathbf{x})$$

Total mass density

Primordial potential with local PNG

$$+ \frac{b_2(z)}{2}\delta_m(\mathbf{x}, z)^2 + b_{K^2}(z)K_{ij}(\mathbf{x}, z)^2 + b_{\phi\delta}(z)f_{\text{NL}}\phi(\mathbf{x})\delta_m(\mathbf{x}, z)$$

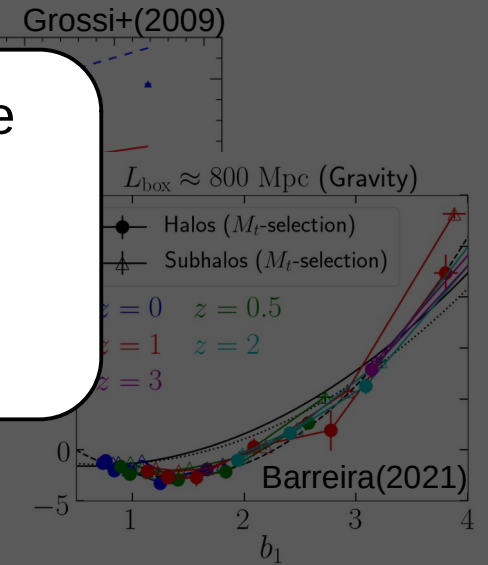
Tidal field



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Most of the existing knowledge of **biasing** is limited to the case of **dark matter halos in gravity-only simulations**.

**Not quite sufficient** since we don't observe halos, but the galaxies inside them.

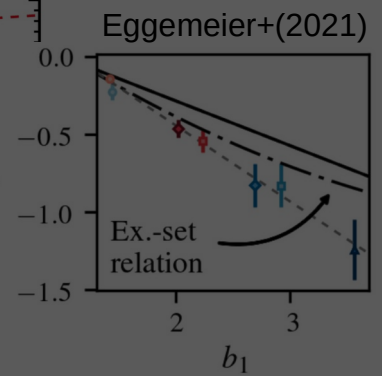
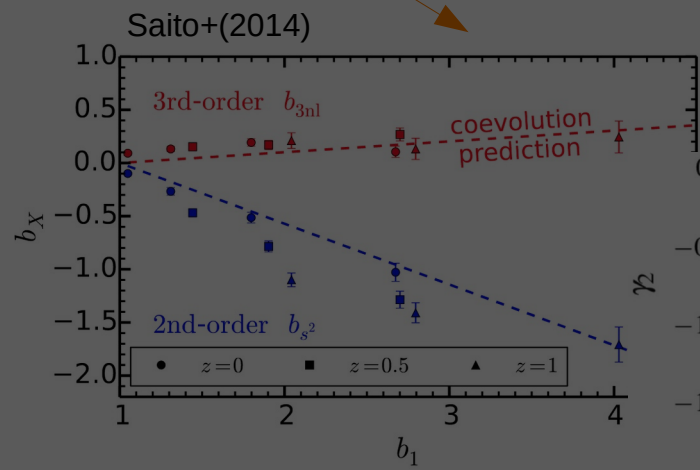
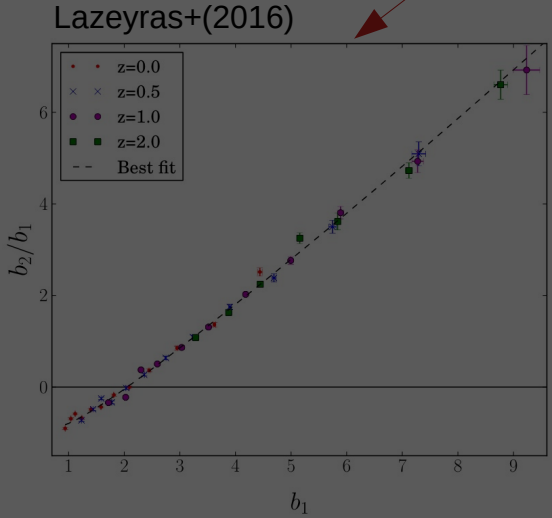


Density contrast  $\delta_g(\mathbf{x}, z)$   $\stackrel{\text{det.}}{=} b_1(z)\delta_m(\mathbf{x}, z) + b_\phi(z)f_{\text{NL}}\phi(\mathbf{x})$

**Total mass density**  $b_1(z)\delta_m(\mathbf{x}, z)$       **Primordial potential with local PNG**  $b_\phi(z)f_{\text{NL}}\phi(\mathbf{x})$

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**Tidal field**  $b_{K^2}(z)K_{ij}(\mathbf{x}, z)^2$

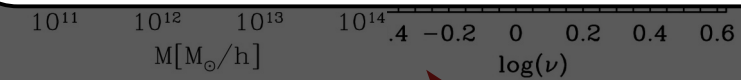
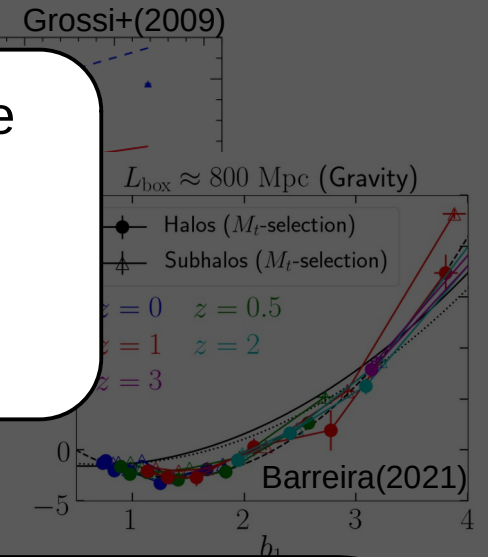




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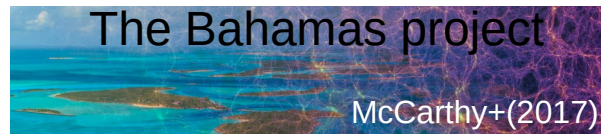
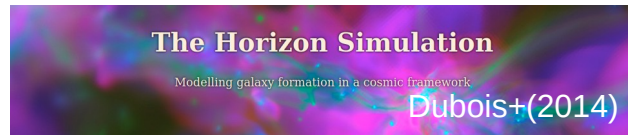
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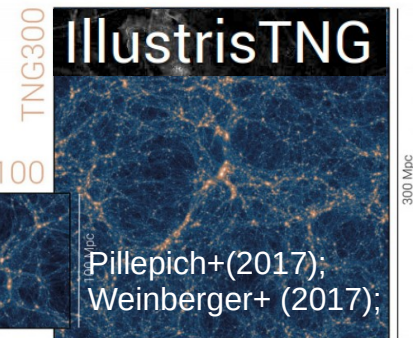
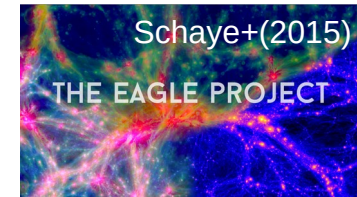
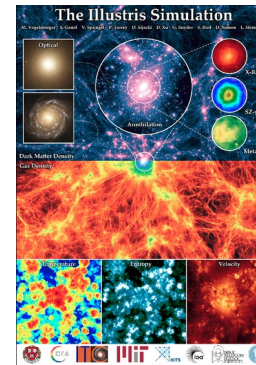


Density contrast  $\delta_g(\mathbf{x}, z) \stackrel{\text{det.}}{=} b_1 \delta_{\text{DM}}$

Galaxy bias studies made possible only recently as **galaxy formation simulations approach cosmological volumes**.

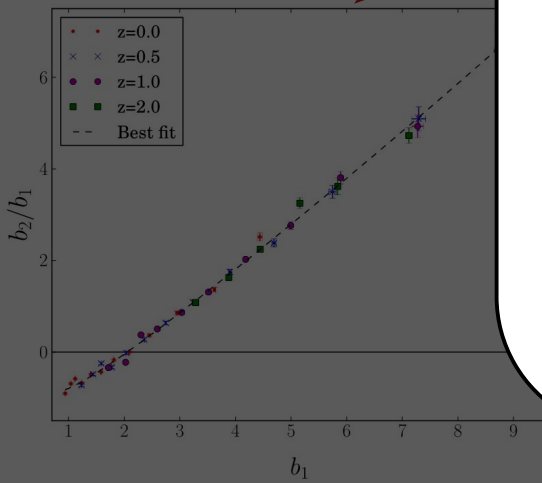


Vogelsberger+(2014)



Pillepich+(2017); Weinberger+(2017);

Lazeyras+(2016)



# Outline

- What is galaxy bias?
- Recent results from galaxy formation simulations.
- Consequences for tests of inflation using galaxy data.

# The IllustrisTNG galaxy formation model

## The IllustrisTNG galaxy formation model

<http://www.tng-project.org>; Pillepich+(2017); Weinberger+ (2017);

- Gravity + hydrodynamical cosmological simulations of galaxy formation.  
( $L_{\text{box}} = 205\text{Mpc}/h, 75\text{Mpc}/h$ )
- Includes modeling of: gas cooling; star formation, stellar feedback, black hole growth & feedback.
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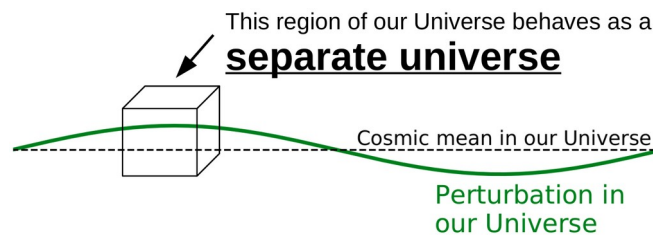
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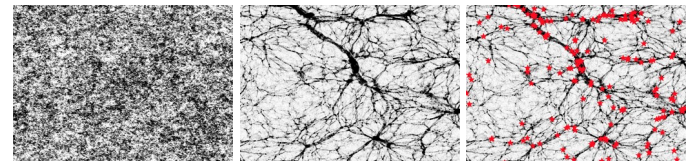
### Separate Universe simulations



Sirko+(2005), Wagner+(2014), Li+(2014, 2016)  
Schmidt+(2018), Barreira+(2019)

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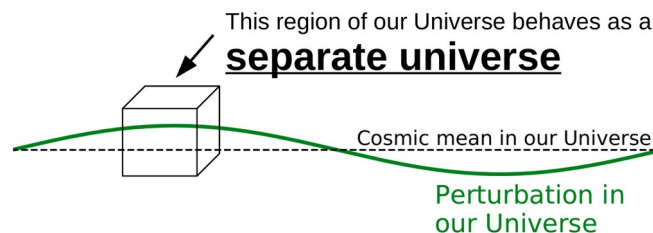
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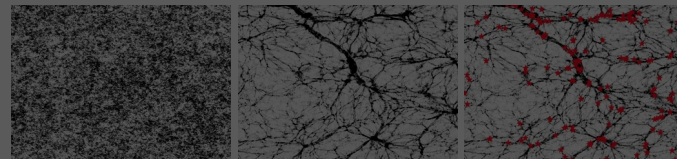
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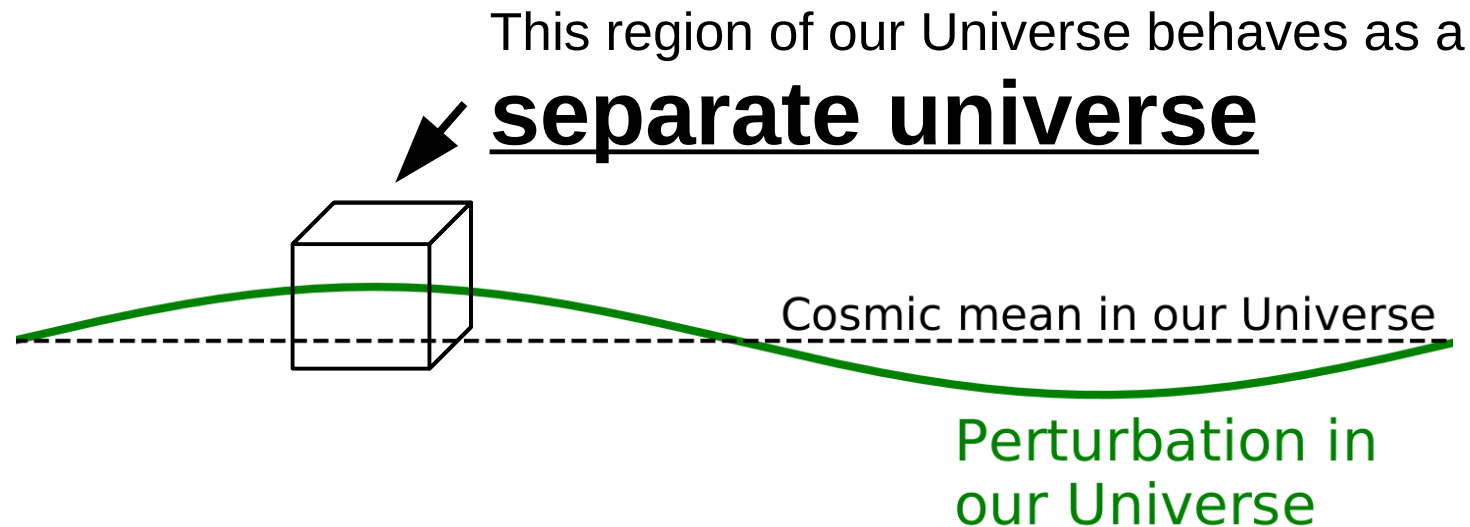
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# Separate Universe simulations

Local structure formation inside long-wavelength perturbations in a Fiducial cosmology is equivalent to global structure formation in a modified cosmology.



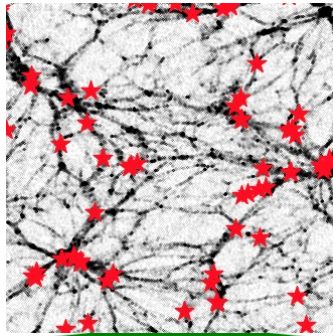
## Significant advantage of this method:

simulation does not have to have a large volume to encompass large-scale perturbations!  
(especially significant for hydrodynamical simulations, which are more expensive)

# Separate Universe simulations

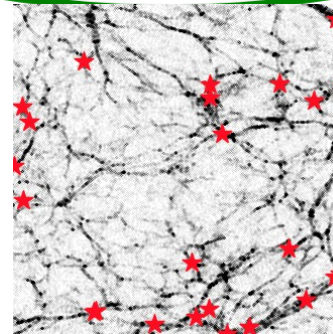
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Simulation of **one cosmology**



Bias as the response of the galaxy abundance to changes in the **cosmological parameters**.

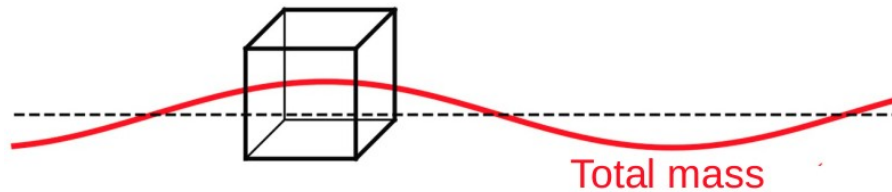
**VS.**



Simulation of **another cosmology**

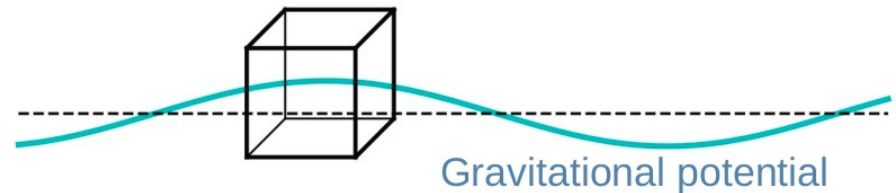
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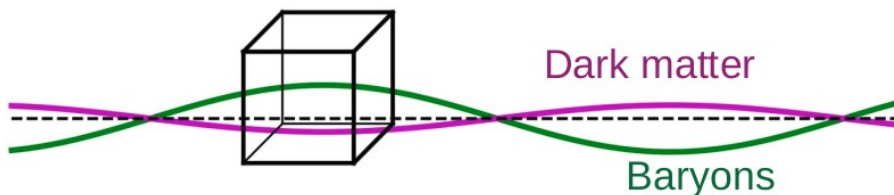
**Separate Universe:** modified cosmic matter density.

Sirko+(2005), Wagner+(2014), Li+(2014), Lazeyras+(2016)



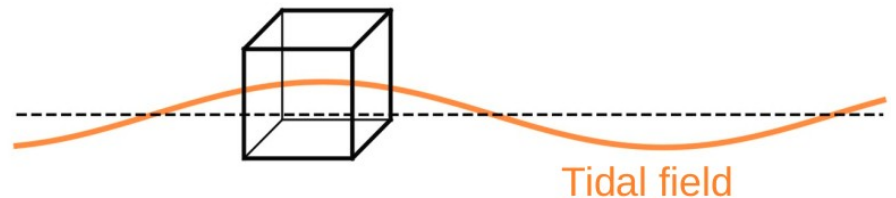
**Separate Universe:** modified amplitude of the primordial power spectrum.

Biagetti+(2016), Barreira+(2020), Barreira (2021)



**Separate Universe:** modified cosmic baryon and dark matter densities.

Barreira+(2020), Voivodic&Barreira (2021), Khoraminezhad+(2021)



**Separate Universe:** anisotropic spacetime (e.g. Bianchi cosmology).

Schmidt+(2018), Stuecker+(2020), Masaki+(2020), Akitsu+(2021)

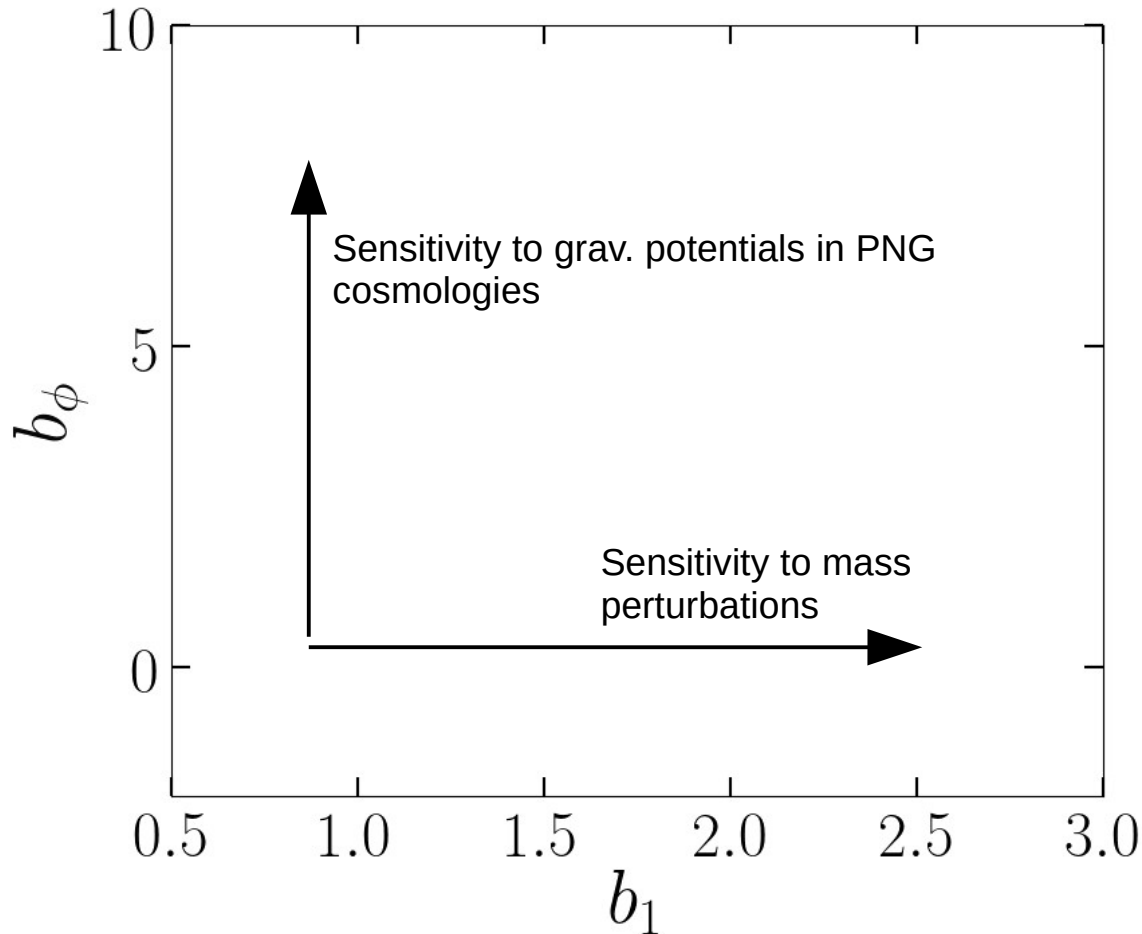


# Results: PNG bias parameters

From separate universe simulations w/ IllustrisTNG

Barreira+(2020)  
Barreira(2021)

$$\delta_g(\mathbf{x}, z) \supset \underbrace{b_1(z)\delta_m(\mathbf{x}, z)}_{\text{Mass density}} + \underbrace{b_\phi(z)f_{\text{NL}}\phi(\mathbf{x})}_{\text{Gravitational potential}}$$



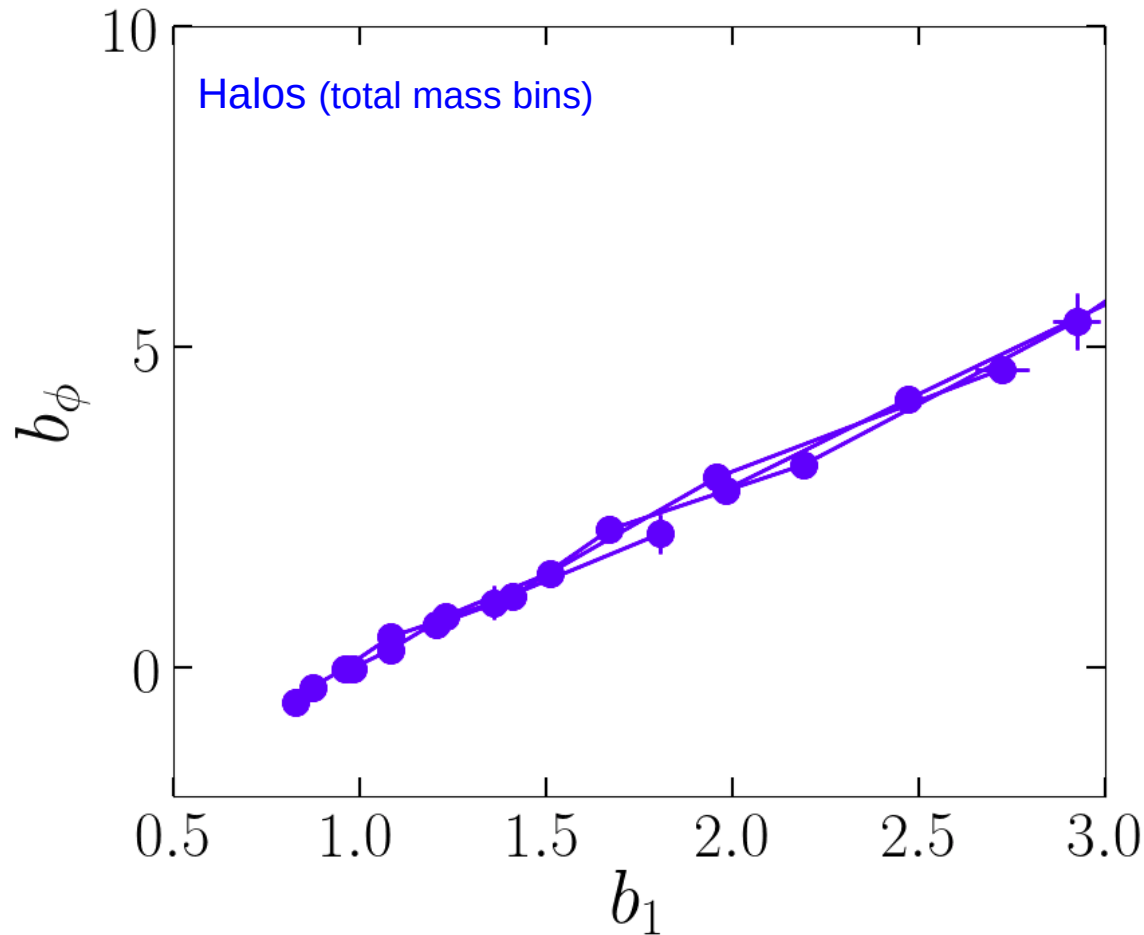
**Recall:** the space of **bias parameters** organizes the tracers by their sensitivity to changes in **their environment**.

