Amplitudes and the Riemann Zeta Function

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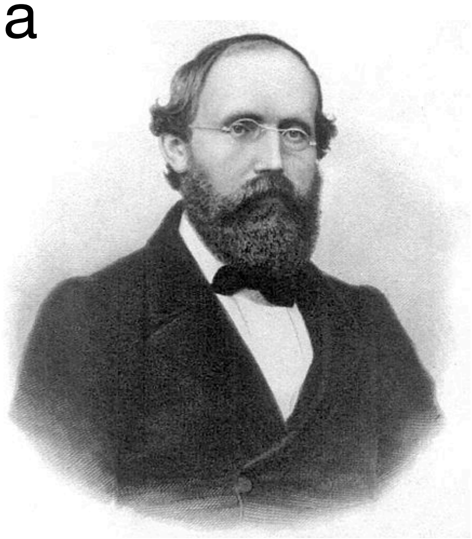
Motivation

The Riemann zeta function

The Riemann zeta function

- Introduced by Bernhard Riemann in 1859, a particular function of a single complex variable:



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for $\mathrm{Re}(z) > 1$. Extend to the rest of the complex plane by analytic continuation.

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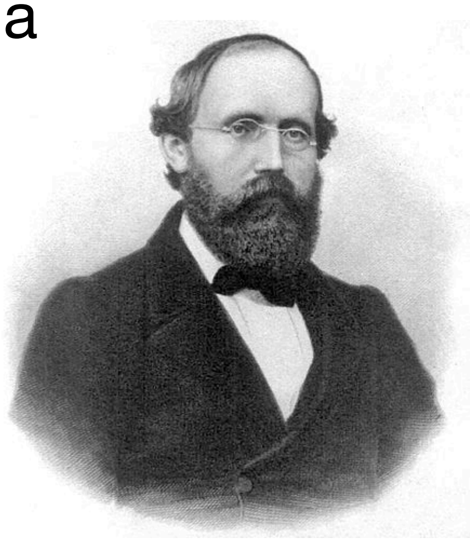


$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

for $\mathrm{Re}(z) > 1$. Extend to the rest of the complex plane by analytic continuation.

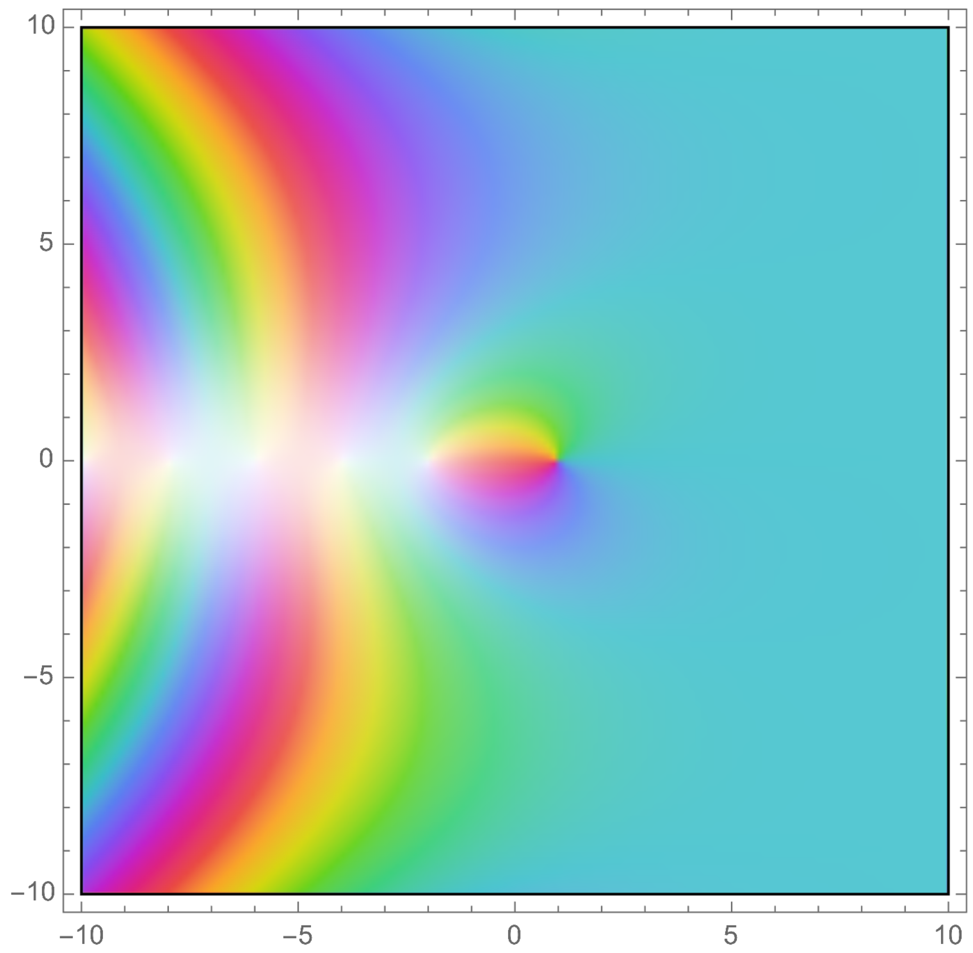
- Many interesting properties, with deep connections to the distribution of the primes:

$$\zeta(z) = \prod_{p \text{ prime}} \frac{1}{1-p^{-z}} \qquad \text{(Euler)}$$

$$\log \zeta(z) = z \int_0^{\infty} \frac{\pi(x)}{x(x^z - 1)} \mathrm{d}x \quad \text{for} \quad \pi(x) = (\#\text{ primes} \leq x)$$

The Riemann zeta function

- The zeta function has been the subject of 150 years of mathematical interest, and its properties have been intensively investigated.