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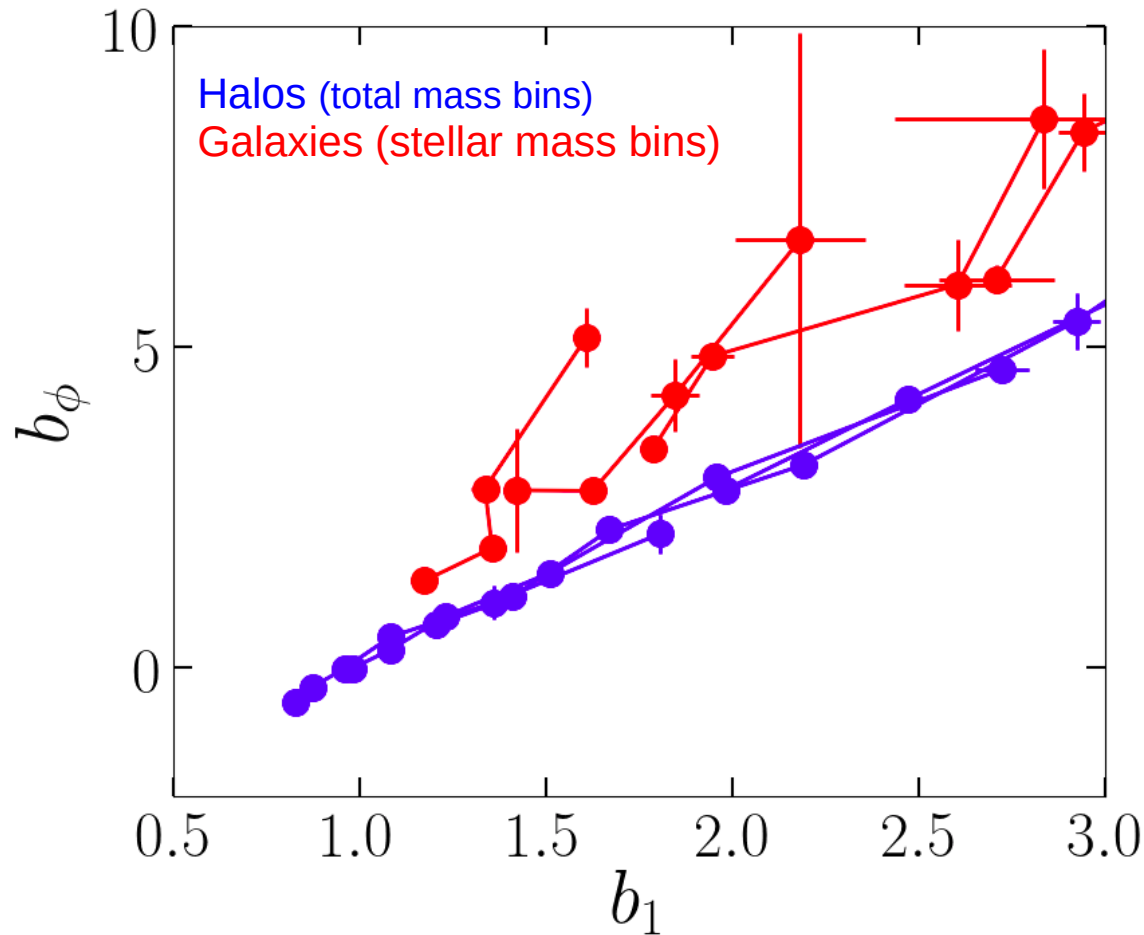


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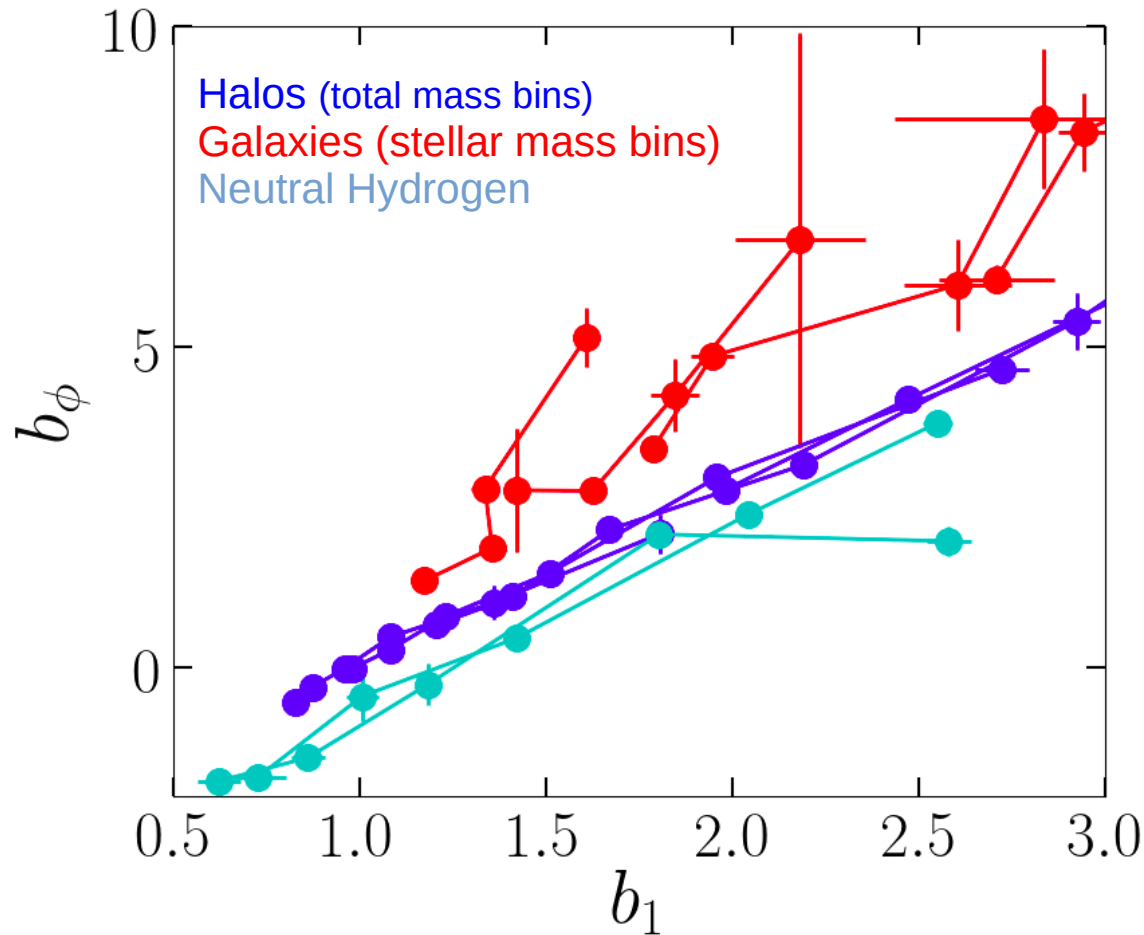


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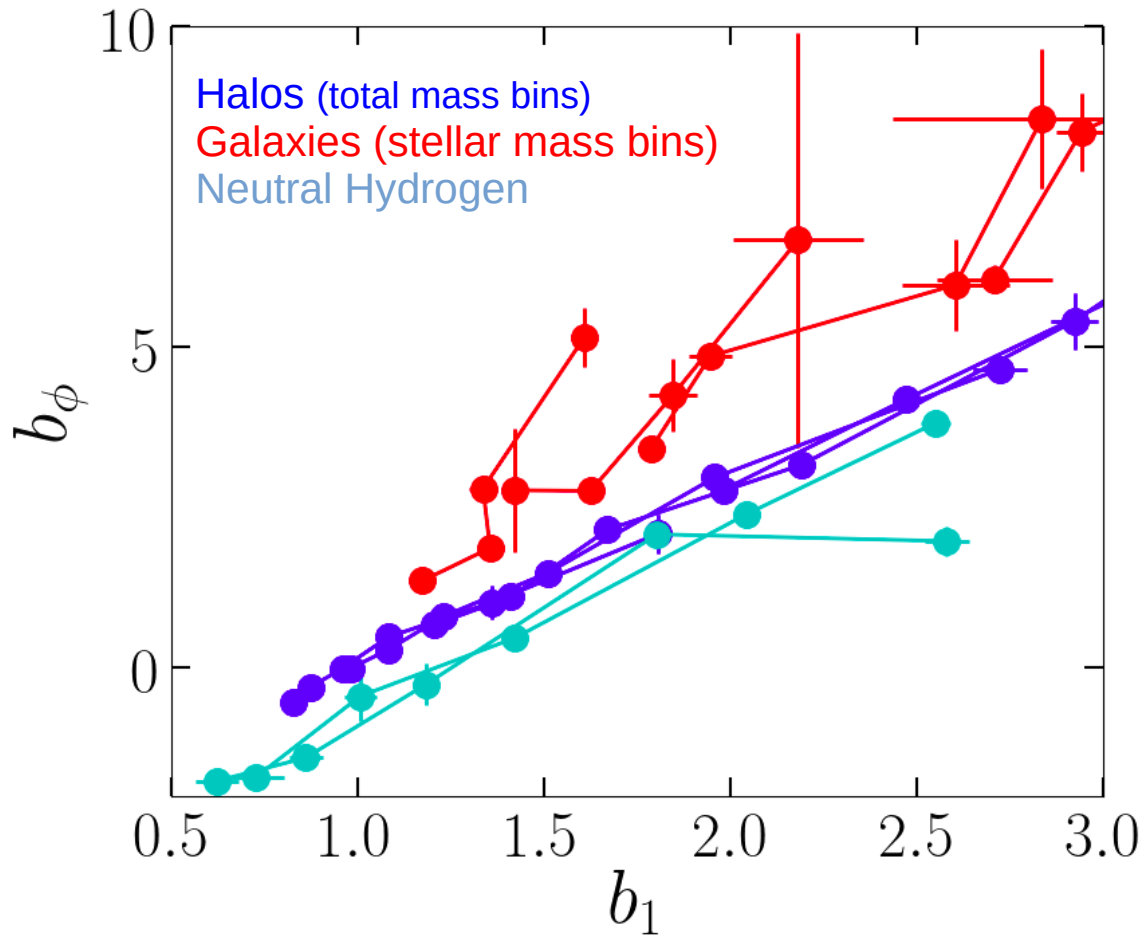


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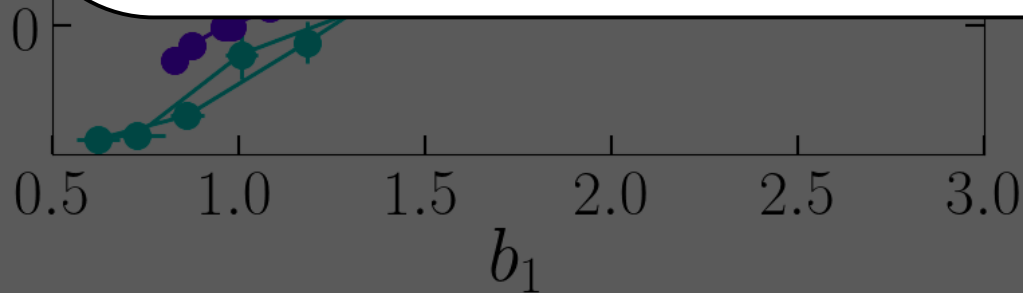
**Galaxy bias**  $\sim$  **Halo bias** + **Occupation bias**

$b_\phi$

Cosmic mean

$\mathcal{O}(x)$

Voivodic & Barreira 2021



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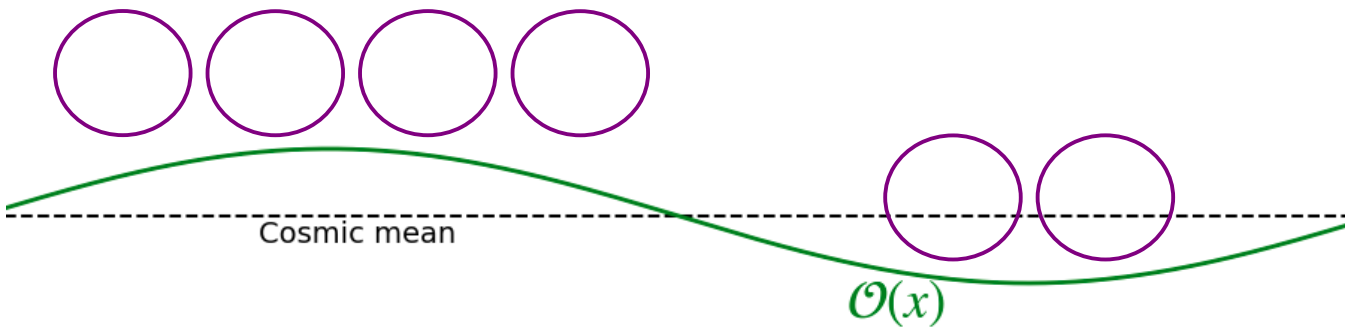
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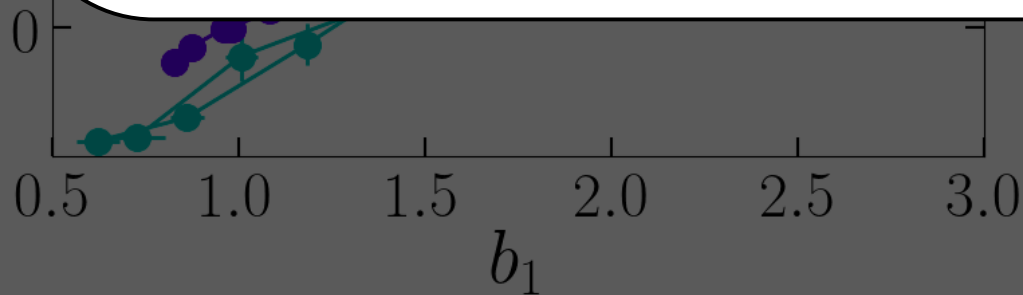
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How many more halos exist inside the perturbations?

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## Galaxy bias

~

## Halo bias

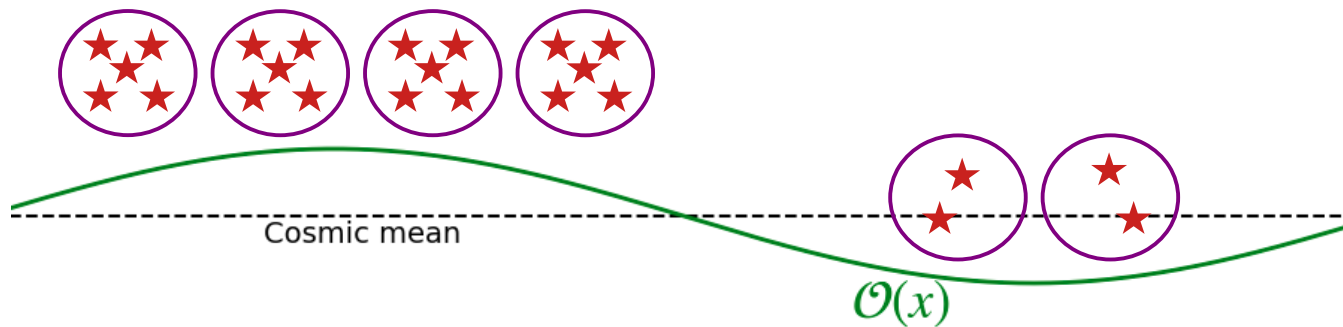
+

## Occupation bias

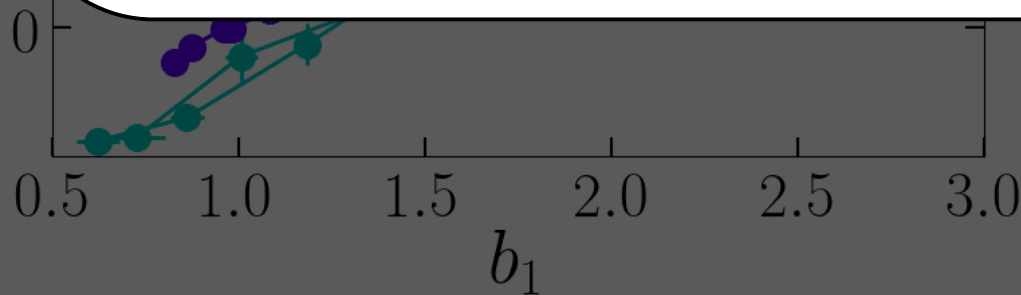
How many more halos exist inside the perturbations?

How many more galaxies each halo hosts inside the perturbations?

$b_\phi$



Voivodic & Barreira 2021





# Halo occupation bias

Voivodic & Barreira 2021

Bias of the galaxies

$$b_g = \frac{1}{\bar{n}_g} \int dM n_h(M) N_g(M) b_h(M)$$

**Halo mass function**      **Galaxy occupation distribution**      **Halo bias**

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Voivodic & Barreira 2021

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Halo mass functionGalaxy occupation distributionHalo biasOccupation bias

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Halo mass functionH I contentHalo biasH I content bias

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Voivodic & Barreira 2021

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for grav. potentials in IllustrisTNG

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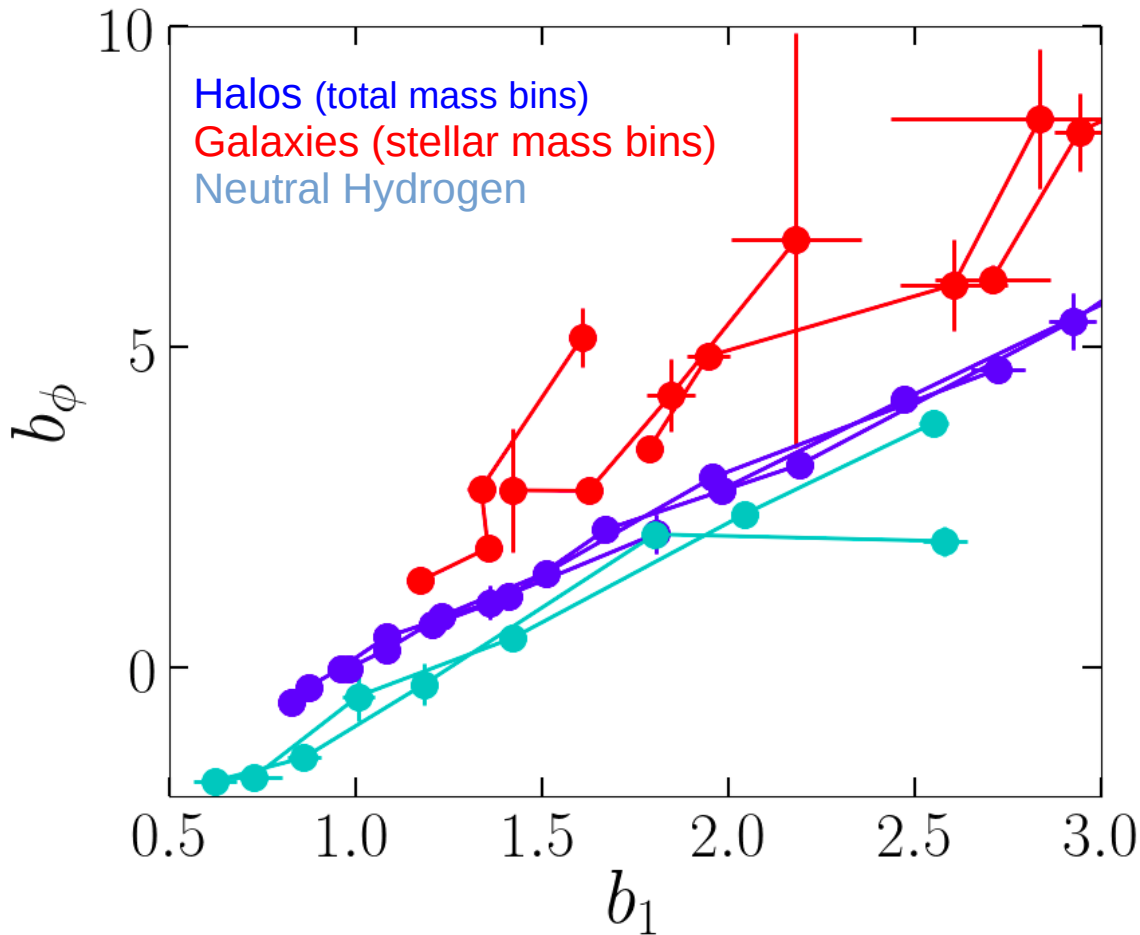
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Because inside  $f_{\text{NL}}\phi$  perturbations, the **halos** contain **more galaxies** and **less neutral hydrogen**.