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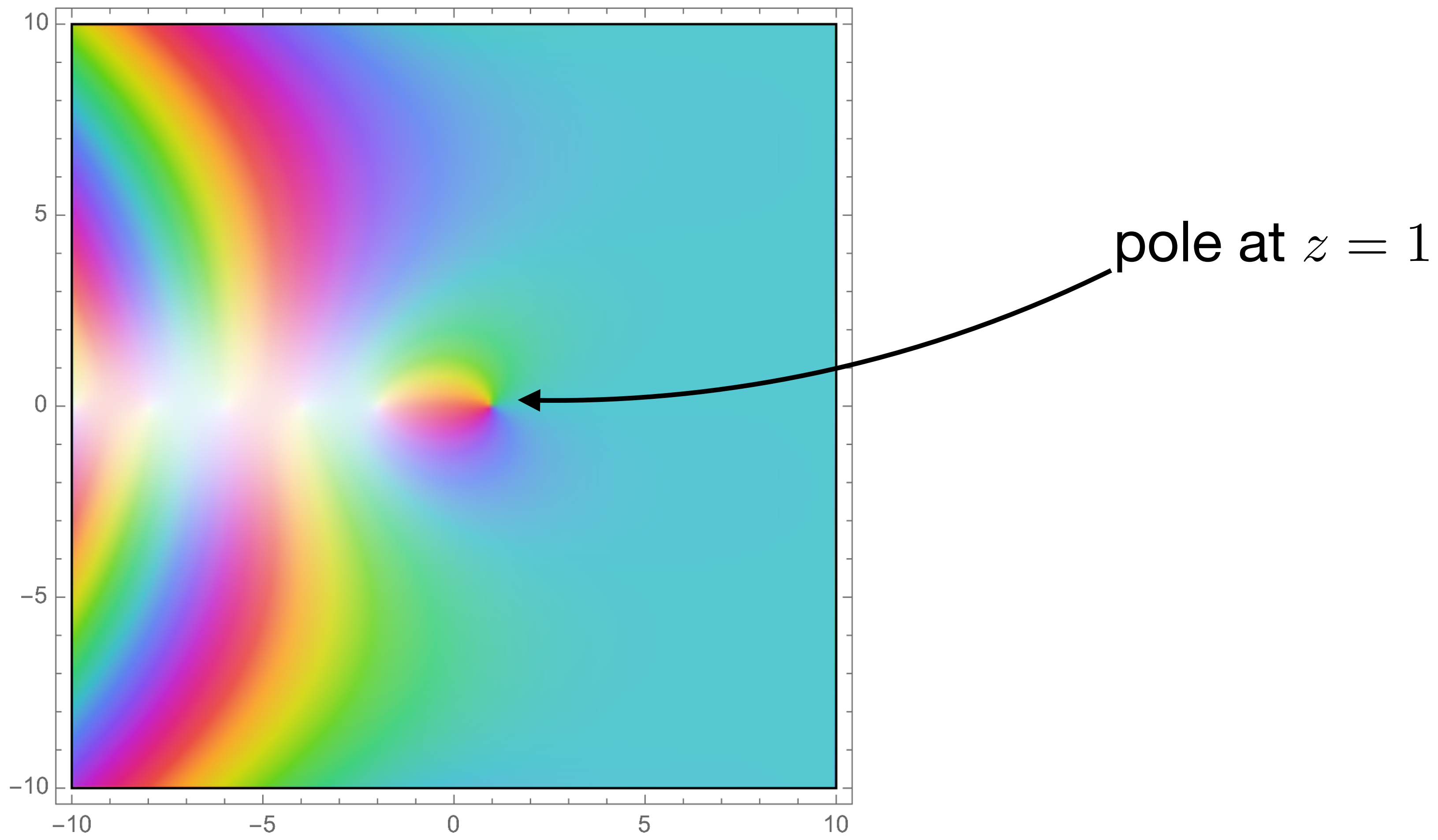