# The IllustrisTNG galaxy formation model

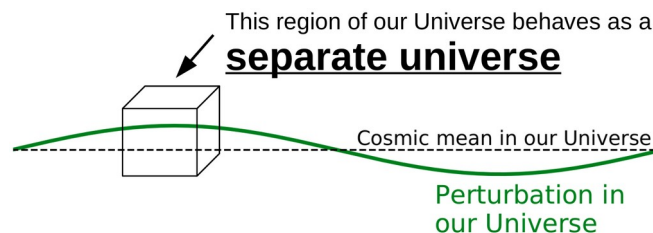
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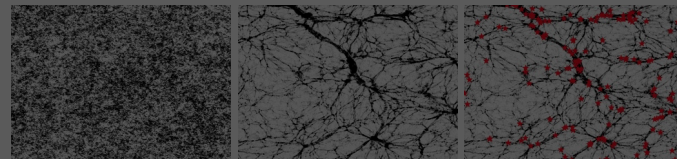
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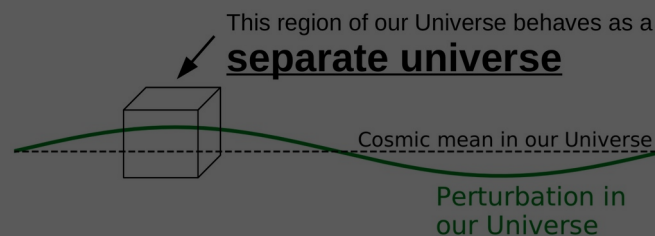
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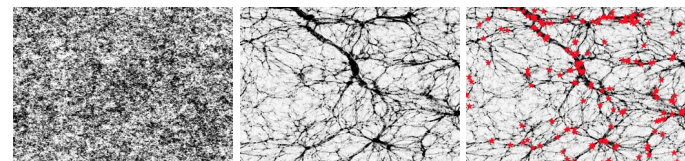
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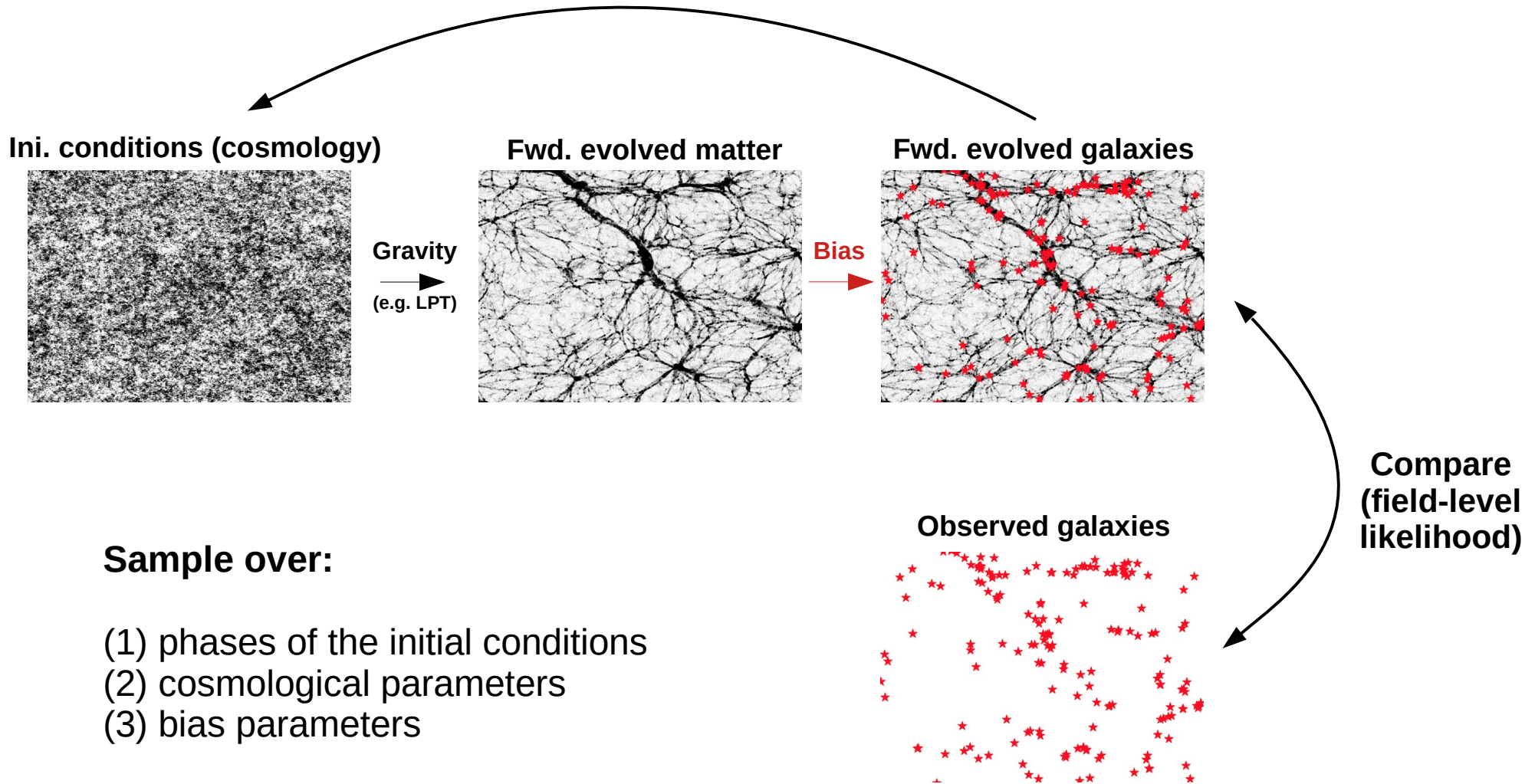
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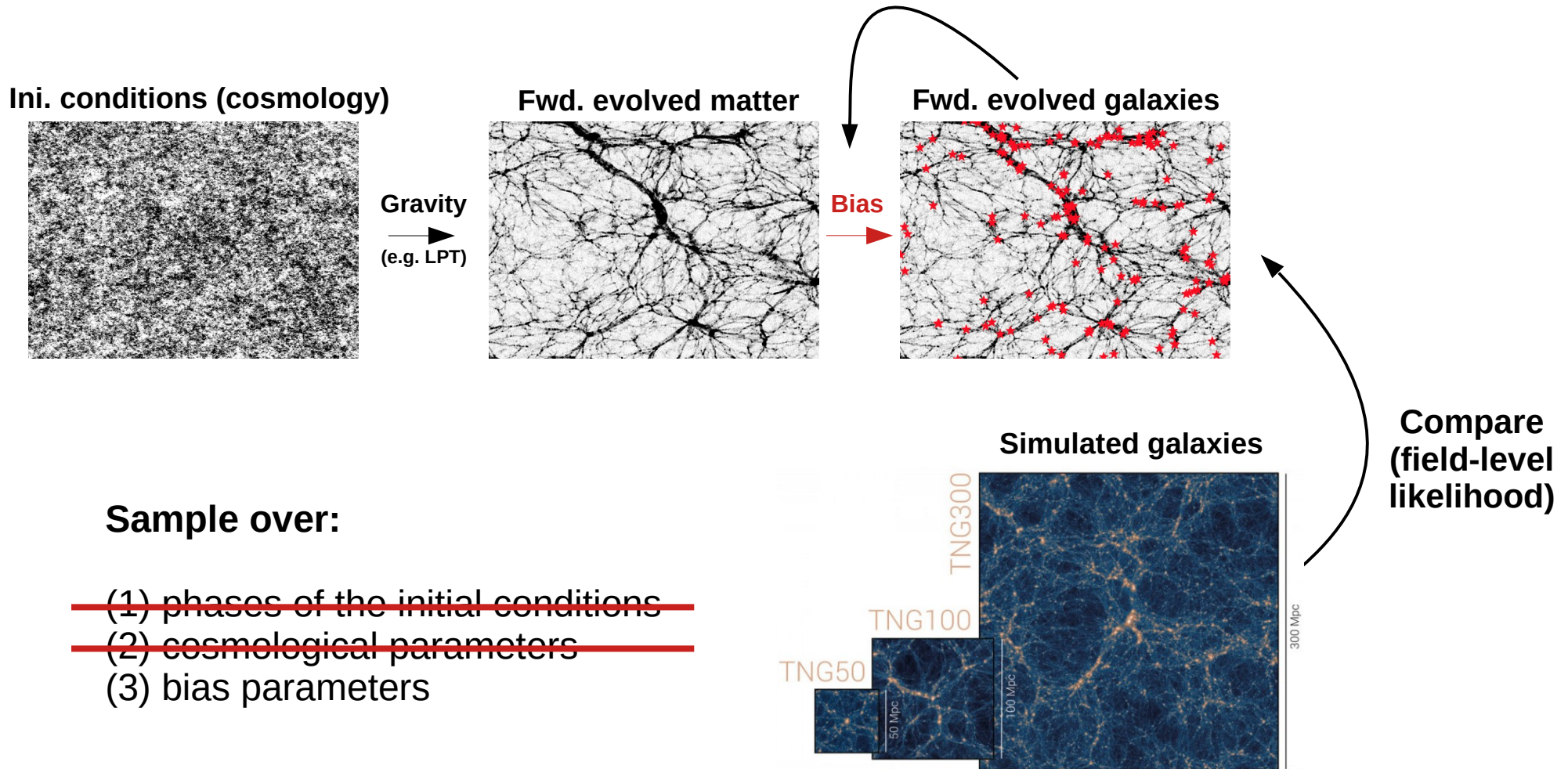


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**Significant advantage:** inference is at the field-level (not a summary statistic), so it uses all available information. **Good signal-to-noise even from ~100Mpc boxes.**



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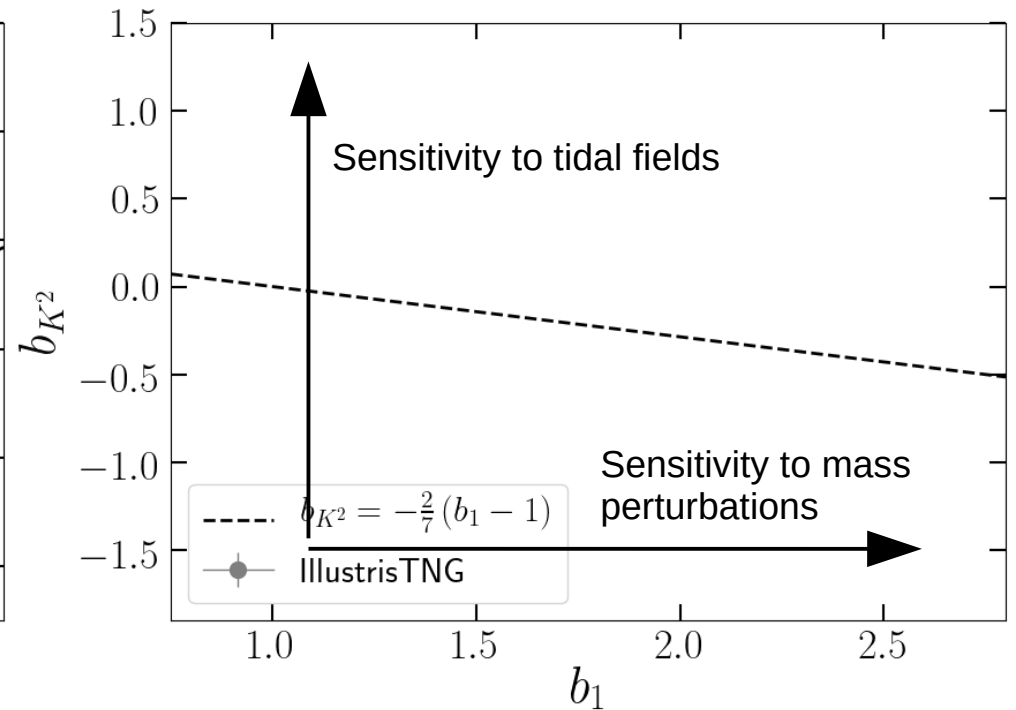
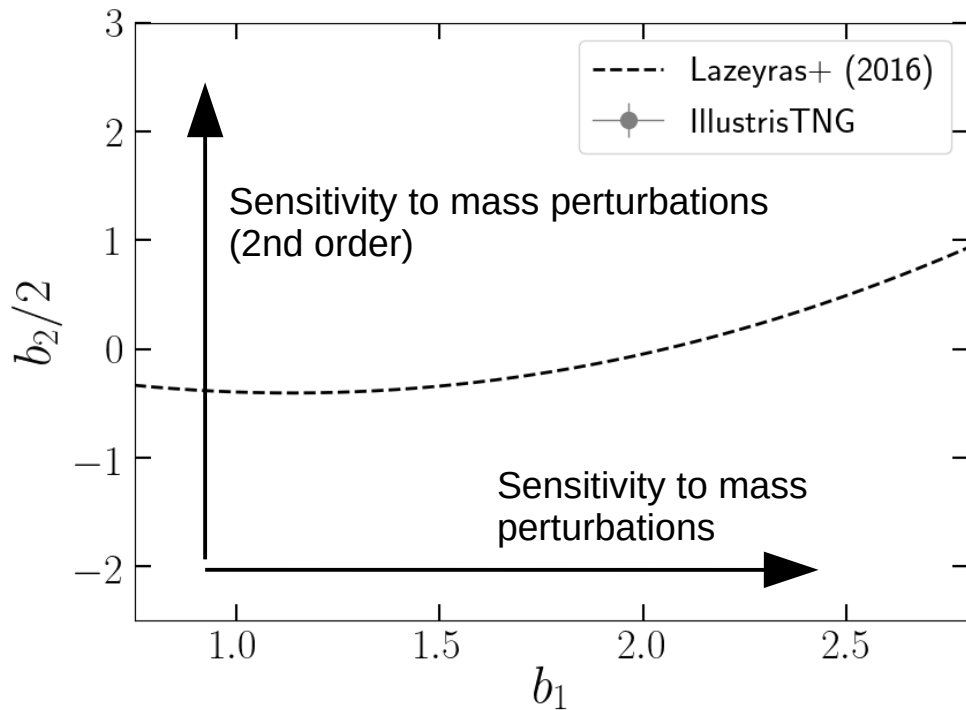
# Results: quadratic bias parameters

From field-level forward models w/ IllustrisTNG

Barreira, Lazeyras & Schmidt (2021)

Lazeyras, Barreira & Schmidt (2021)

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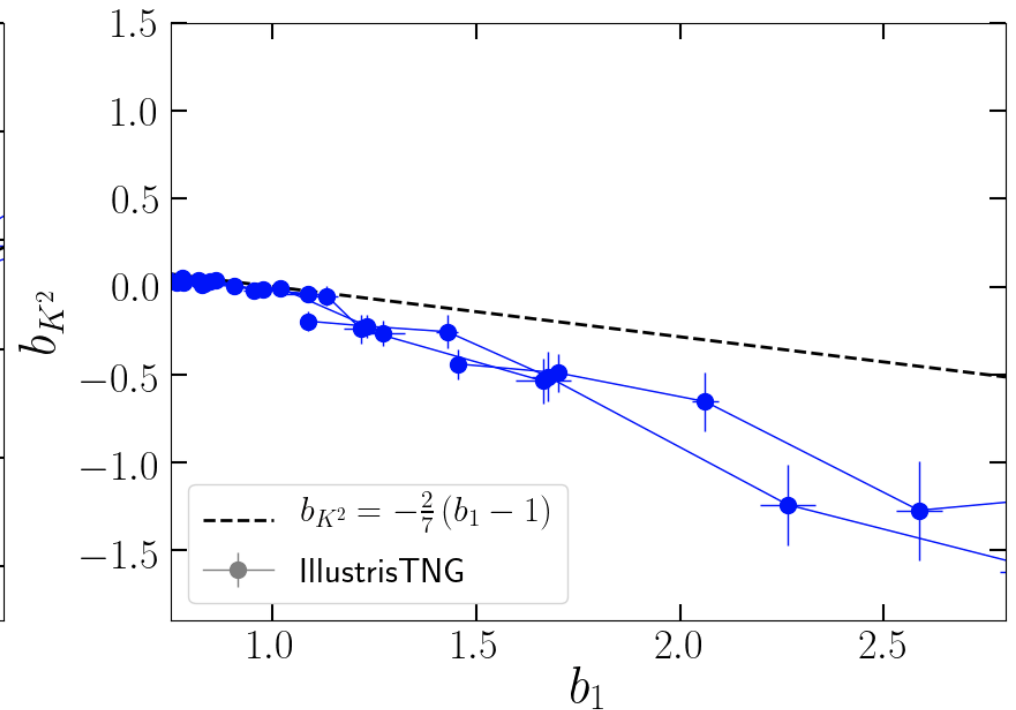
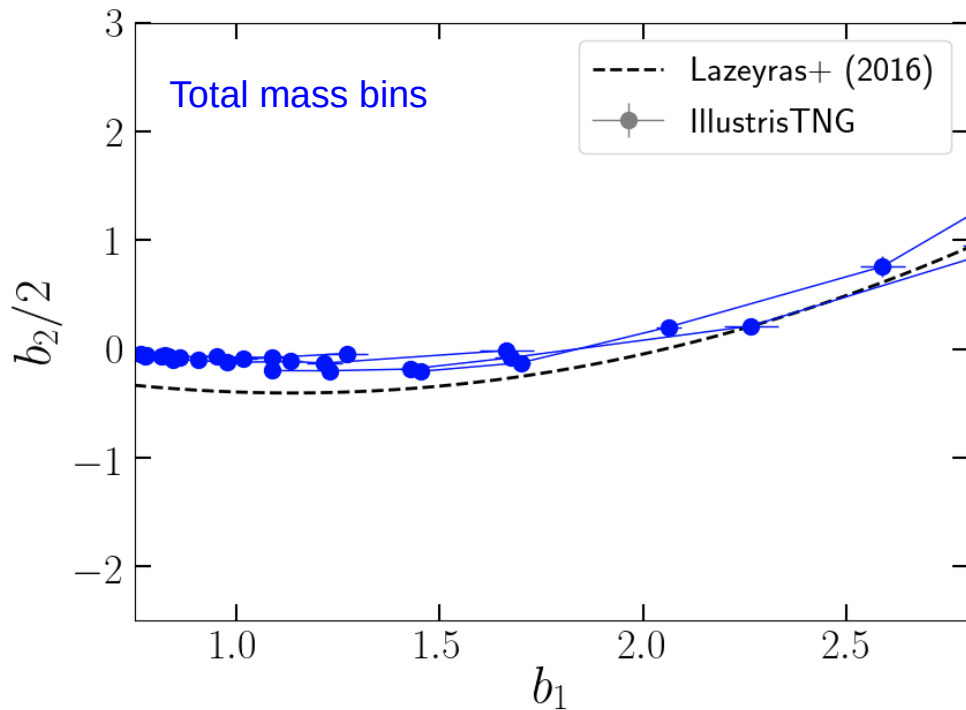
# Results: quadratic bias parameters

From field-level forward models w/ IllustrisTNG

Barreira, Lazeyras & Schmidt (2021)

Lazeyras, Barreira & Schmidt (2021)

$$\delta_g(\mathbf{x}, z) \supset \underbrace{b_1(z)\delta_m(\mathbf{x}, z)}_{\text{Mass density}} + \underbrace{\frac{b_2(z)}{2}\delta_m(\mathbf{x}, z)^2}_{\text{Mass density squared}} + \underbrace{b_{K^2}(z)K_{ij}(\mathbf{x}, z)^2}_{\text{Tidal field squared}}$$