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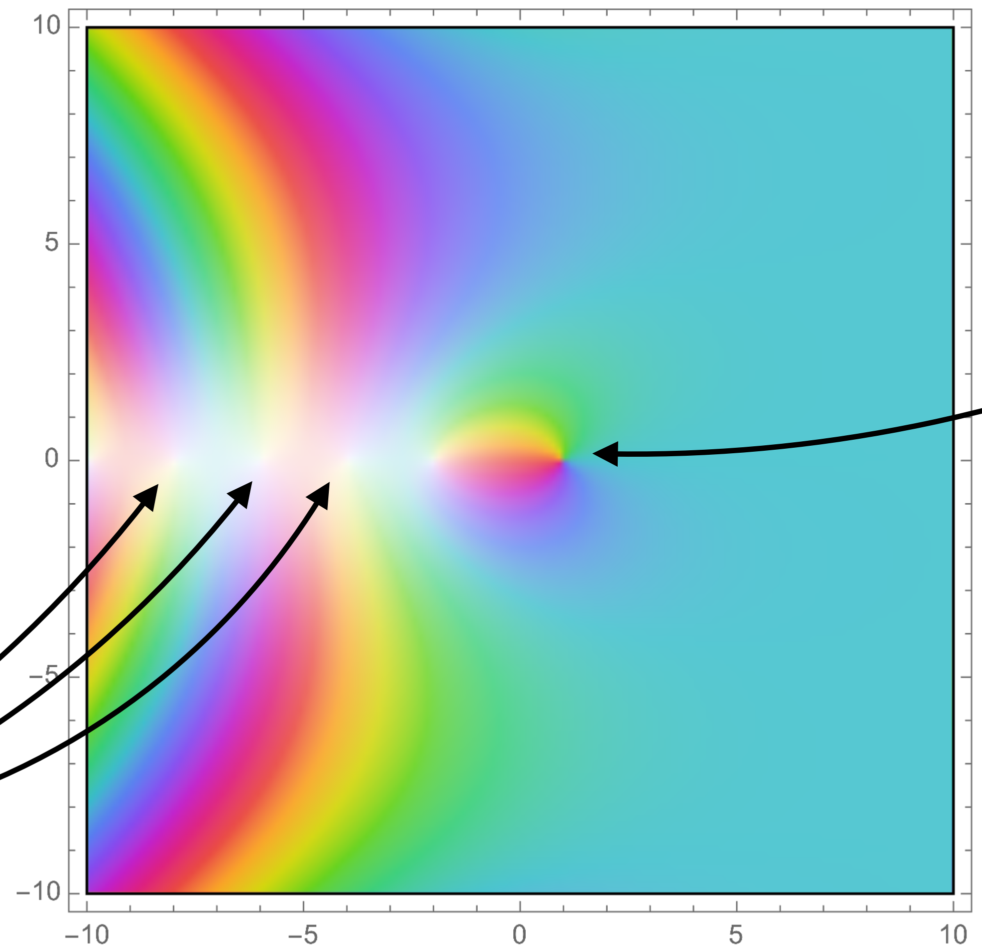

pole at $z = 1$

trivial zeros at $z = -2n$

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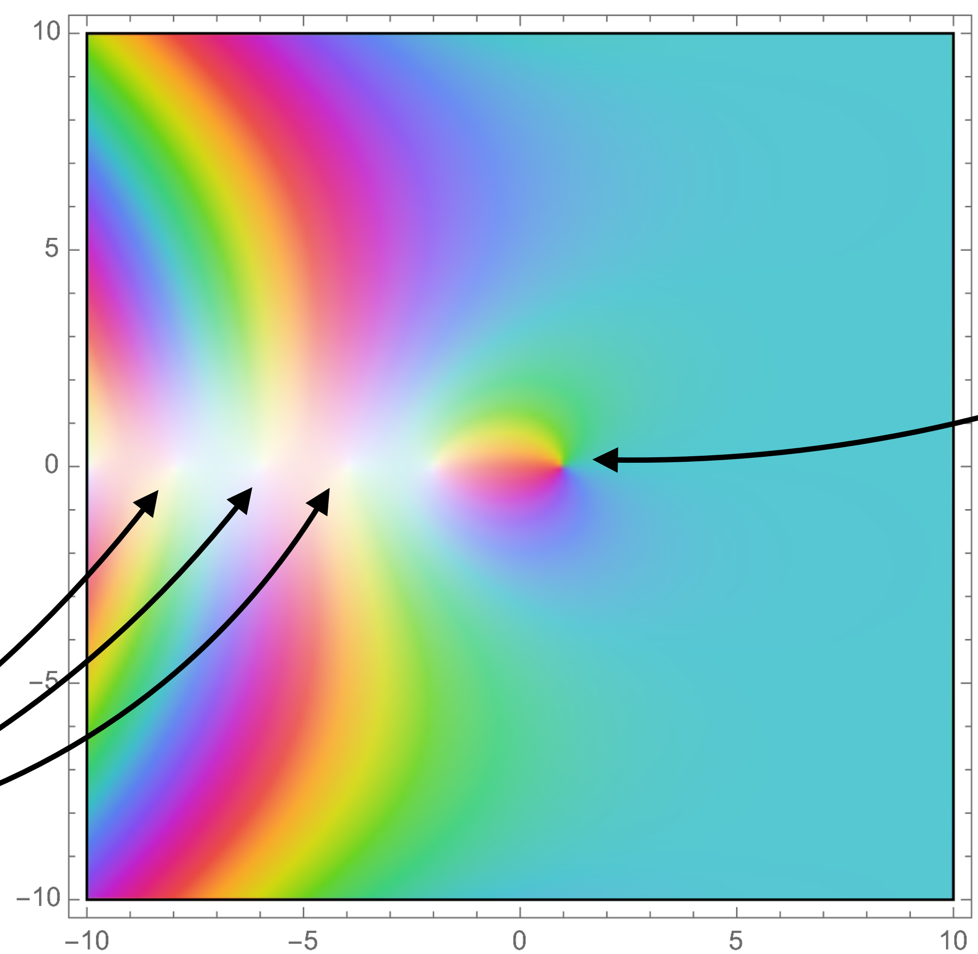