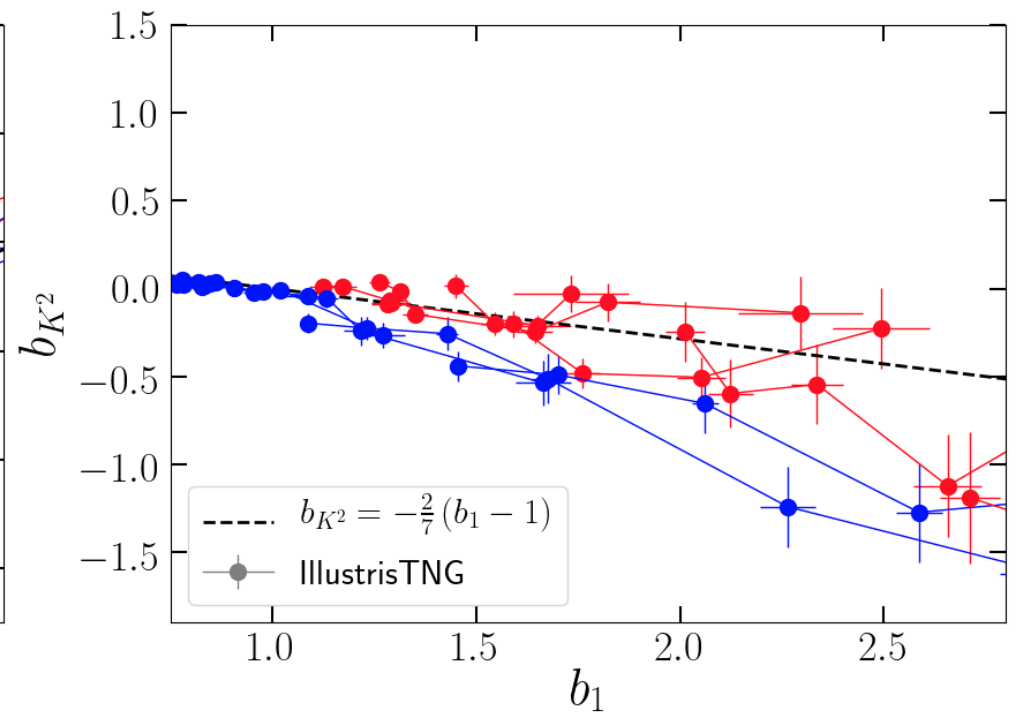
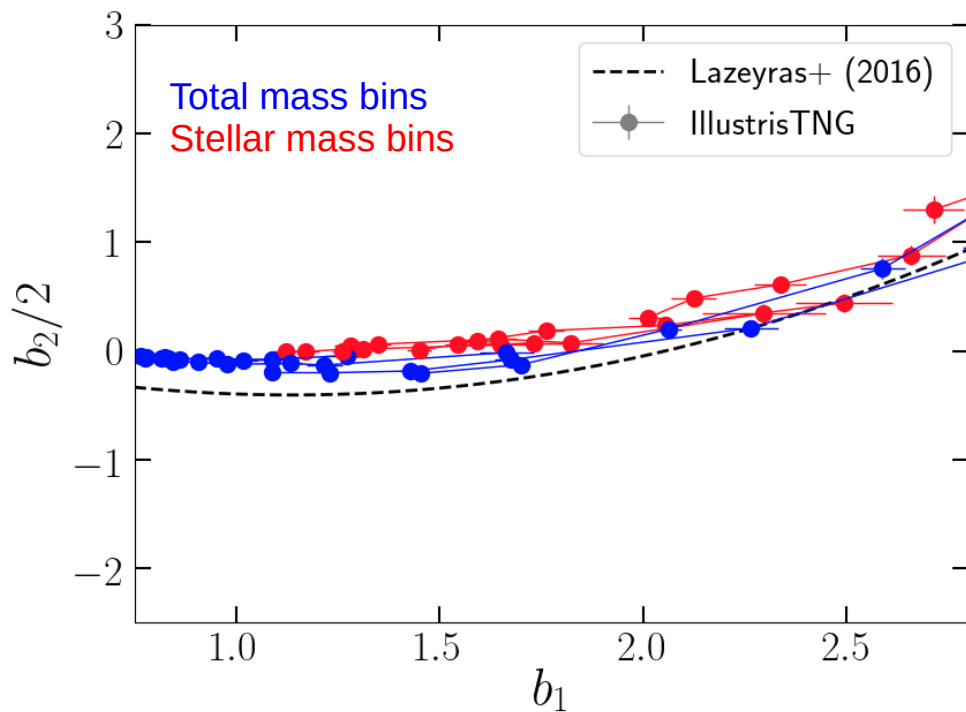
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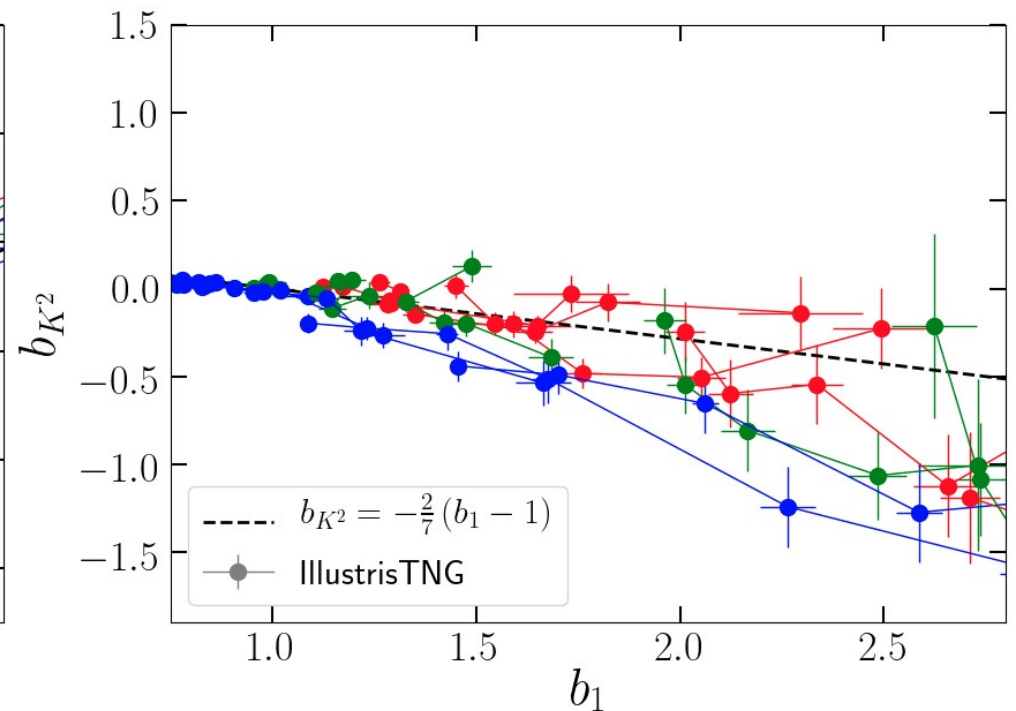
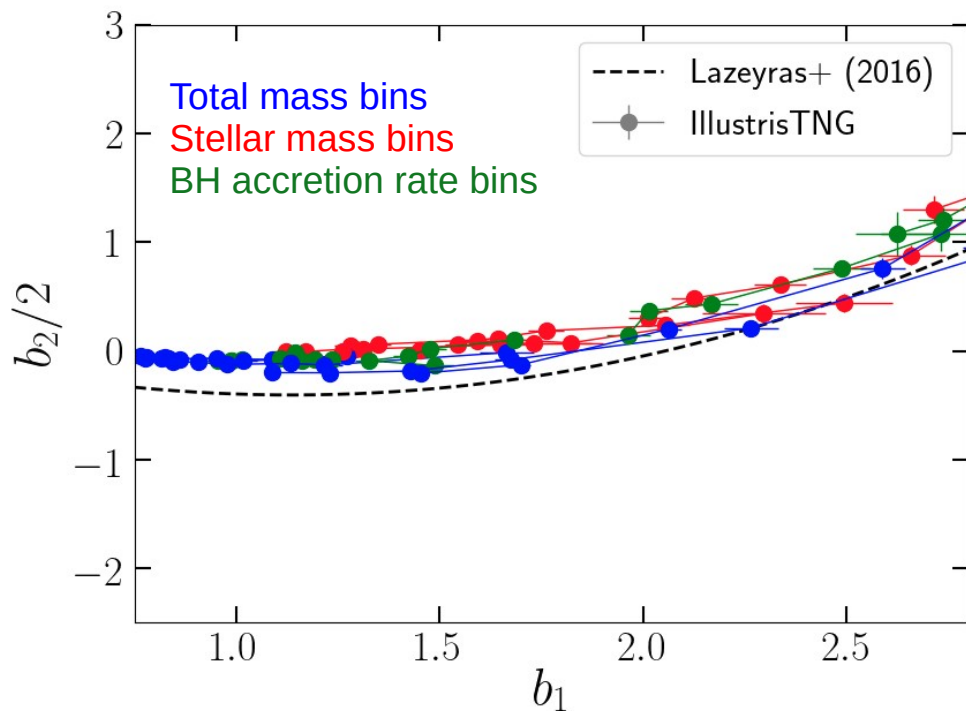
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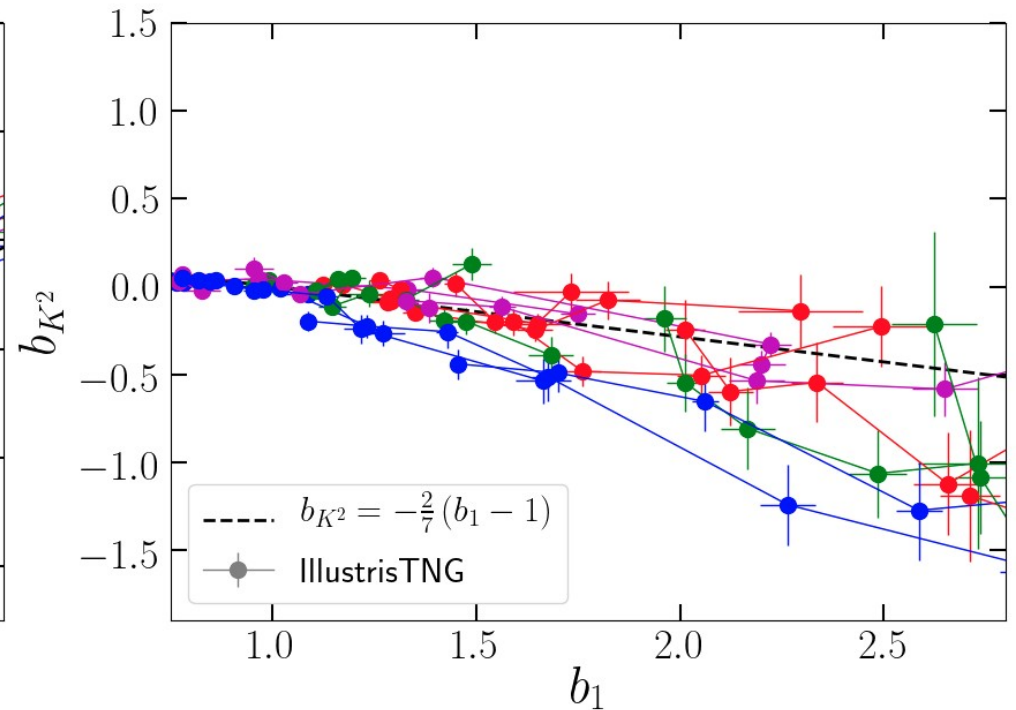
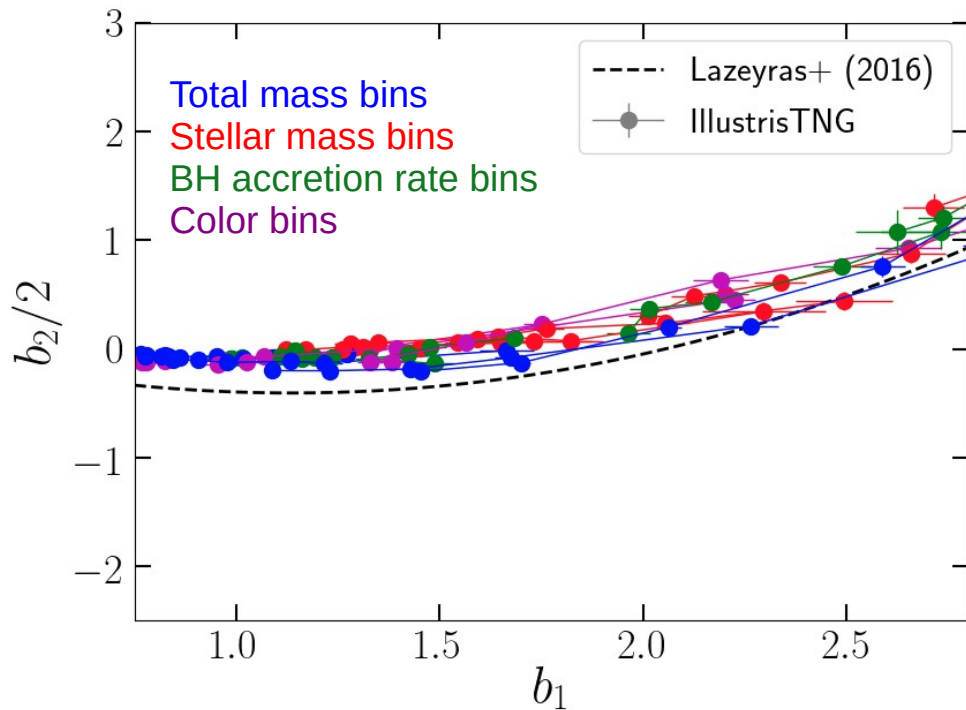
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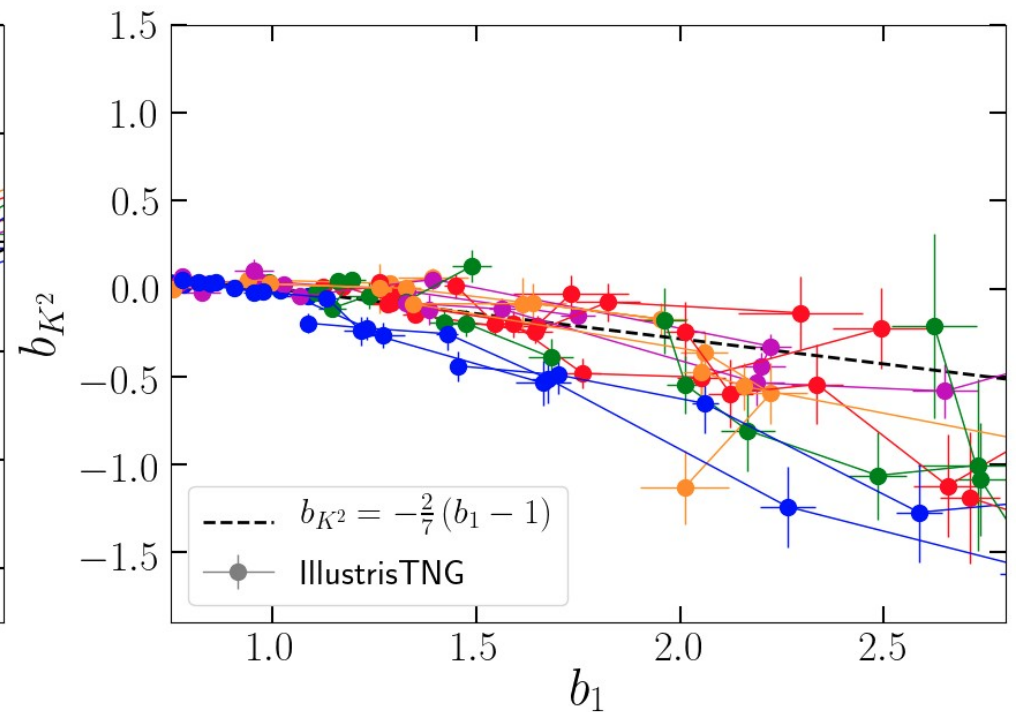
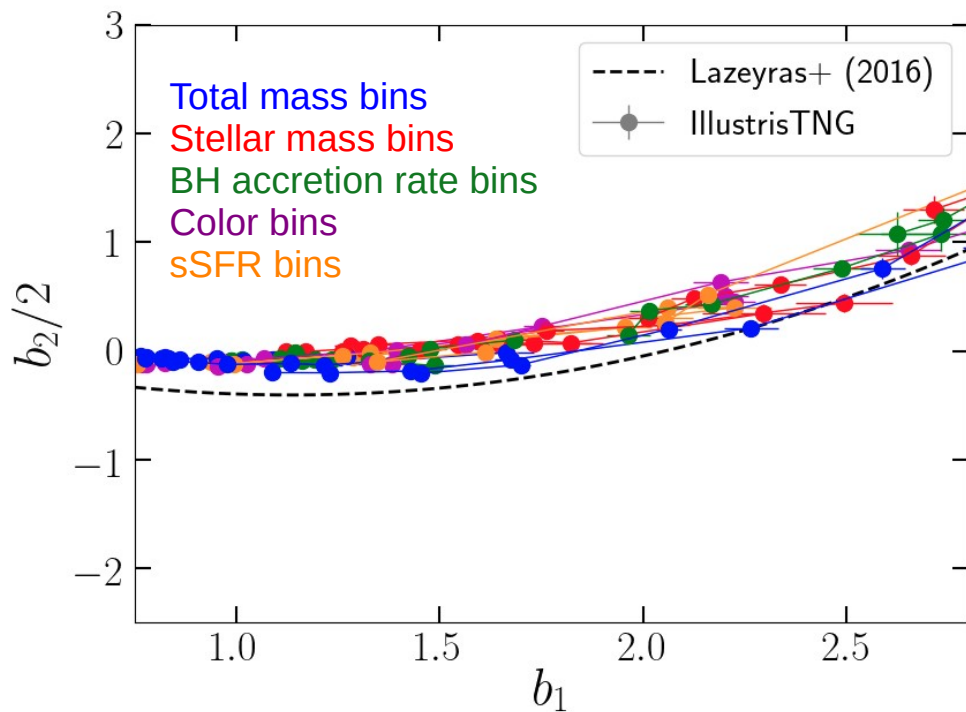
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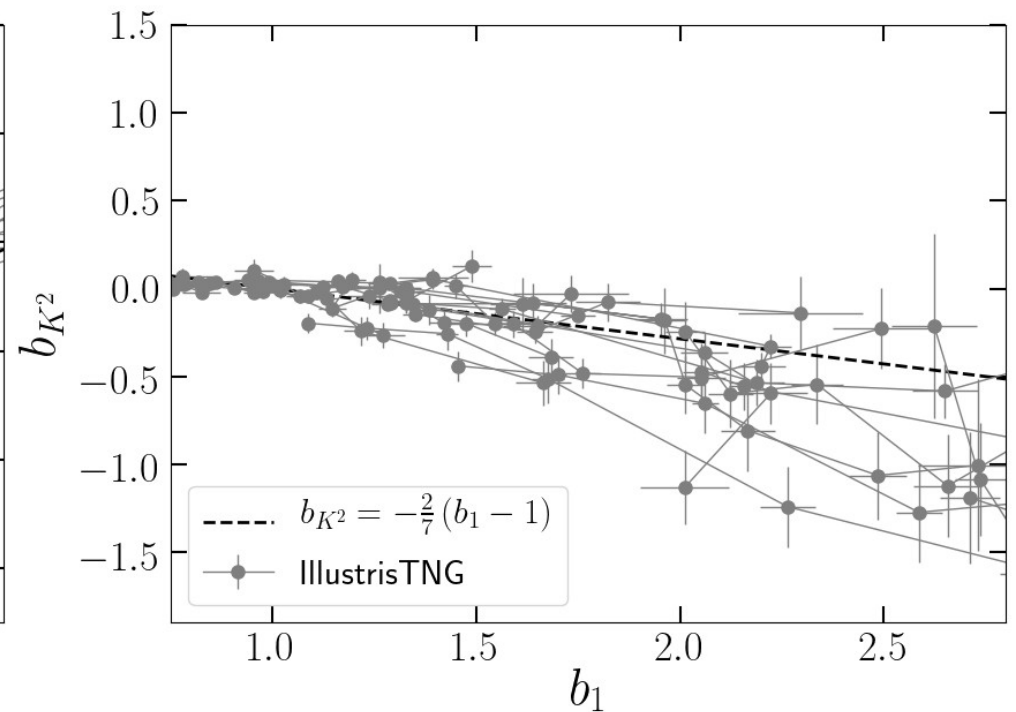
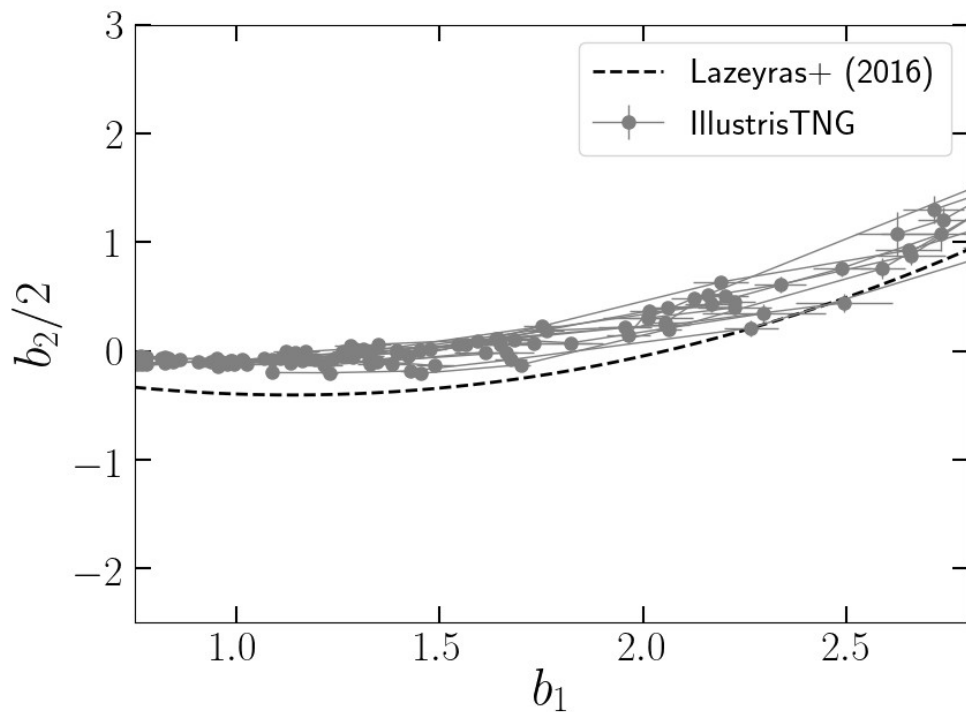
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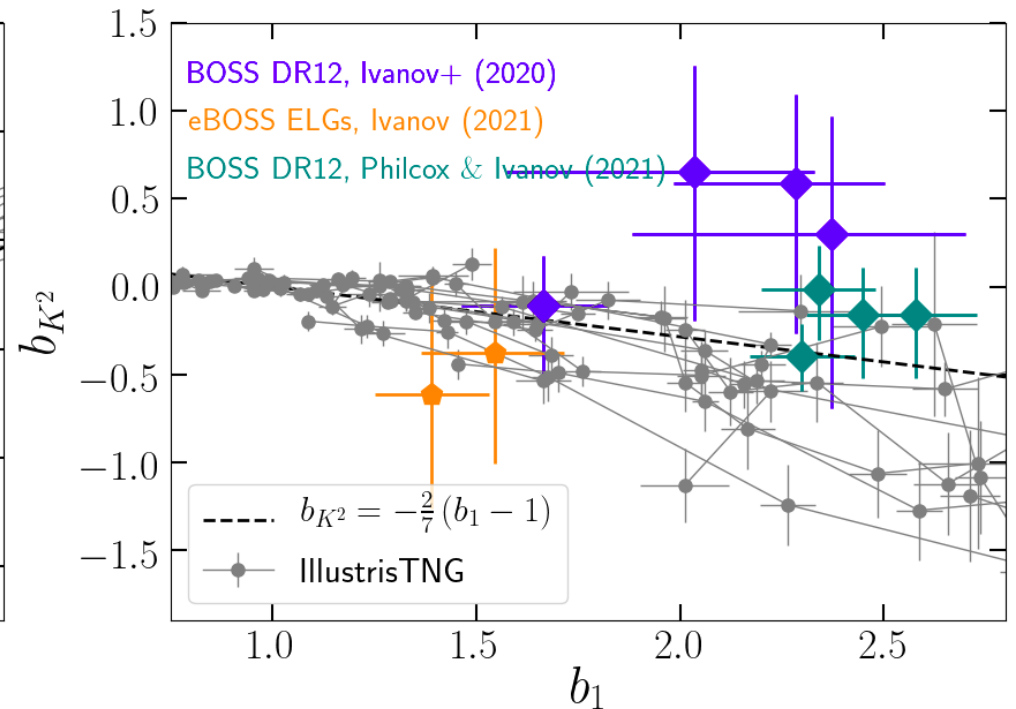
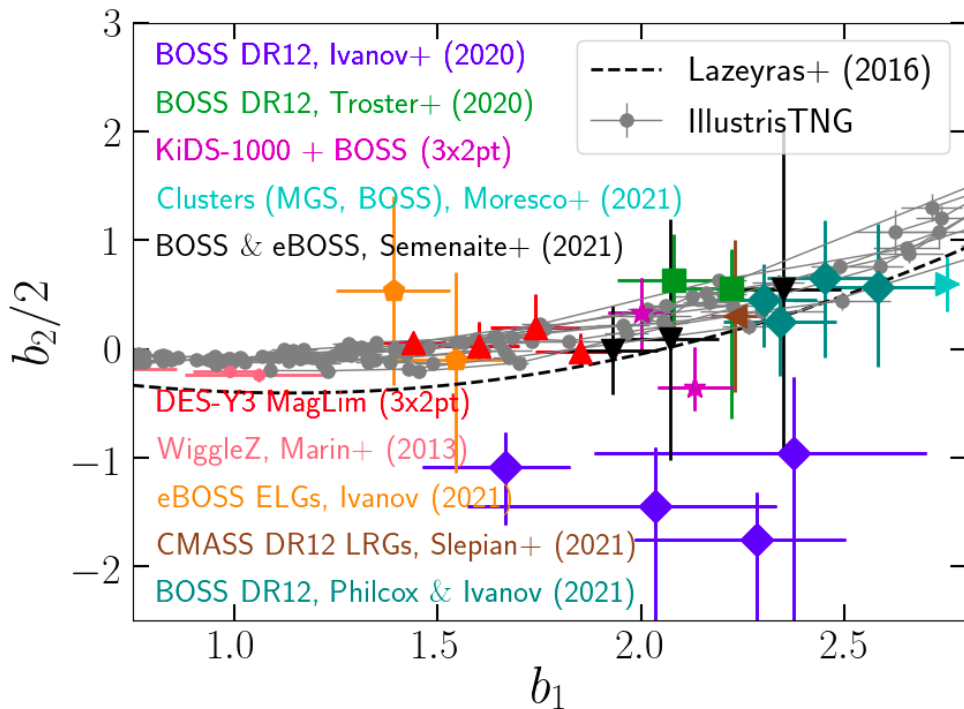
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- Can we **improve cosmological constraints** by assuming priors on galaxy bias?
- Can we **test different galaxy formation models** via their bias predictions?