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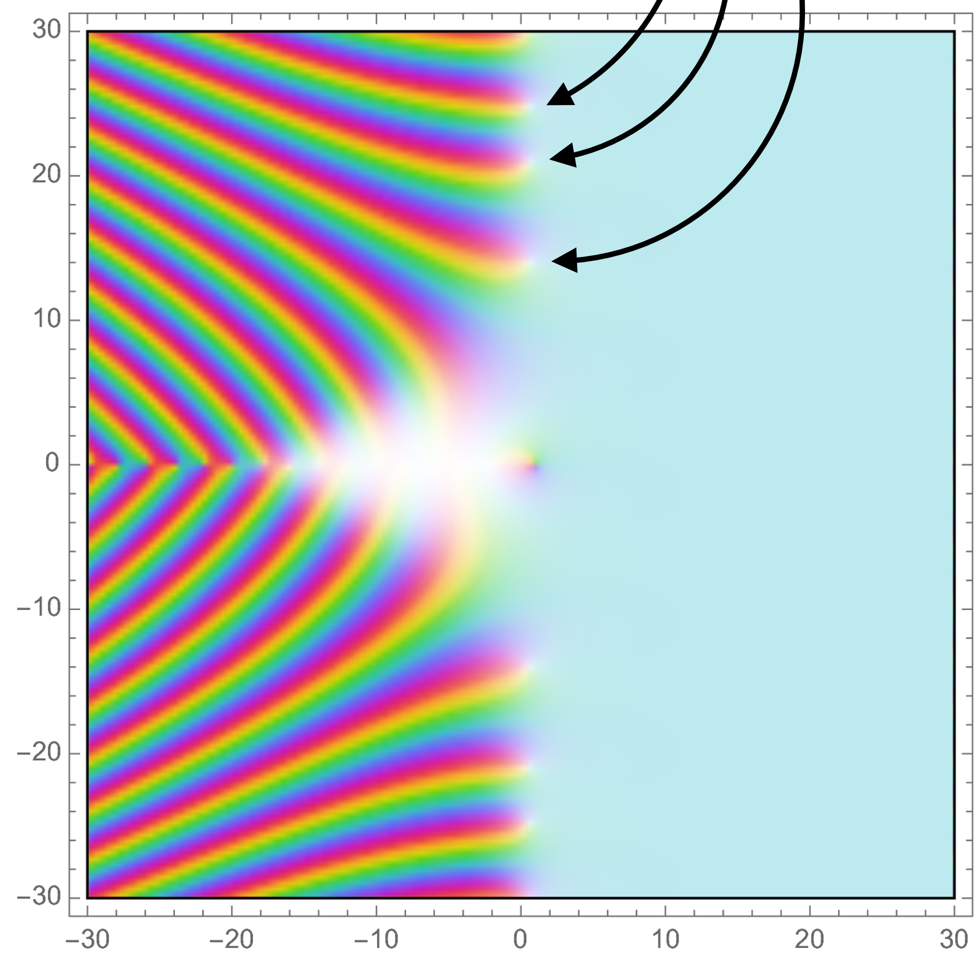
- Functional equation: $\zeta(z) = 2^z \pi^{z-1} \sin(\pi z/2) \Gamma(1-z) \zeta(1-z)$

The Riemann zeta function

- Zooming out, we find a collection of additional zeros that all seem to lie on the "critical line" where $\mathrm{Re}(z) = \frac{1}{2}$.



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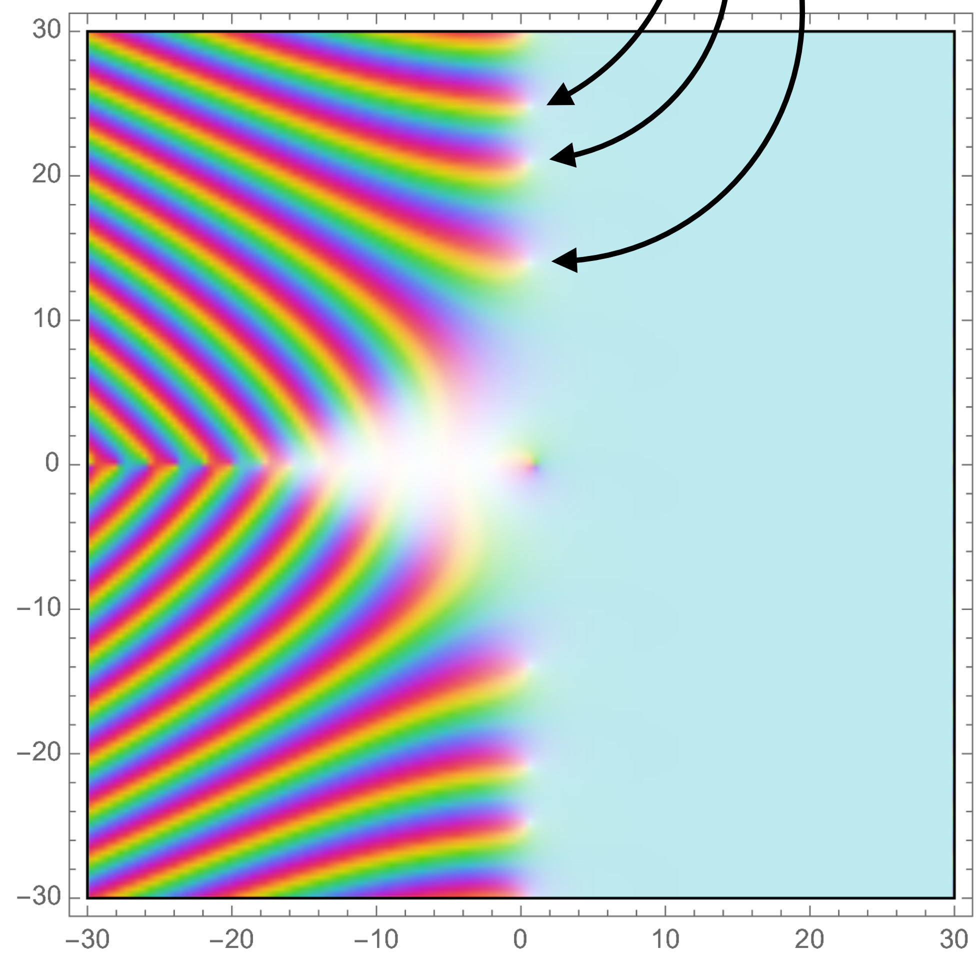

$$\zeta\left(\tfrac{1}{2} \pm i\mu_n\right) = 0$$

$$\mu_1 \simeq 14.135$$

$$\mu_2 \simeq 21.022$$

$$\vdots$$

(We take $\mathrm{Re}(\mu_n) > 0$ throughout.)

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- If true it would have various nice number theory consequences, e.g.,

$$\left| \pi(x) - \int_0^x \frac{\mathrm{d}t}{\log t} \right| < \frac{1}{8\pi}\sqrt{x}\log x \quad \text{for} \quad x \geq 2657$$

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- Other open questions:
  - Are all the zeros simple ones?
  - What can be be proven about the statistical properties of the zeros?
  - What is the asymptotic behavior of ζ on the critical line?

Connections to physics

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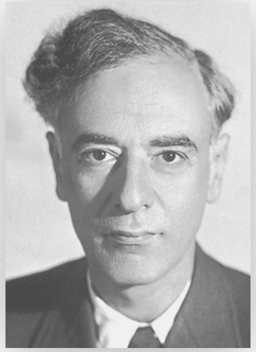

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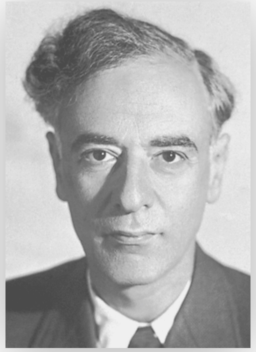

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- Montgomery's pair correlation conjecture: Montgomery (1973)
  The correlation function for the normalized spacings of the nontrivial zeros is:

$$1 - \left(\frac{\sin \pi u}{\pi u}\right)^2 + \delta(u)$$

This is the same as the two-point function for a Gaussian unitary ensemble. Dyson

- Other work in quantum chaotic nonrelativistic scattering includes
  Gutzwiller (1983); Bhaduri, Khare, Law [chaodyn/9406006]; see also Srednicki [1105.2342]

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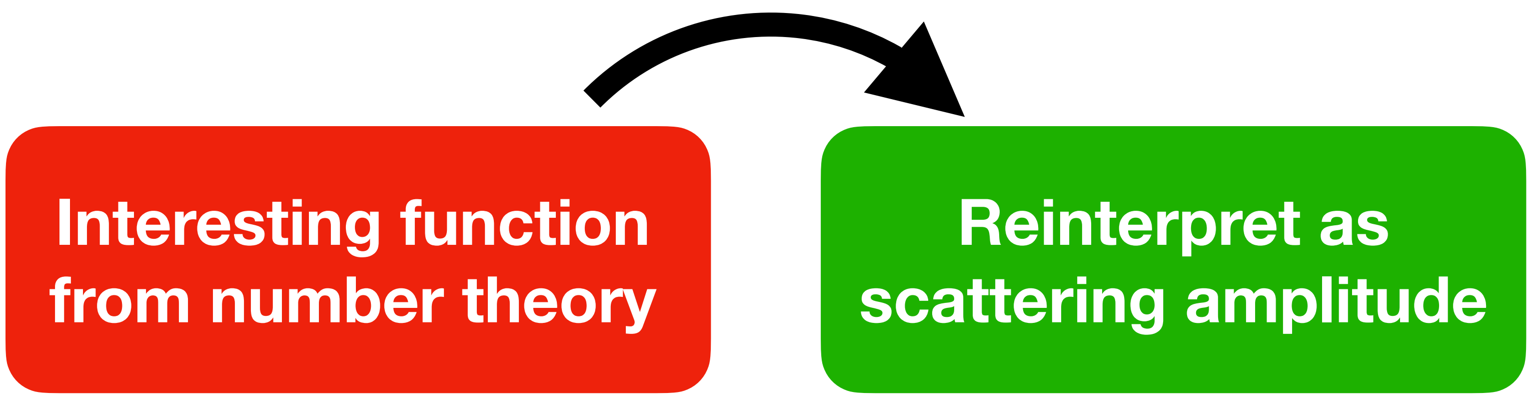

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Previous illustrious results from this process!

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Veneziano (1968):

Euler beta function $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ → **Veneziano amplitude** → **String theory**

What about amplitudes?