Galaxy data = **Cosmology** × **Galaxy bias**  
(dark energy, gravity, inflation, etc) (galaxy-environment connection)

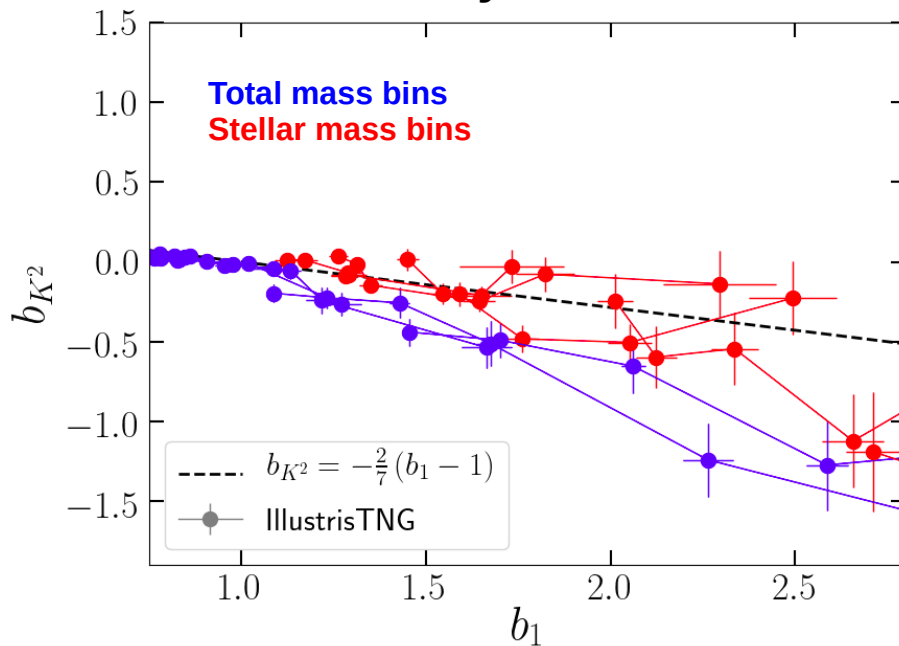


# Results: connection to assembly bias

Barreira, Lazeyras & Schmidt (2021)  
Lazeyras, Barreira & Schmidt (2021)

$$b_g = \frac{1}{\bar{n}_g} \int dM \overset{\text{Halo abundance}}{n_h(M, \dots)} \overset{\text{Galaxy occupation}}{N_g(M, \dots)} \left( \overset{\text{Halo bias}}{b_h(M, \dots)} + \overset{\text{Occupation bias}}{R_{N_g}(M, \dots)} \right)$$

### Galaxy tidal bias

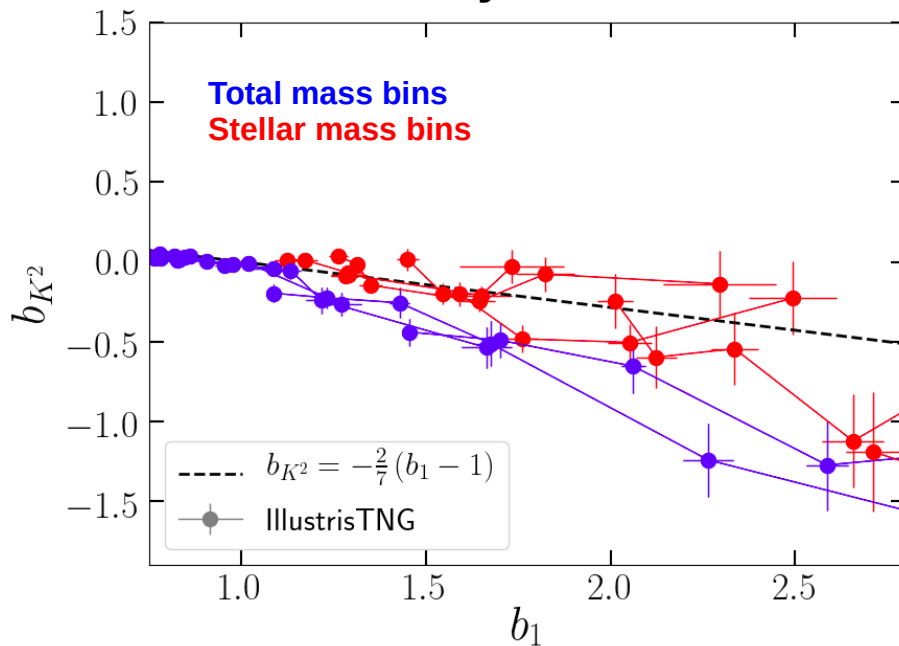


# Results: connection to assembly bias

Barreira, Lazeyras & Schmidt (2021)  
Lazeyras, Barreira & Schmidt (2021)

$$b_g = \frac{1}{\bar{n}_g} \int dc \int dM \overset{\text{Halo concentration}}{n_h(M, c)} \overset{\text{Halo abundance}}{N_g(M, c)} \overset{\text{Galaxy occupation}}{\left( \overset{\text{Halo bias}}{b_h(M, c)} + \overset{\text{Occupation bias}}{R_{N_g}(M, c)} \right)}$$

### Galaxy tidal bias

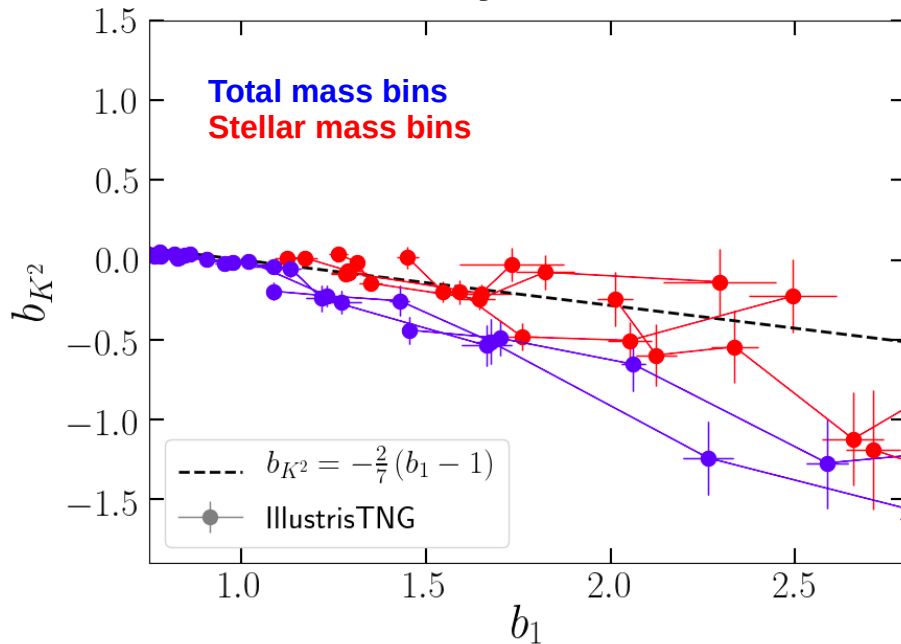


# Results: connection to assembly bias

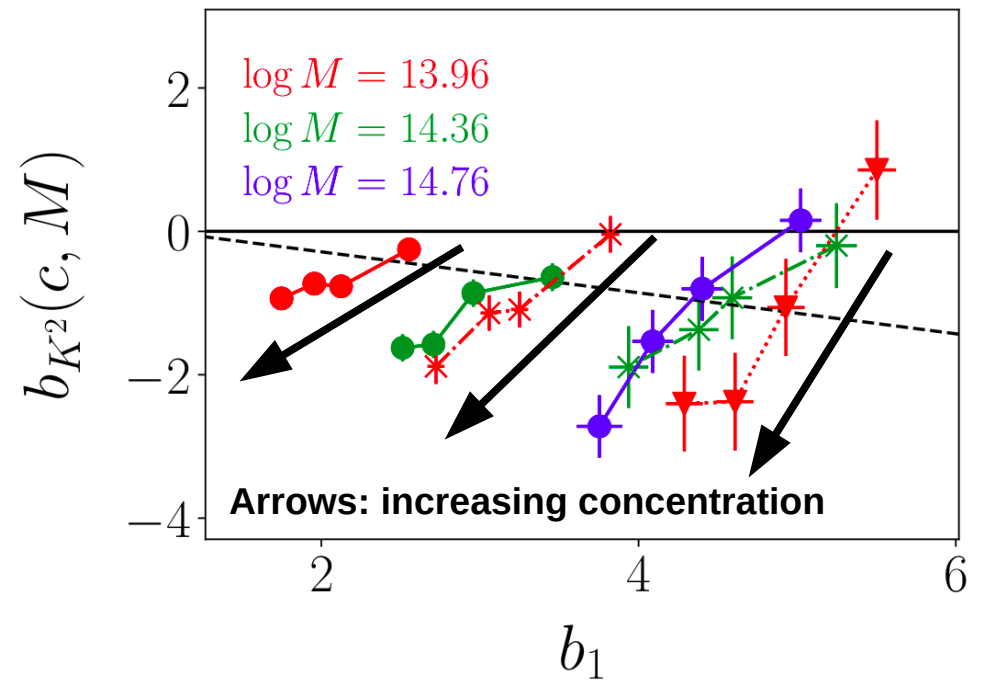
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Galaxy tidal bias



Halo assembly tidal bias



Imprints of halo assembly bias on the galaxy bias relations?

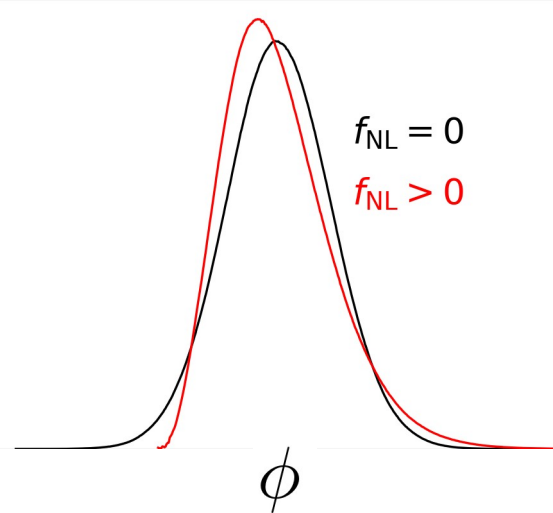
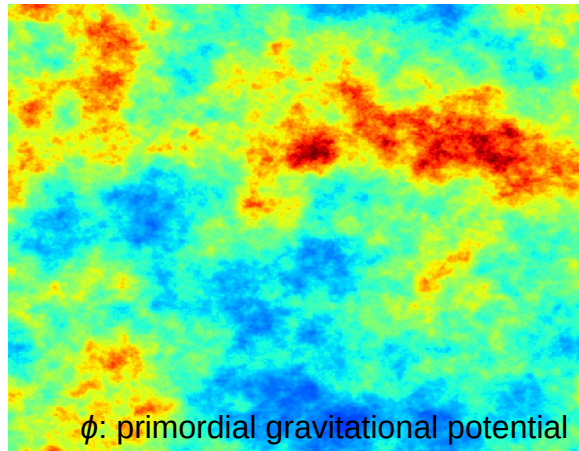
# Outline

- What is galaxy bias?
- Recent results from galaxy formation simulations.
- Consequences for tests of inflation using galaxy data.

# The search for **local-type PNG**, $f_{\text{NL}}$

Komatsu&Spergel(2001)

$$\phi = \phi_G + f_{\text{NL}} [\phi_G^2 - \langle \phi_G^2 \rangle]$$



**Detecting  $f_{\text{NL}}$  rules out the popular single-field inflation scenarios!**

(Creminelli&Zaldarriaga 2004, Creminelli+ 2011, Tanaka&Urakawa 2011, Pajer+ 2013)

**Current tightest bound (Planck 2018):**

$$f_{\text{NL}} = -0.9 \pm 5.1 \text{ (68\%)}$$

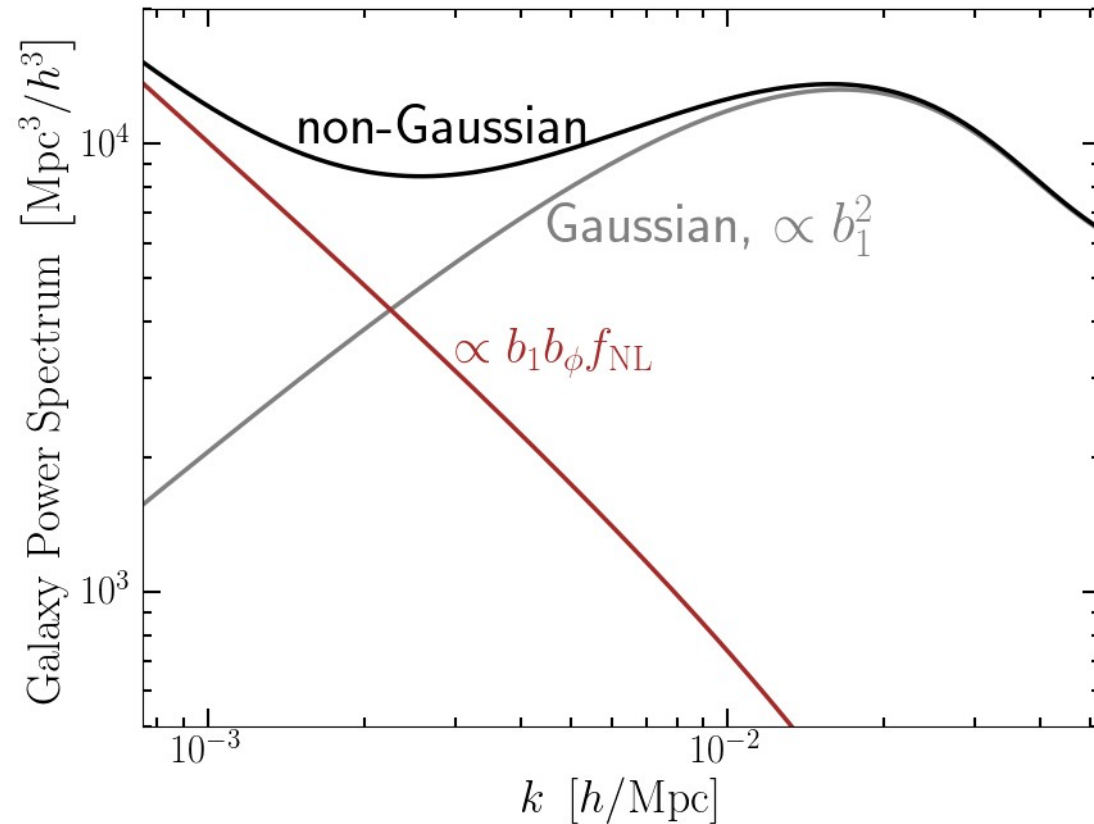
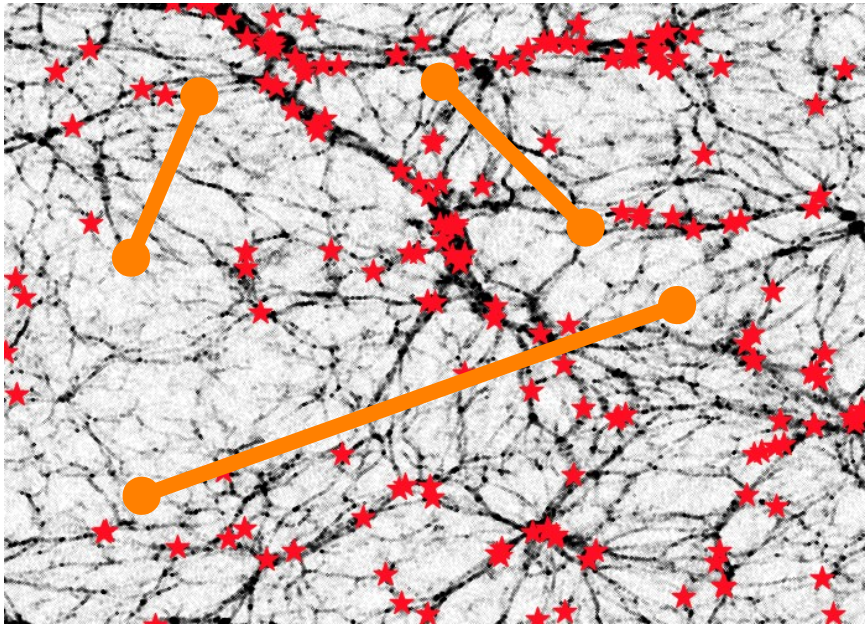
**Future galaxy surveys aiming at:**

$$\sigma_{f_{\text{NL}}} \lesssim 1$$

# The search for **local-type PNG**, $f_{\text{NL}}$

Local PNG leaves a distinct scale-dependent signature on the large-scale galaxy power spectrum (2pt function)

(Dalal+ 2008)



# The search for **local-type PNG**, fNL

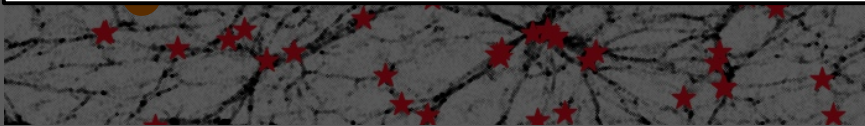
Local PNG leaves a distinct scale-dependent signature on the large-scale galaxy power spectrum (2pt function)

Signature  $\propto$   $b_1 b_\phi f_{\text{NL}}$

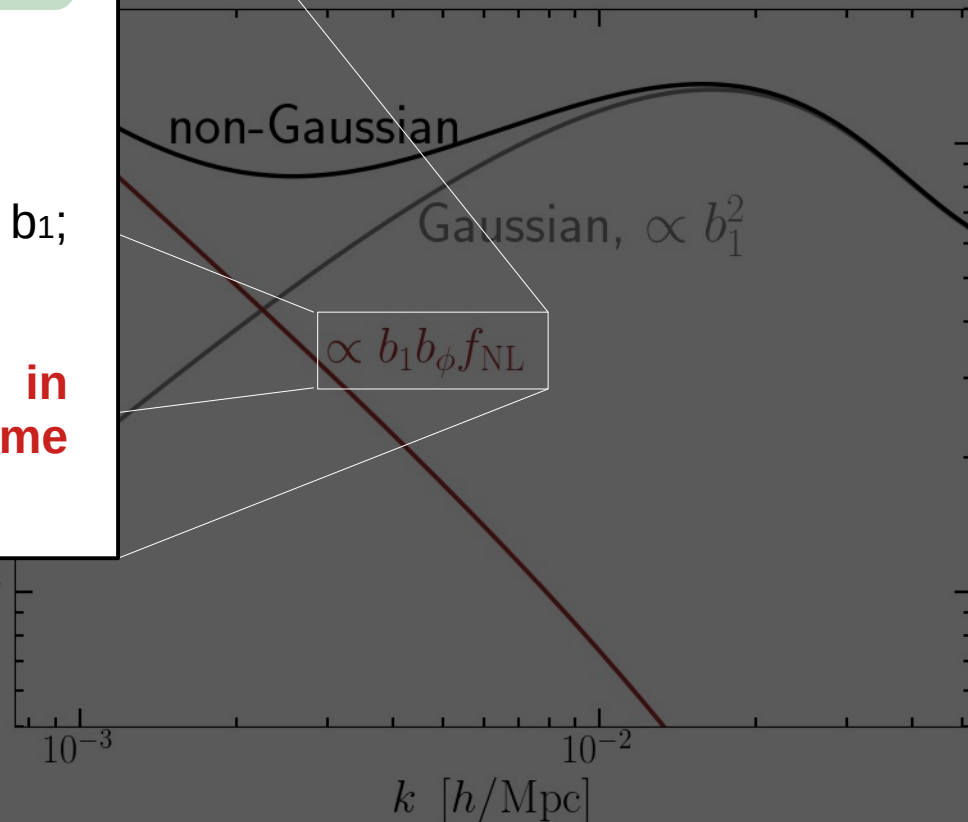
Cosmology

Galaxy formation  
(galaxy bias parameters)

- The data on small scales can determine  $b_1$ ;
- A perfect degeneracy with  $b_\phi$  remains, so **in order to constrain  $f_{\text{NL}}$  we need to assume something about galaxy formation.**



Galaxy  $10^3$

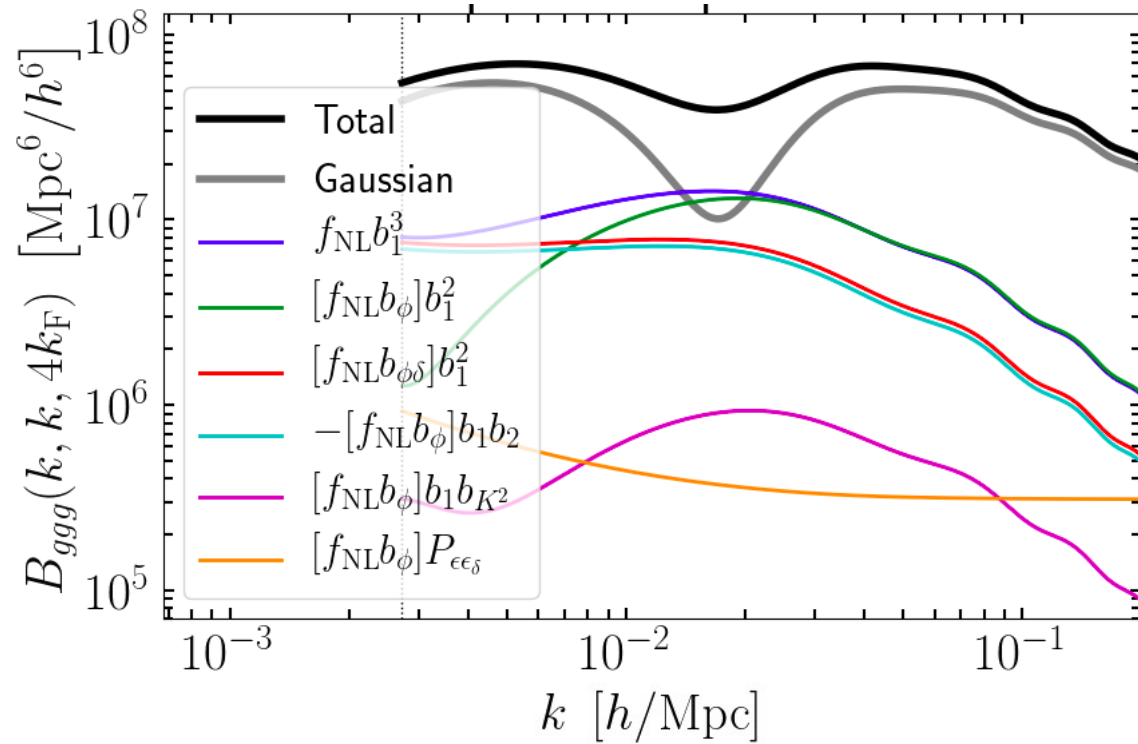
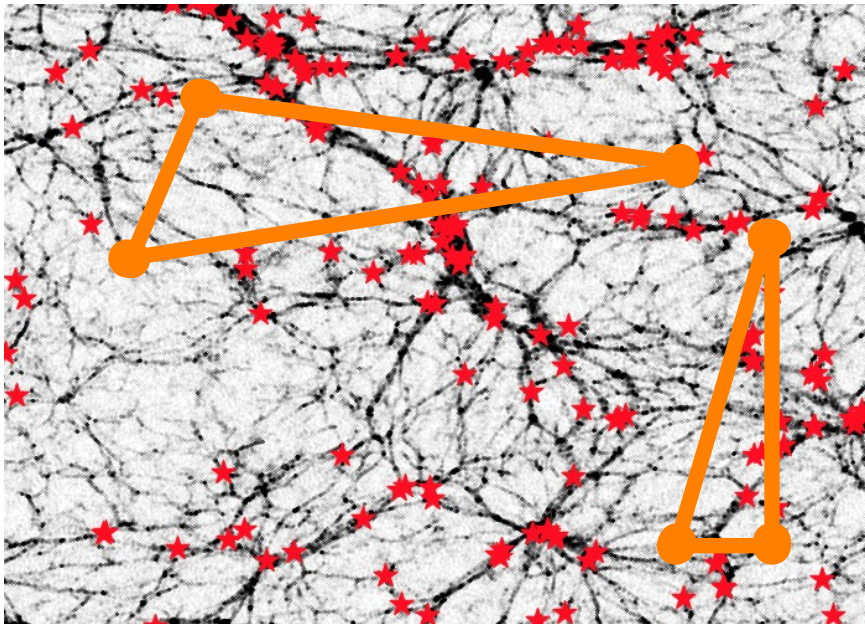




# The search for **local-type PNG**, fNL

**Local PNG contributes also sizeably to the galaxy bispectrum (3pt function)**

(Scoccimarro+(2003), Sefusatti&Komatsu(2007), Jeong&Komatsu(2011))



# The search for **local-type PNG**, fNL

Local PNG contributes also sizeably to the galaxy bispectrum (3pt function)

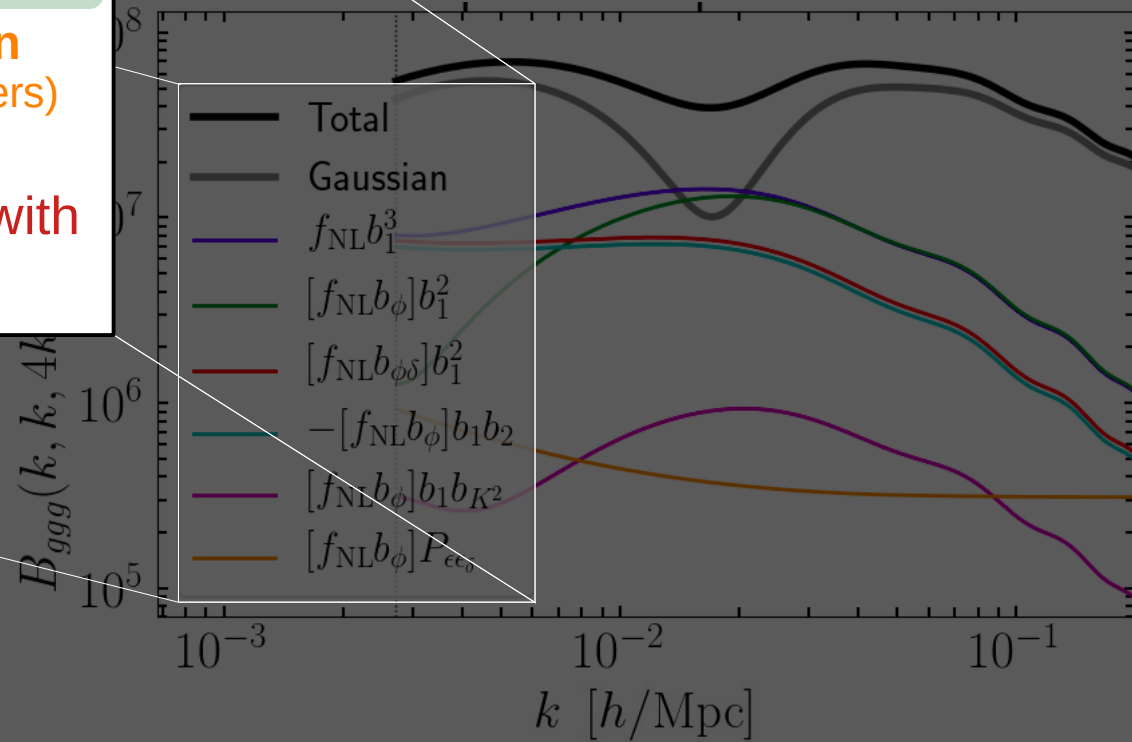
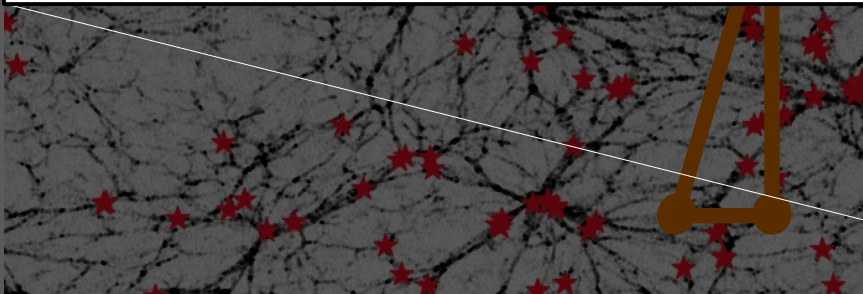
(Matsushita(2007), Jeong&Komatsu(2011))

Signatures  $\propto$   $b_\phi f_{\text{NL}}$ ,  $b_{\phi\delta} f_{\text{NL}}$

**Cosmology**

**Galaxy formation**  
(galaxy bias parameters)

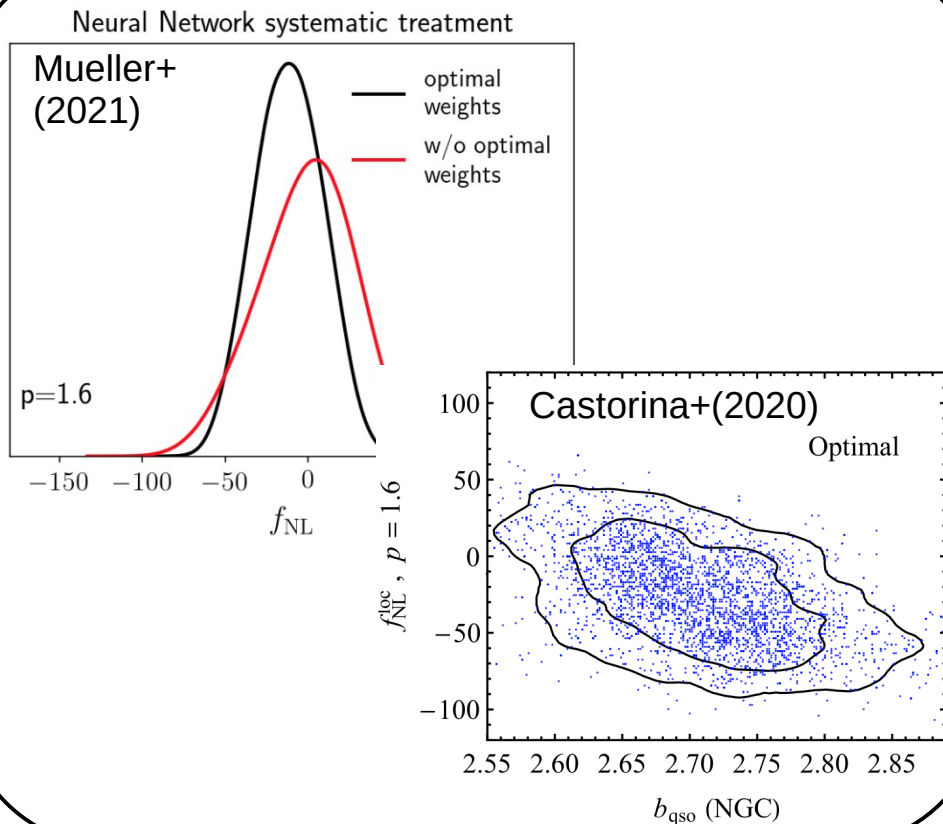
**Galaxy bias is largely degenerate with fNL in the bispectrum too.**



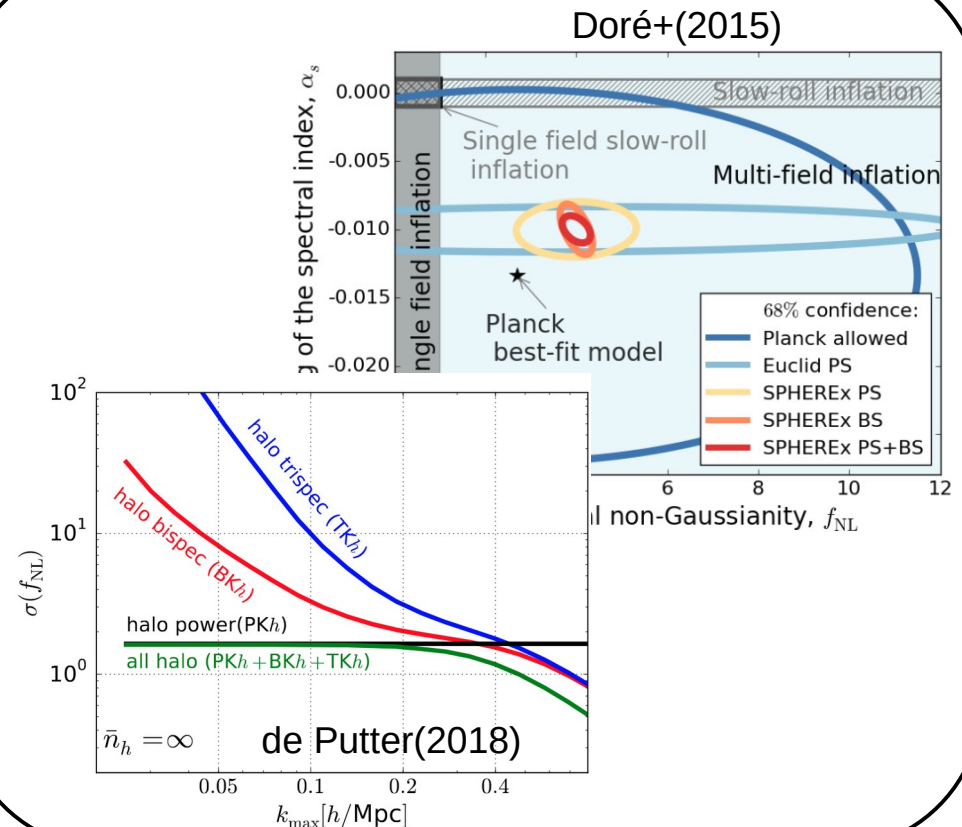
# The search for **local-type PNG**, f<sub>NL</sub>

How did all past works constrain/forecast f<sub>NL</sub> then?

## Many real data constraints



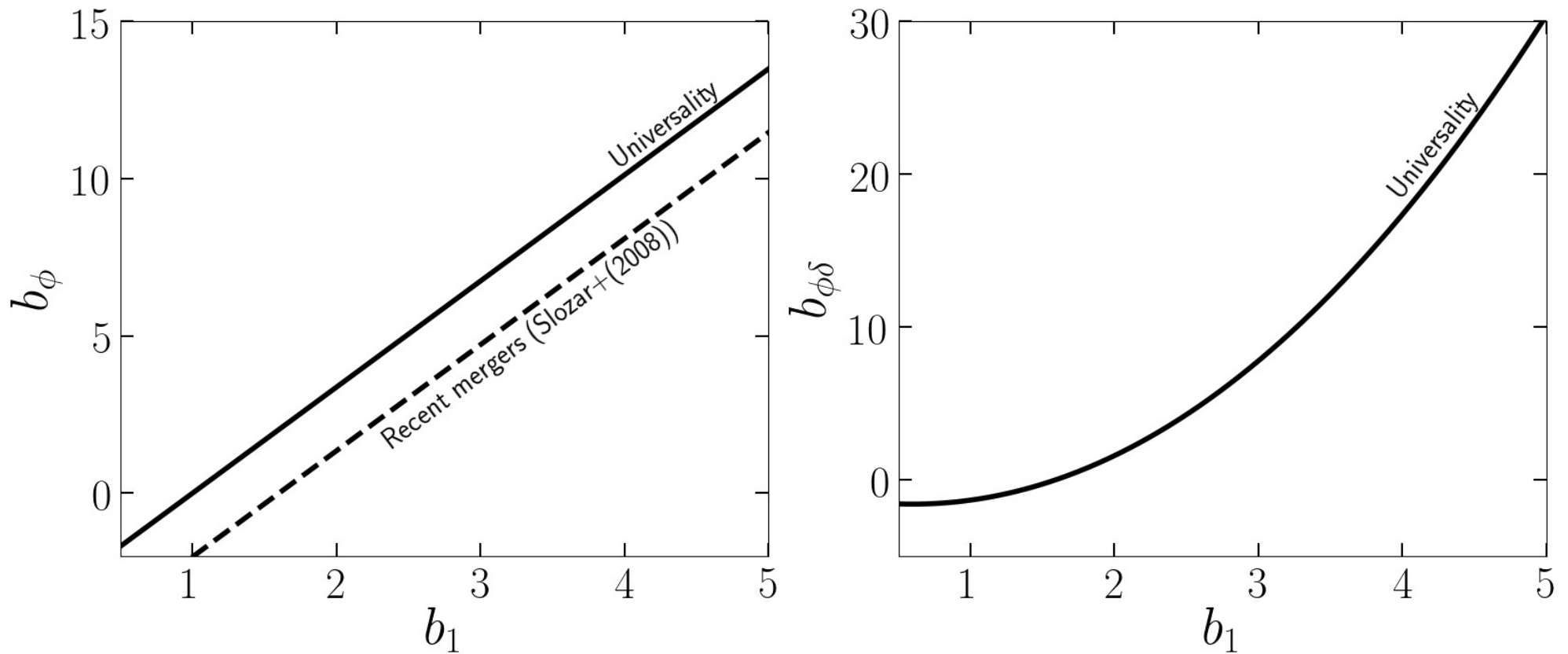
## Many more future data forecasts



# The PNG bias relations

## The universality relations

(fix  $b_\phi$  and  $b_{\phi\delta}$  in terms of  $b_1$ , which can be fitted for)

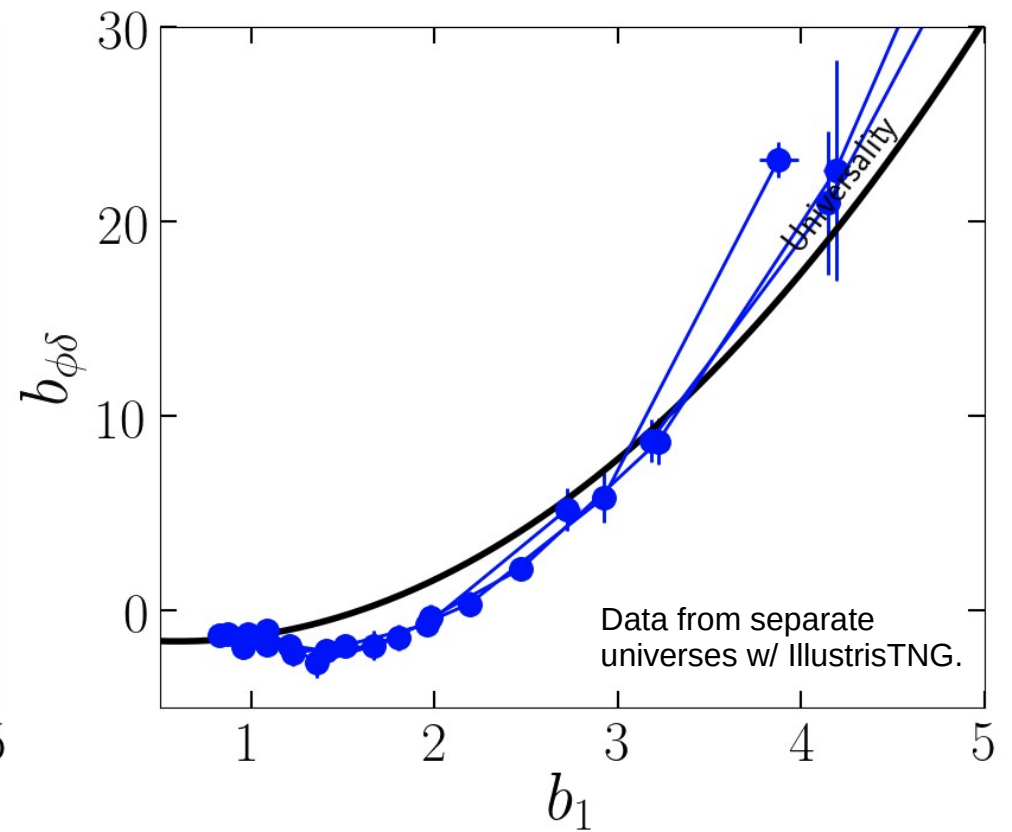
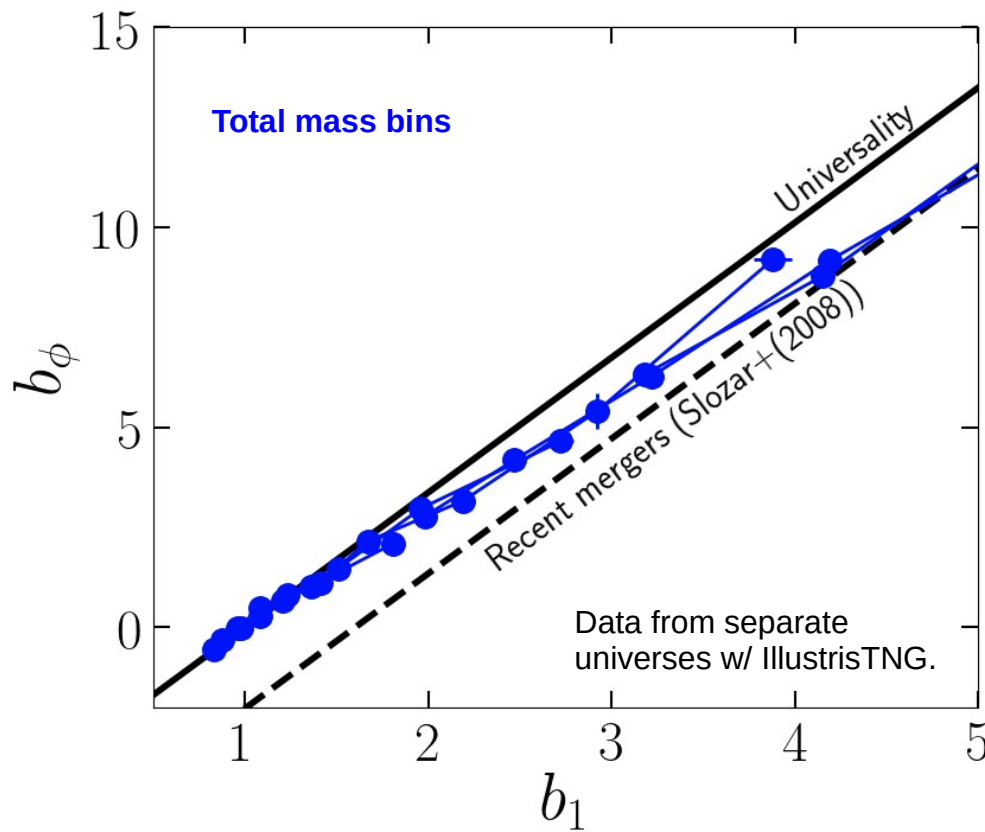


Despite being routinely used, there is no reason for these relations to hold for real galaxies/tracers, and they do not!

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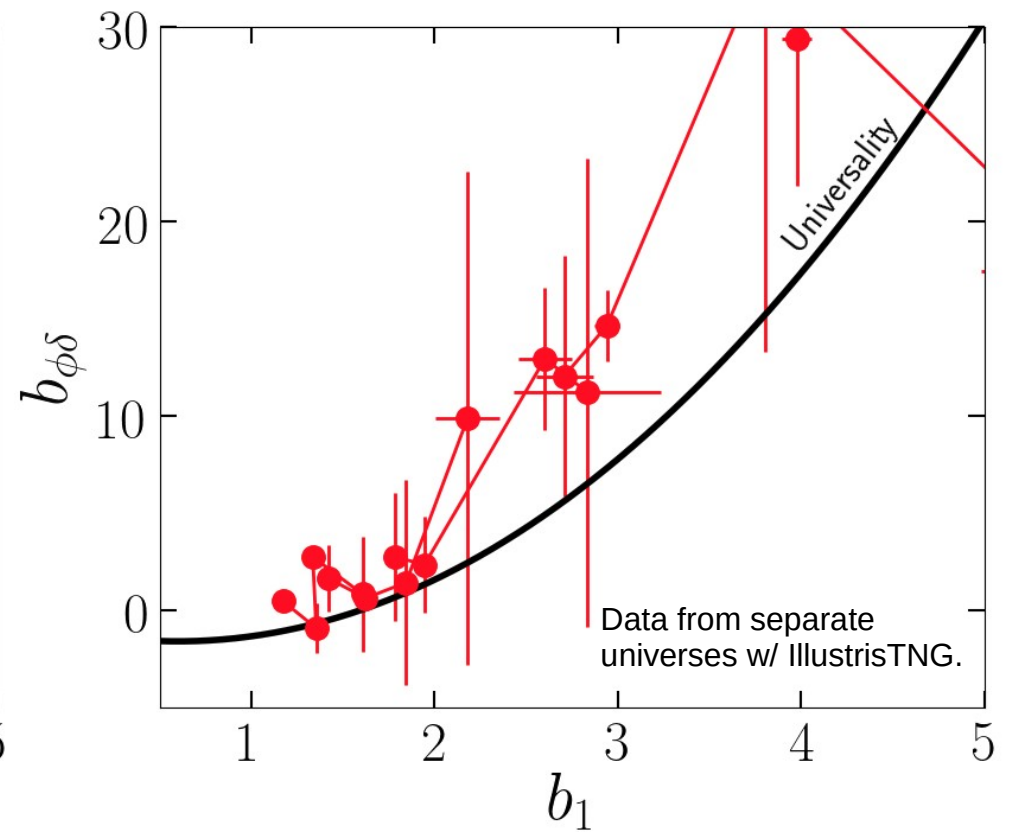
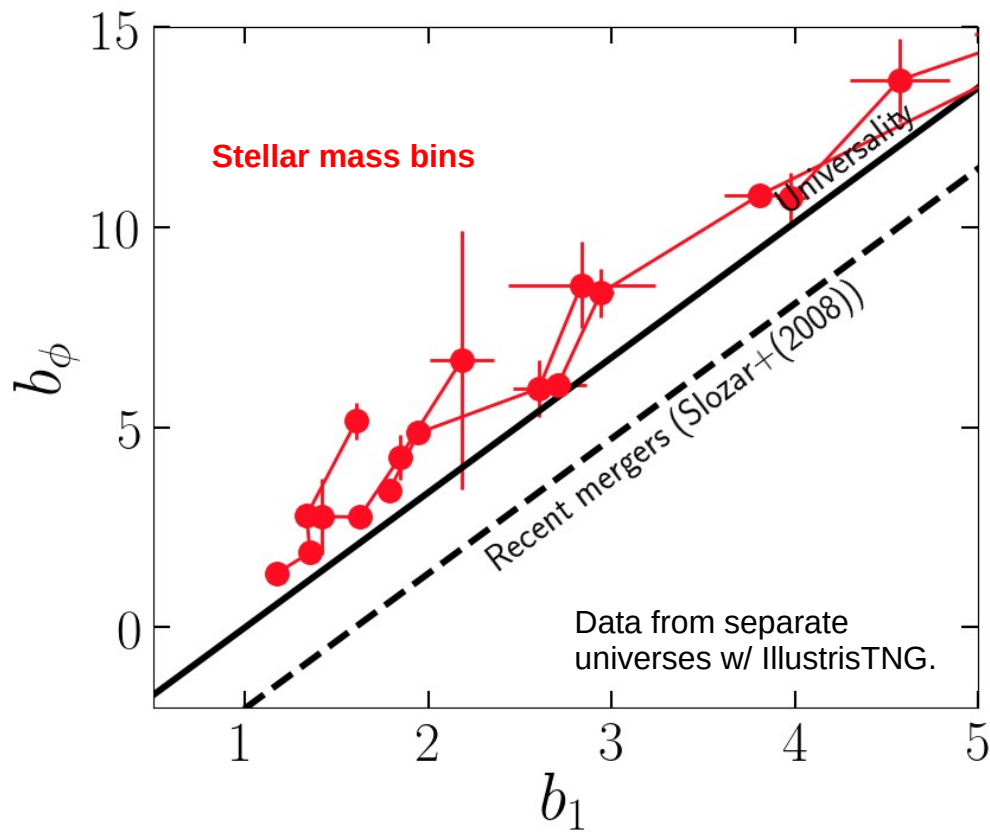