- Rather than try to prove the Riemann hypothesis, can we gain any insight if we somehow recast the zeta function as a relativistic scattering amplitude?
- Indeed, the Veneziano amplitude itself can be written in terms of ζ: Freund, Witten (1987)

$$A_4(s,t,u) = B(-\alpha(s),-\alpha(t)) + B(-\alpha(t),-\alpha(u)) + B(-\alpha(s),-\alpha(u)) = \prod_{x=s,t,u} \frac{\zeta(1+\alpha(x))}{\zeta(-\alpha(x))}$$

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However, this is somewhat illusory: the nontrivial zeros cancel out entirely.

He, Jejjala, Minic [1501.01975]

$$\frac{\zeta(1+z)}{\zeta(-z)} = \pi^{\frac{1}{2}+z} \frac{\Gamma\left(-\frac{z}{2}\right)}{\Gamma\left(\frac{1+z}{2}\right)}$$

Zeta/amplitudes correspondence

- Rather than try to prove the Riemann hypothesis, can we gain any insight if we somehow recast the zeta function as a relativistic scattering amplitude?

In this talk, we will construct a relativistic four-point scattering amplitude $\mathcal{M}(s,t)$ that truly captures the nontrivial properties of the zeta function.

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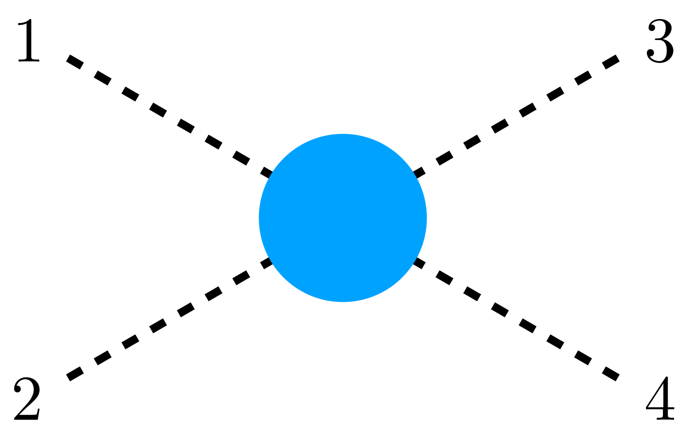
On-shell constructibility	$\longleftrightarrow$	Hadamard product expansion
CPT invariance	$\longleftrightarrow$	Reflection of zeros across critical line

Building the amplitude

Bottom-up approach

- Most important feature: ζ has nontrivial zeros that (appear to) all lie on a line

  Connection with amplitudes: poles all lie on lines corresponding to real kinematics, $s, t, u = m^2$



$$s = -(p_1 + p_2)^2$$
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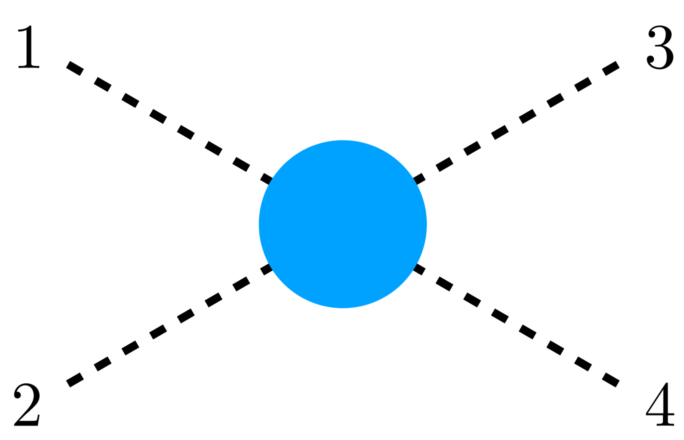
  Connection with amplitudes: poles all lie on lines corresponding to real kinematics, $s, t, u = m^2$

- Let's use this as a guiding principle to design our zeta-amplitude. We'll start by trying to build a forward amplitude ($t = 0$) with poles corresponding to zeros of ζ.



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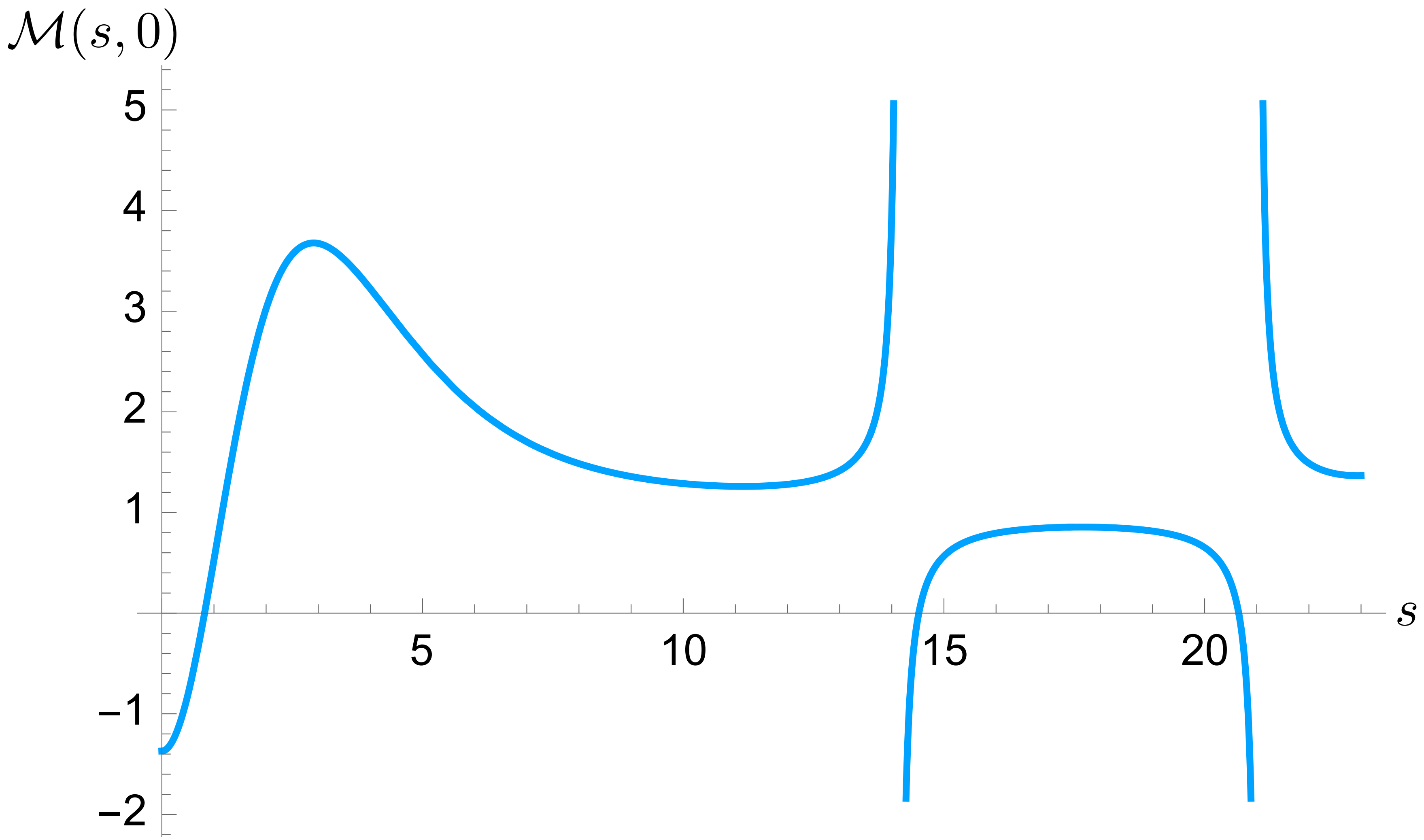
- What can $\mathcal{A}(s)$ be?

Bottom-up approach

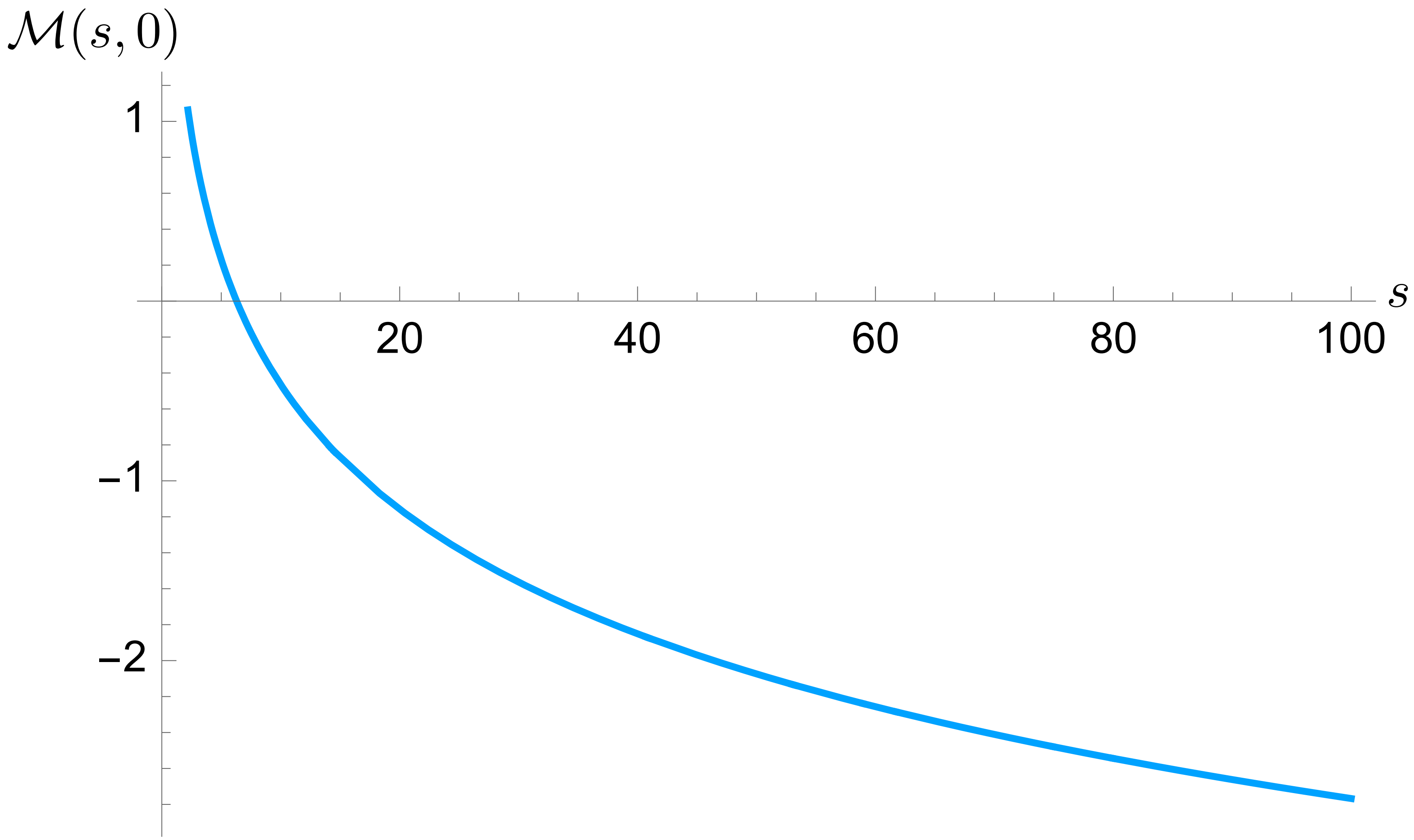
- What about $\mathcal{A}(s) = 1/\zeta\left(\frac{1}{2} + is\right)$?