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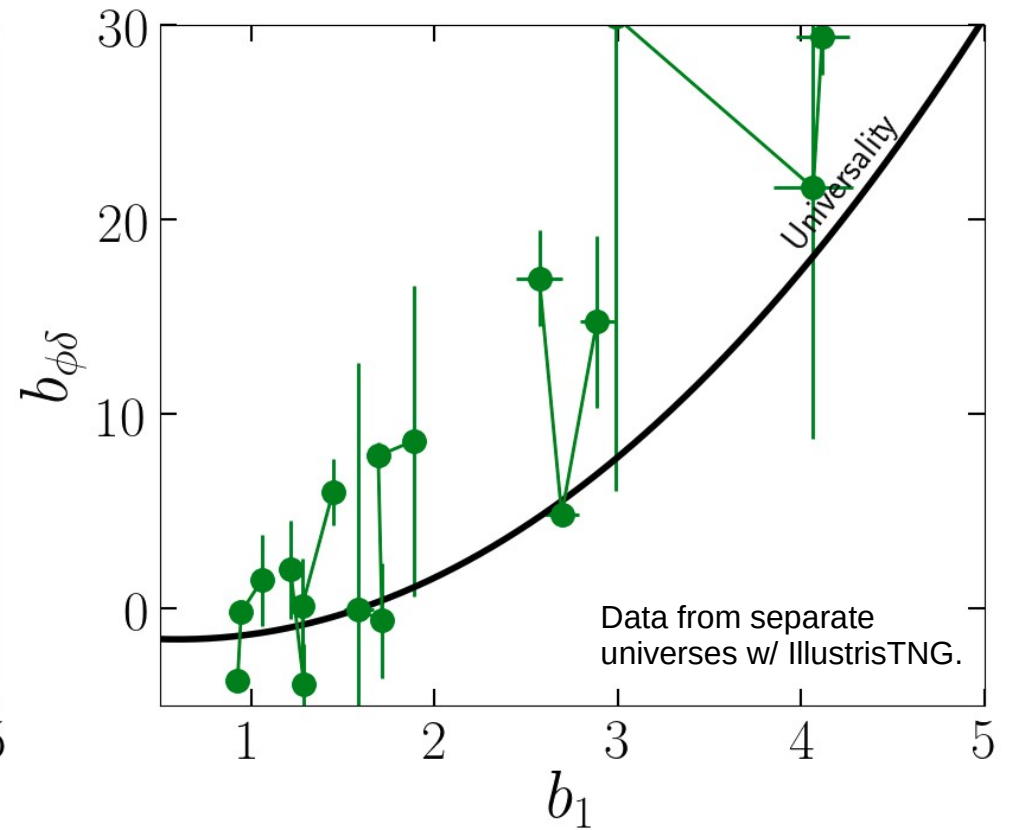
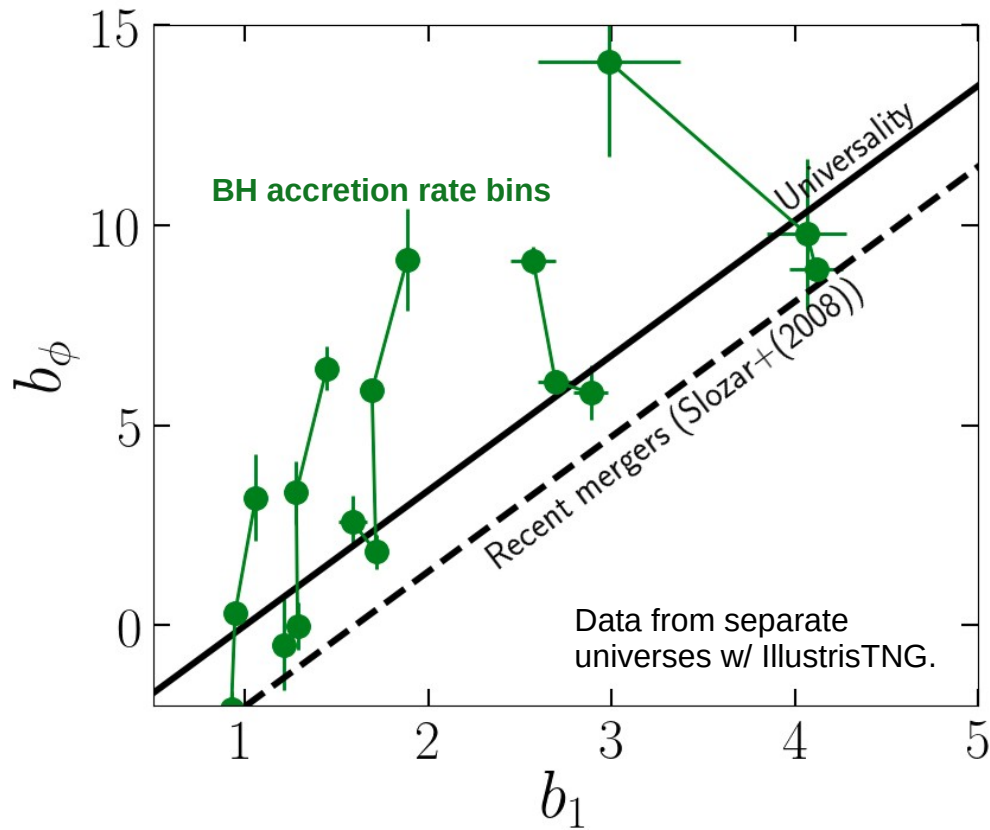


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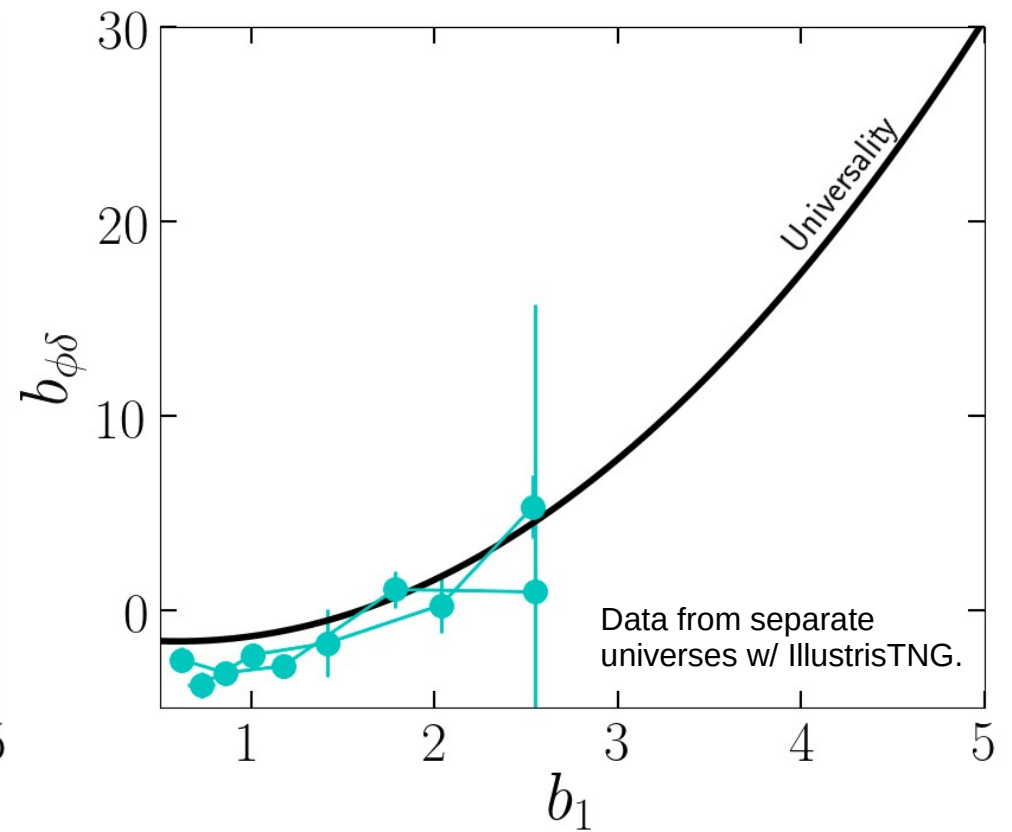
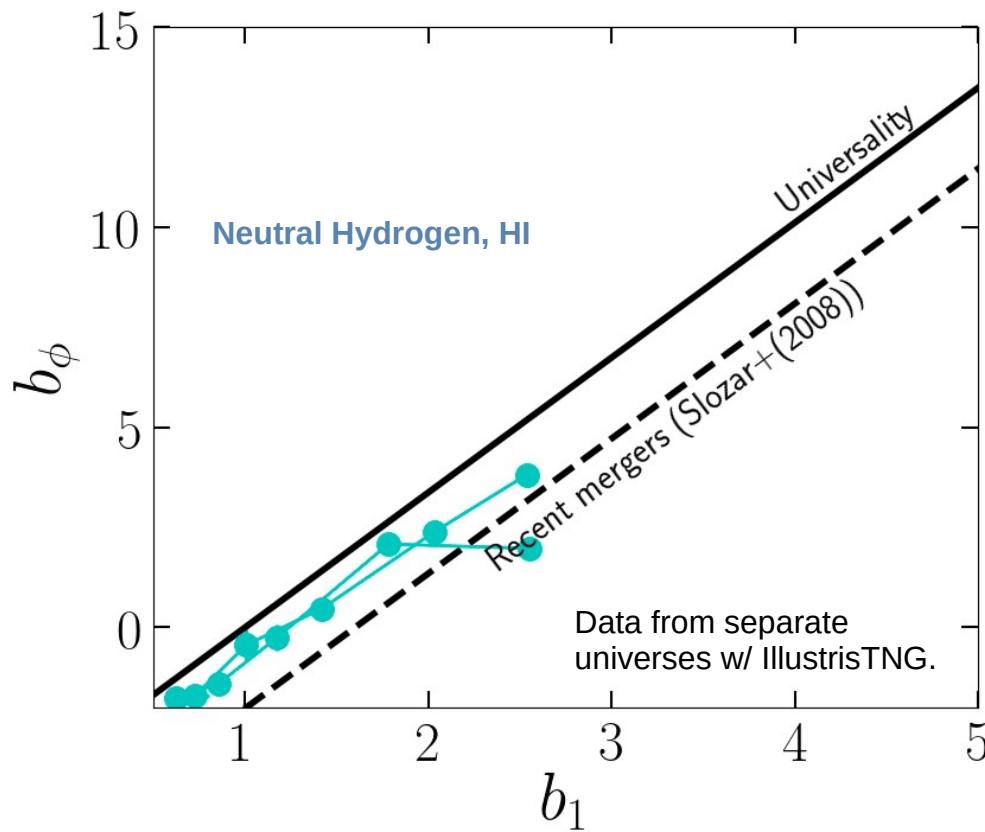


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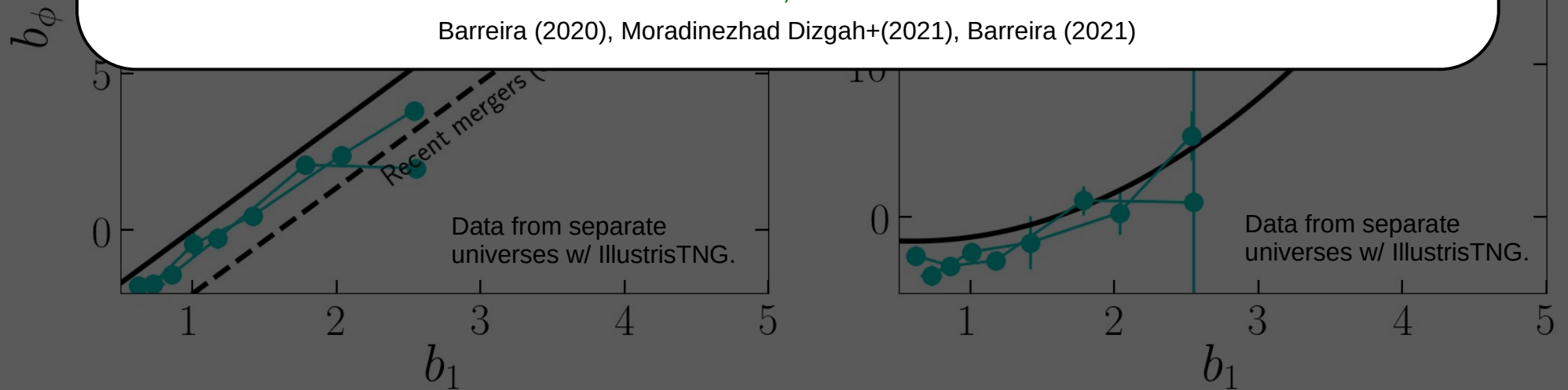
(fix  $b_\phi$  and  $b_{\phi\delta}$  in terms of  $b_1$ , which can be fitted for)

To which extent are **all our constraints/forecasts on  $f_{NL}$  incorrect** because we have been using the wrong bias relations?

**More work on:** more simulations, better selection of simulated galaxies, different galaxy formation models.

We will have to live with bias uncertainties, so **how does that affect  $f_{NL}$  constraints?**

Barreira (2020), Moradinezhad Dizgah+(2021), Barreira (2021)



Despite being routinely used, there is no reason for these relations to hold for real galaxies/tracers, and they do not!

# The impact of bias uncertainties

Barreira 2021

Simulated likelihood analysis for an idealized fictitious survey.

## Forecast setup

(not survey specific)

- **Model:** tree-level power spectrum and bispectrum; No RSD, relativistic effects or obs. systematics.
- **Data:** Multitracer power spectrum, and bispectrum (noiseless realization of the model at the fiducial);
- **Covariance:** Gaussian + partial non-Gaussian;
- **Galaxies:** 2 stellar mass selected samples;
- **Volume, redshift, fiducial,  $k_{\max}$ :**
  - $V = 100 \text{ Gpc}^3/h^3$ ;
  - $z = 1$
  - $f_{\text{NL}} = 5$ ;
  - $k_{\max} = 0.2 \text{ h/Mpc}$
- **Cosmology:** fixed, except  $f_{\text{NL}}$  and  $A_s$ ;
- **Shot-noise:** leading-order (4 parameters);

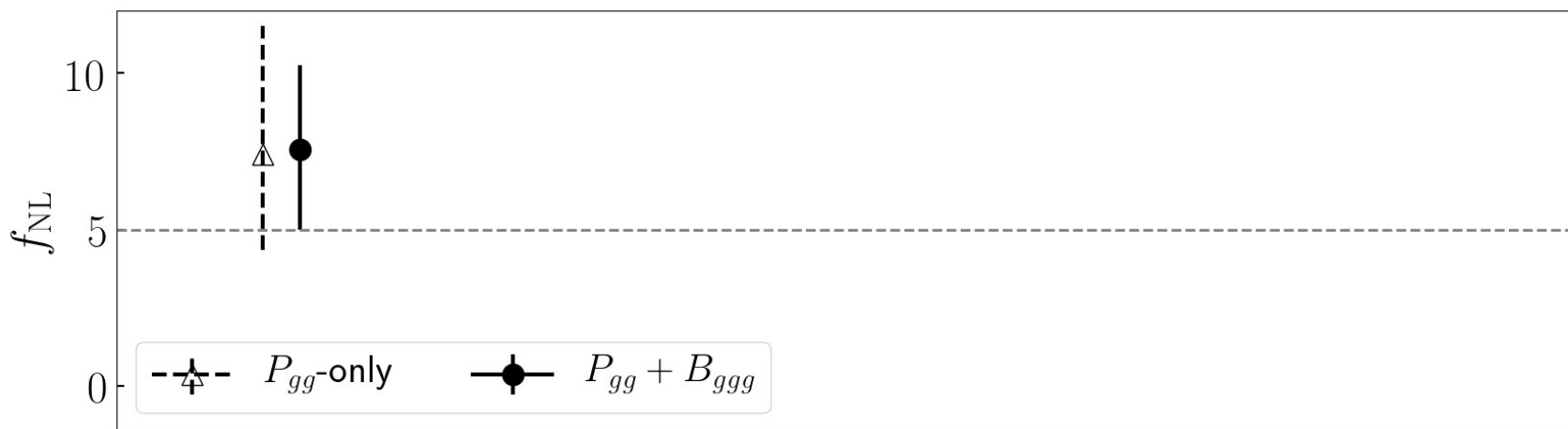
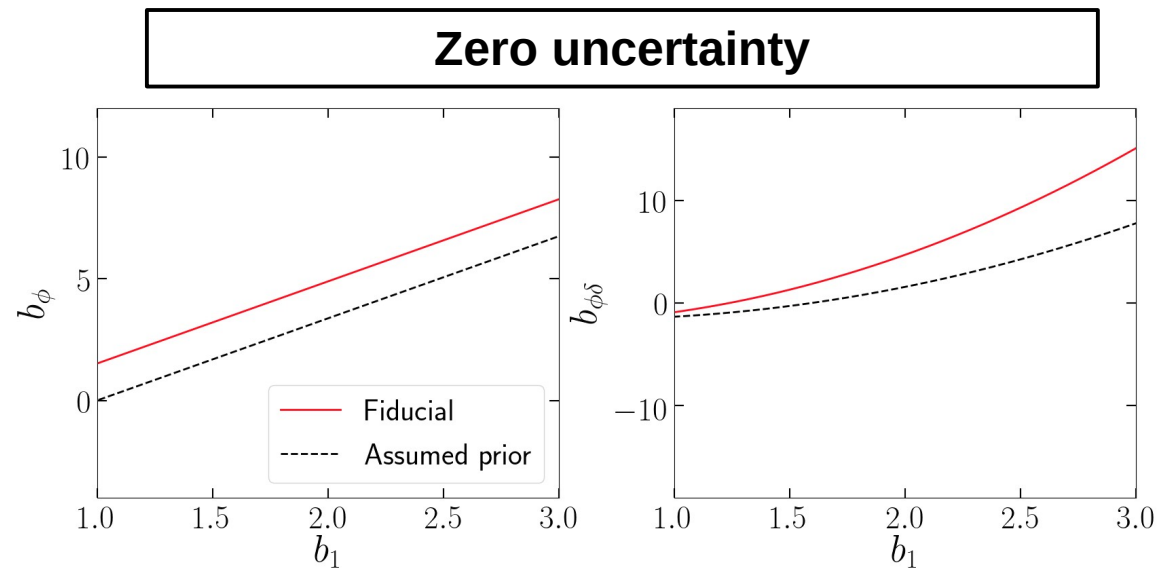
# Priors on PNG bias parameters

Barreira 2021

## Approach 1:

marginalize over Gaussian priors on the bias parameter relations

Prior centered on the wrong relation



Precise knowledge  
of wrong relations

# Priors on PNG bias parameters

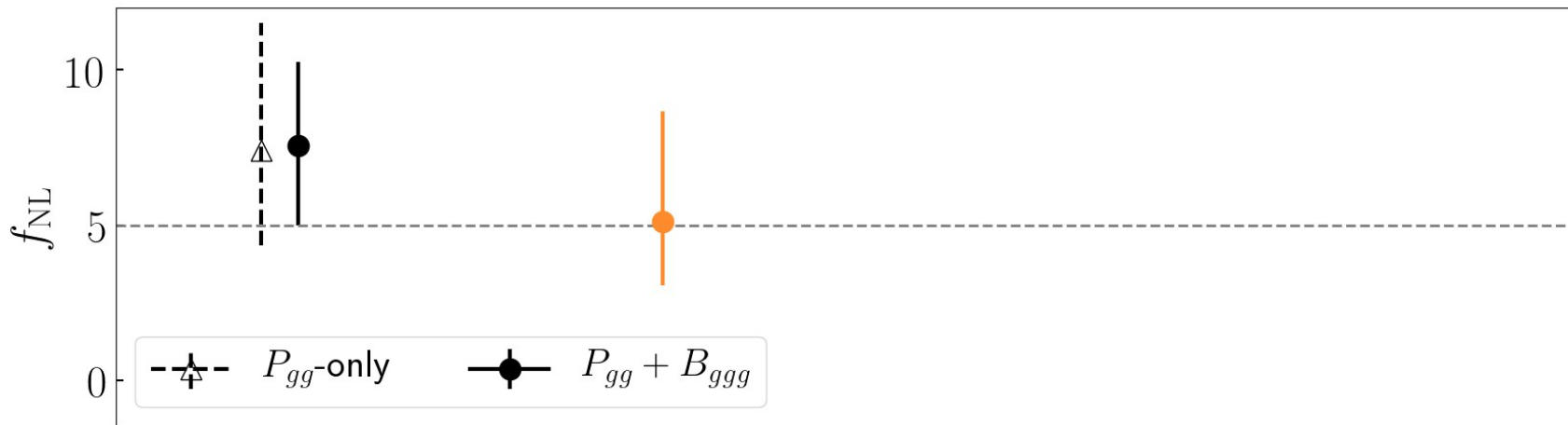
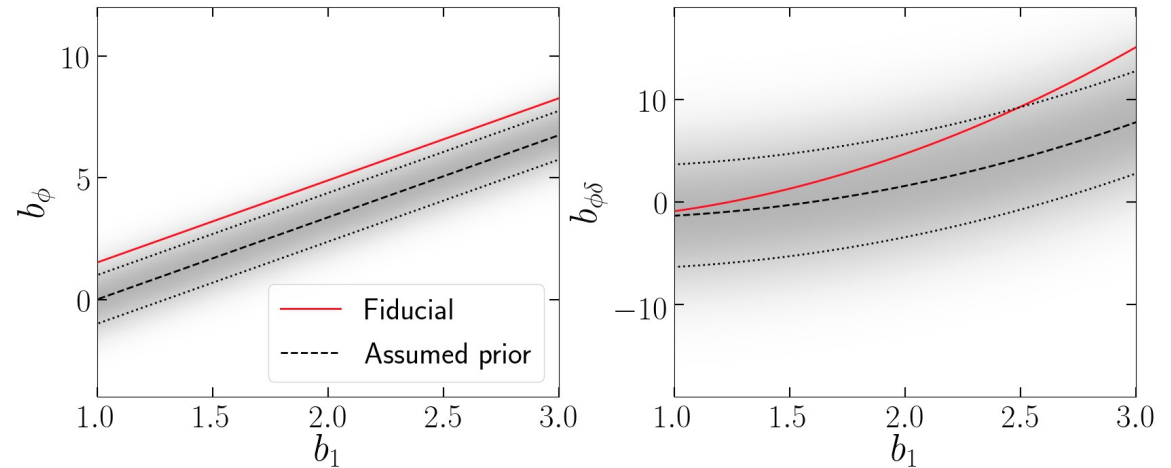
Barreira 2021

Prior centered on the wrong relation

## Approach 1:

marginalize over Gaussian priors on the bias parameter relations

Uncertainty of  $\Delta b_\phi = 1$ ,  $\Delta b_{\phi\delta} = 5$



Precise knowledge of wrong relations

Gaussian prior on  $b_\phi(b_1)$ ,  $b_{\phi\delta}(b_1)$

# Priors on PNG bias parameters

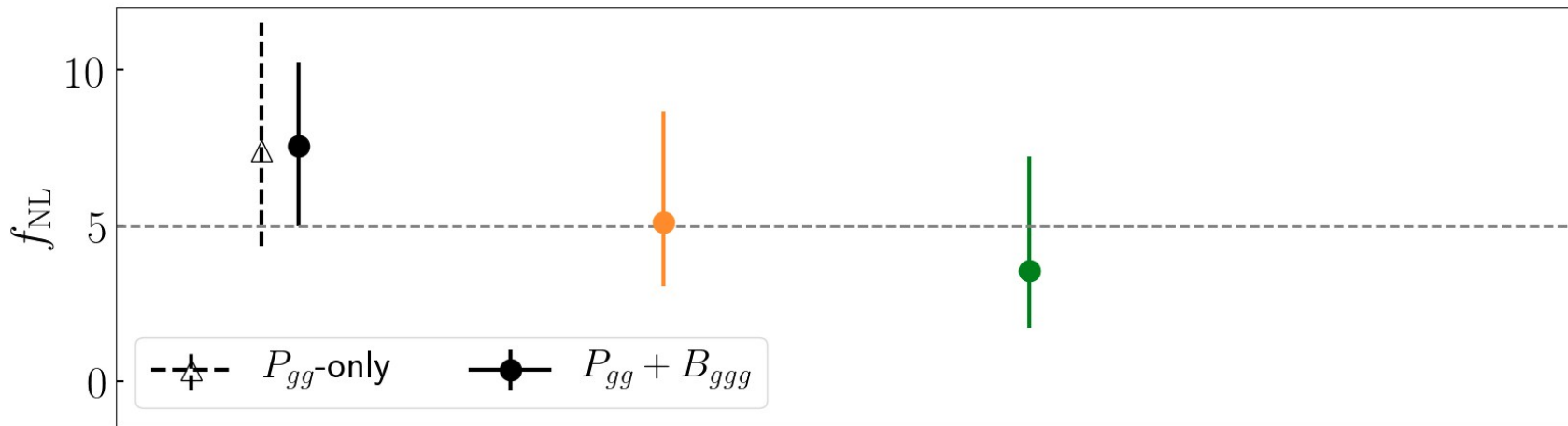
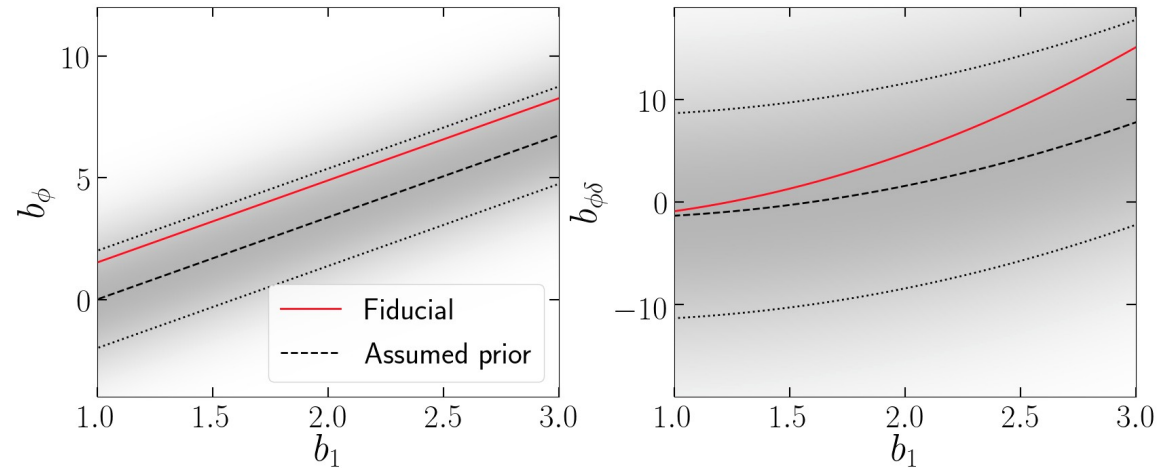
Barreira 2021

Prior centered on the wrong relation

Uncertainty of  $\Delta b_\phi = 2$  ,  $\Delta b_{\phi\delta} = 10$

## Approach 1:

marginalize over Gaussian priors on the bias parameter relations



Precise knowledge of wrong relations

Gaussian prior on  $b_\phi(b_1)$ ,  $b_{\phi\delta}(b_1)$

2x wider prior on  $b_\phi(b_1)$ ,  $b_{\phi\delta}(b_1)$

# Priors on PNG bias parameters

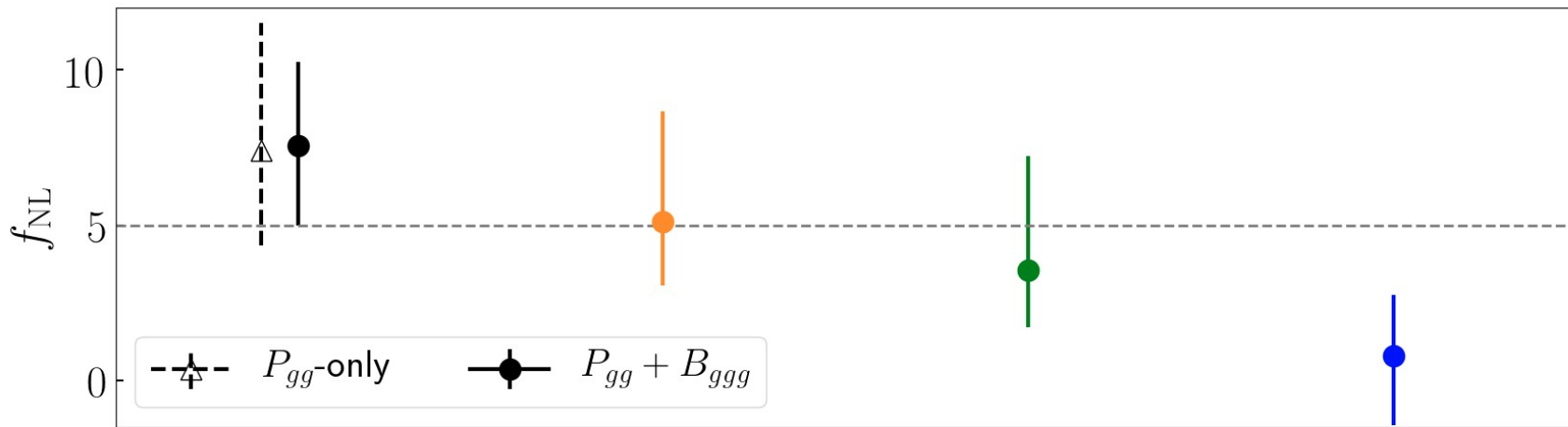
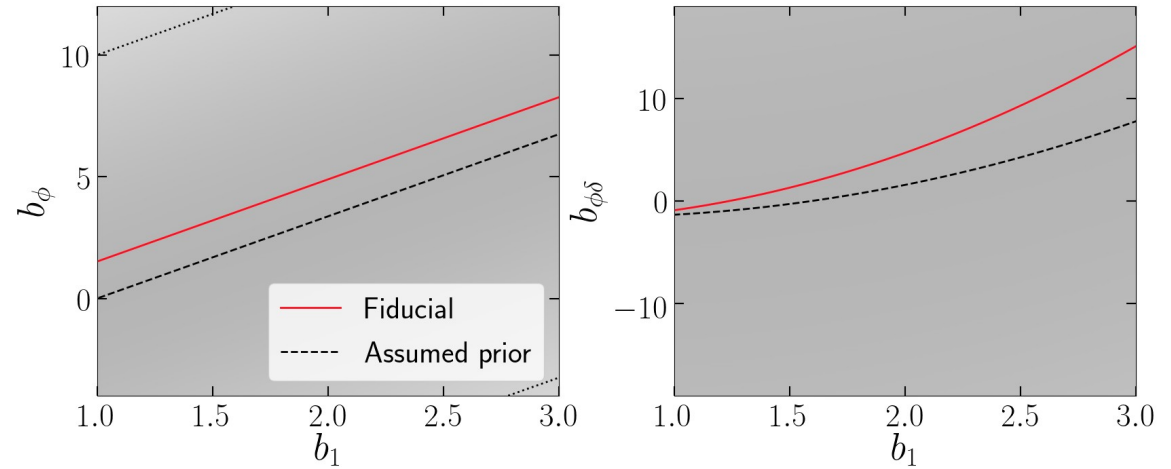
Barreira 2021

Prior centered on the wrong relation

Uncertainty of  $\Delta b_\phi = 10$  ,  $\Delta b_{\phi\delta} = 50$

## Approach 1:

marginalize over Gaussian priors on the bias parameter relations



Precise knowledge of wrong relations

Gaussian prior on  $b_\phi(b_1)$ ,  $b_{\phi\delta}(b_1)$

2x wider prior on  $b_\phi(b_1)$ ,  $b_{\phi\delta}(b_1)$

10x wider prior on  $b_\phi(b_1)$ ,  $b_{\phi\delta}(b_1)$

# Priors on PNG bias parameters

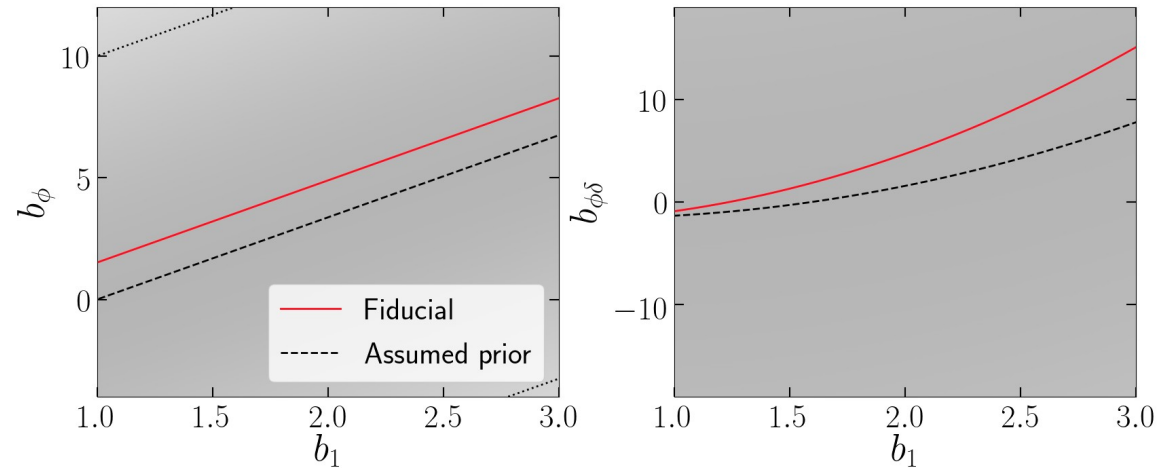
Barreira 2021

## Approach 1:

marginalize over Gaussian priors on the bias parameter relations

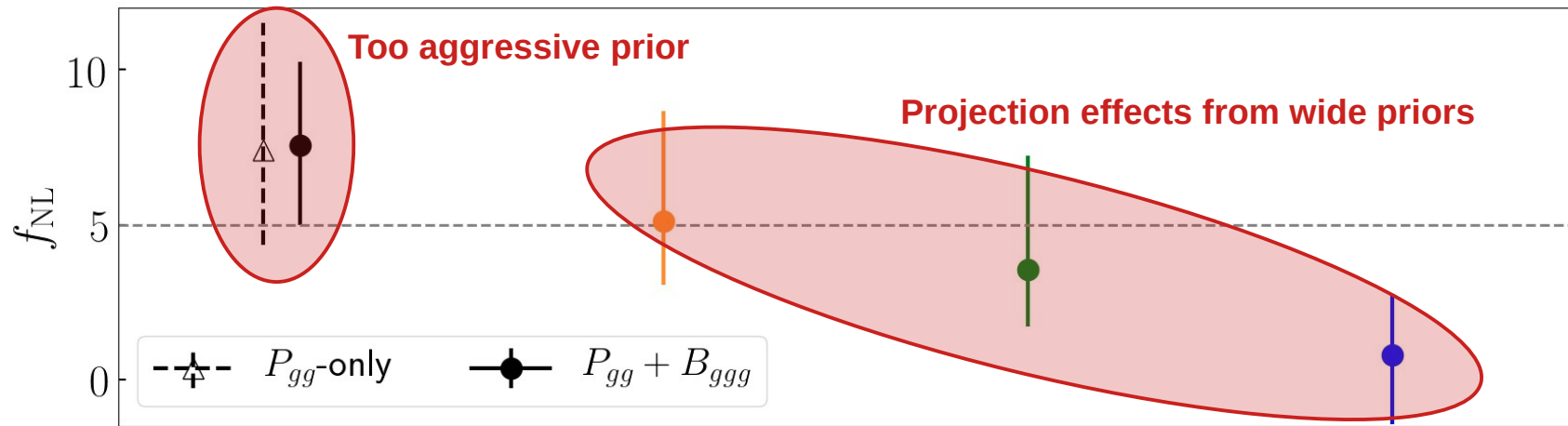
Prior centered on the wrong relation

Uncertainty of  $\Delta b_\phi = 10$  ,  $\Delta b_{\phi\delta} = 50$



## Wide priors are not conservative!

We really need accurate and precise priors from simulations to constrain  $f_{NL}$ .



Precise knowledge of wrong relations

Gaussian prior on  $b_\phi(b_1)$ ,  $b_{\phi\delta}(b_1)$

2x wider prior on  $b_\phi(b_1)$ ,  $b_{\phi\delta}(b_1)$

10x wider prior on  $b_\phi(b_1)$ ,  $b_{\phi\delta}(b_1)$

# The impact of bias uncertainties

Barreira 2021

**Approach 2:** fits for the products of  $f_{NL}b_{\phi}$  and  $f_{NL}b_{\phi\delta}$  .

## **The advantage**

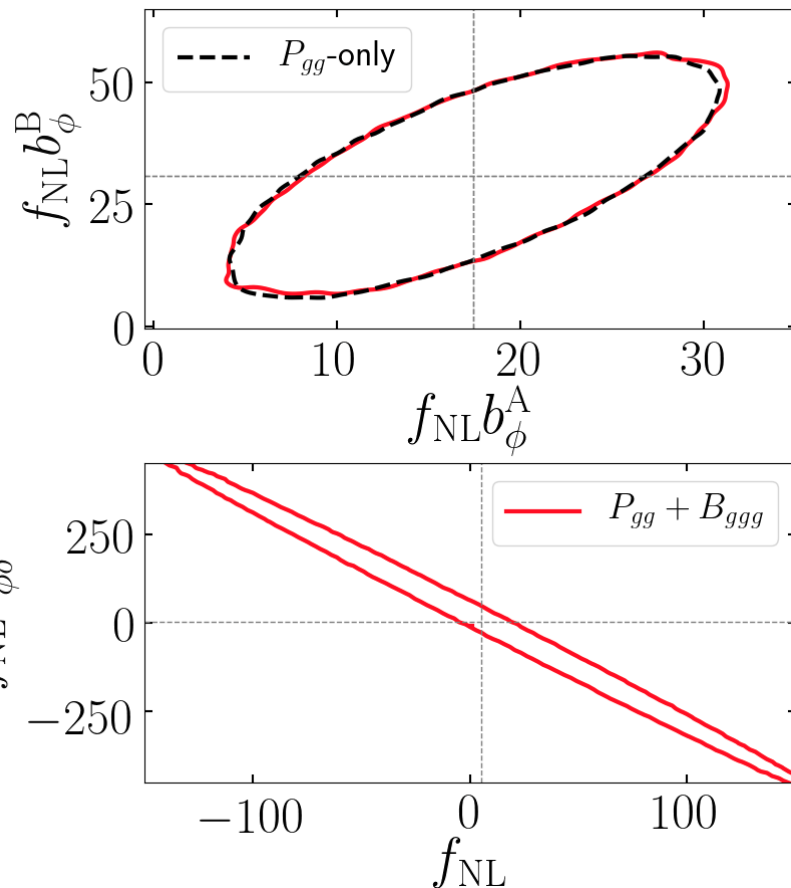
Bypasses any assumptions on PNG bias, while still allowing to distinguish  $f_{NL}$  from zero (we just won't know the exact value).



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## The advantage

Bypasses any assumptions on PNG bias, while still allowing to distinguish  $f_{\text{NL}}$  from zero (we just won't know the exact value).

## The disadvantage

The bispectrum does not improve the constraints (new parameter degeneracies arise).

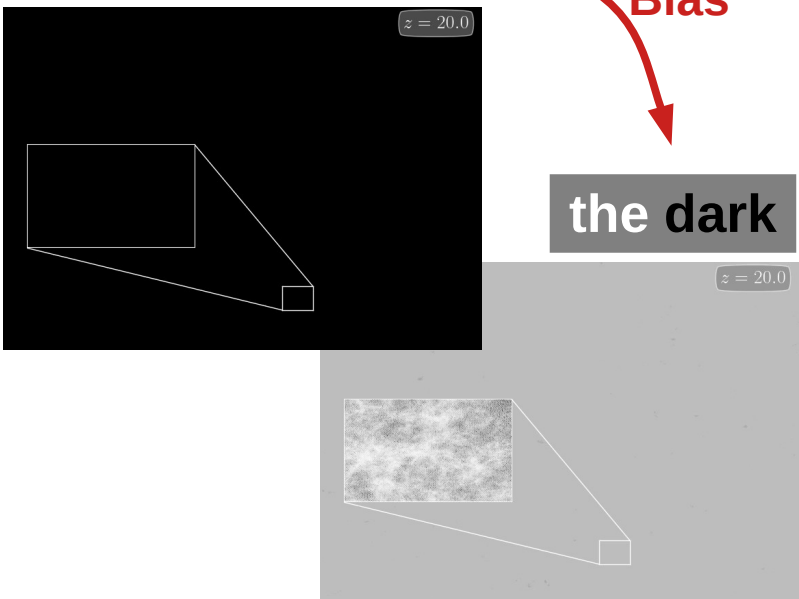
## The recommendation

At least power spectrum analyses should adopt this **bias-assumption-free approach**.

the visible

Bias

the dark



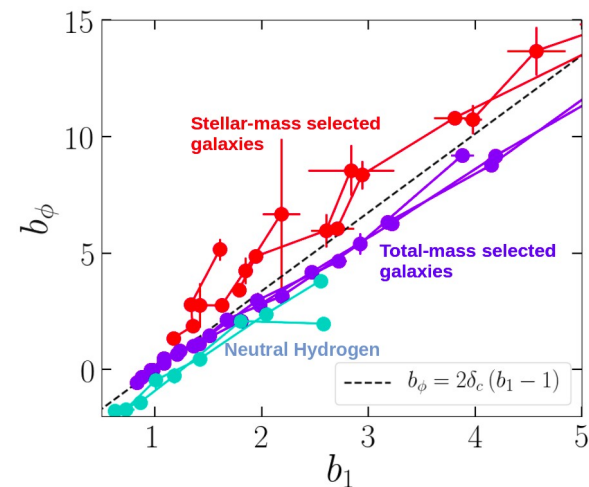
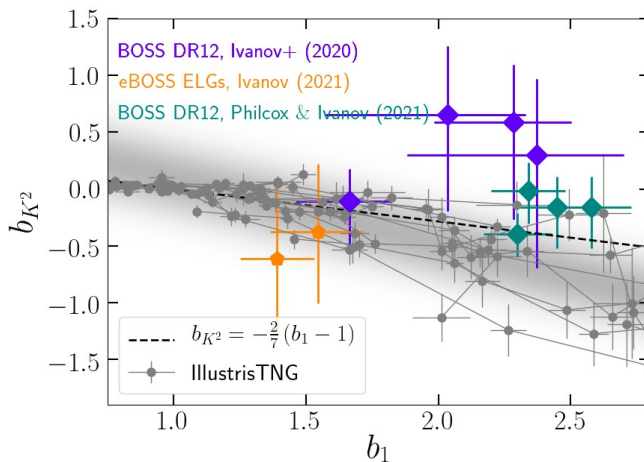
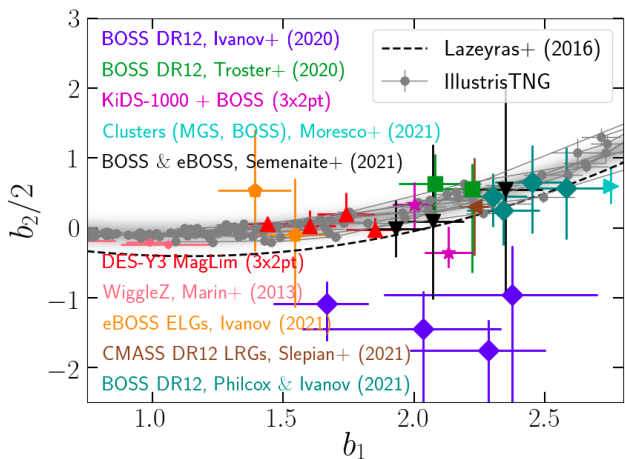
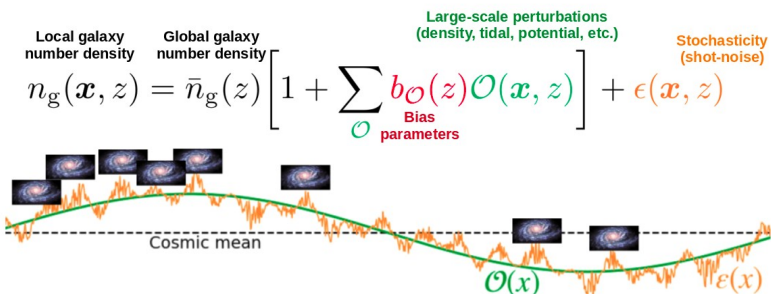
# Summary

- **Galaxy bias** is the relation between the visible and the dark Universe: it is crucial to infer cosmology from galaxy data.

It leads also to new insights about the galaxy-environment connection (eg. halo occupation responses, assembly bias).

- **Separate universe simulations and field-level forward models** are powerful tools to advance our understanding of galaxy bias and the differences to halo bias.

- **Need to revisit local PNG constraints using galaxy data**, which make unjustified assumptions about galaxy bias!



**Extra slides**

# HOD bias derivation

## Origin of the HOD bias:

$$\begin{aligned} b_{g,\mathcal{O}} &= \frac{\partial \ln n_g}{\partial \mathcal{O}} \\ &= \frac{1}{\bar{n}_g} \frac{\partial}{\partial \mathcal{O}} \int dM n_h(M) N_g(M) \\ &= \frac{1}{\bar{n}_g} \int dM n_h(M) N_g(M) \left[ \frac{\partial \ln n_h(M)}{\partial \mathcal{O}} + \frac{\partial \ln N_g(M)}{\partial \mathcal{O}} \right] \\ &= \frac{1}{\bar{n}_g} \int dM n_h(M) N_g(M) \left[ b_{h,\mathcal{O}}(M) + R_{N_g,\mathcal{O}}(M) \right] \end{aligned}$$

## Halo bias expansion:

$$n_h(\mathbf{x}, z) = \bar{n}_h(z) \left[ 1 + b_h(z) \mathcal{O}(\mathbf{x}, z) \right]$$

## HOD bias expansion:

$$N_g(\mathbf{x}, M, z) = \bar{N}_g(M, z) \left[ 1 + R_{N_g}(M, z) \mathcal{O}(\mathbf{x}, z) \right]$$