  ✘ Poles with opposite-sign residues: tachyons



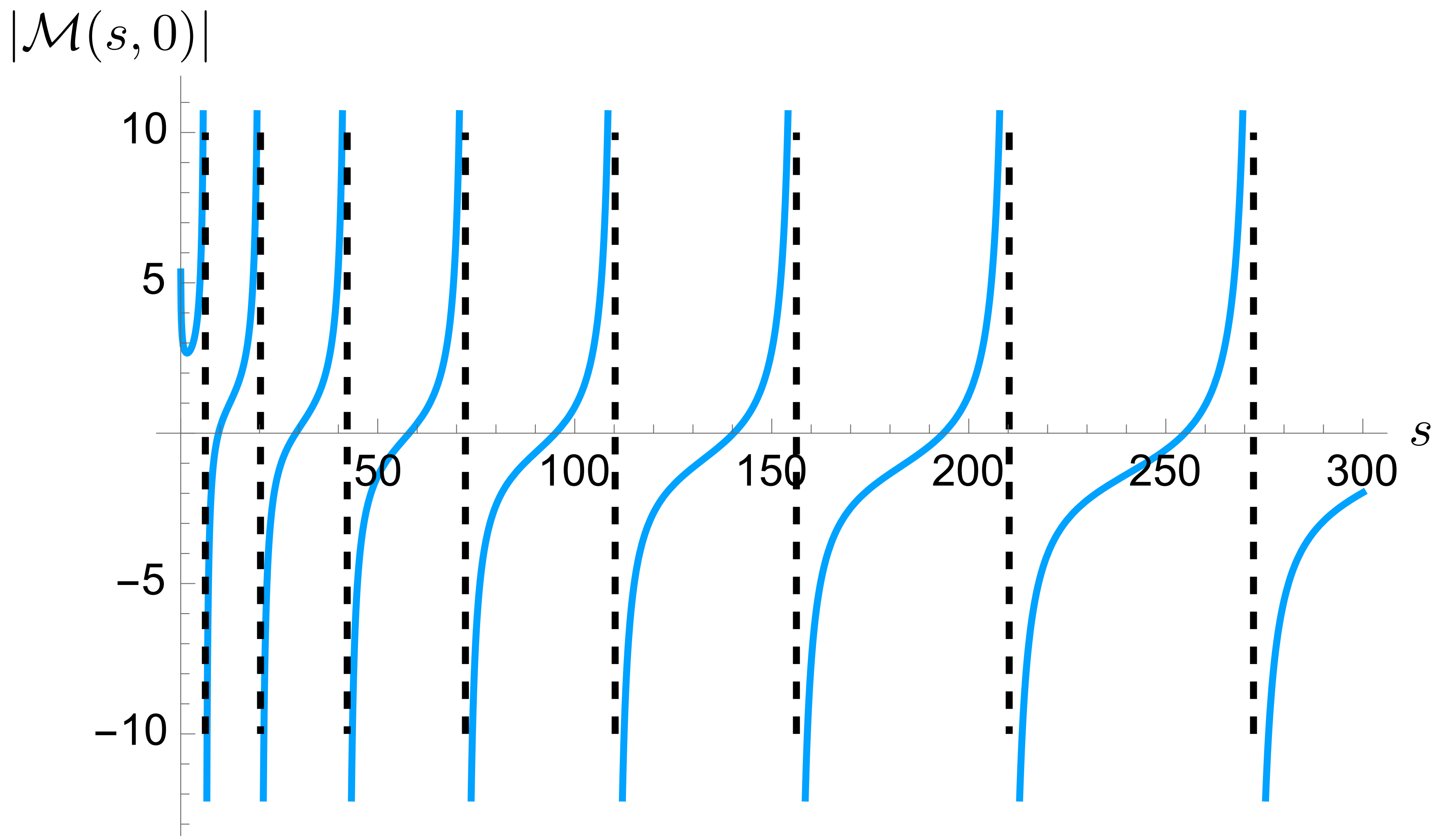
Bottom-up approach

- What about $\mathcal{A}(s) = \dfrac{\zeta'\left(\frac{1}{2}+is\right)}{\zeta\left(\frac{1}{2}+is\right)}$?
    - ✘ No more poles



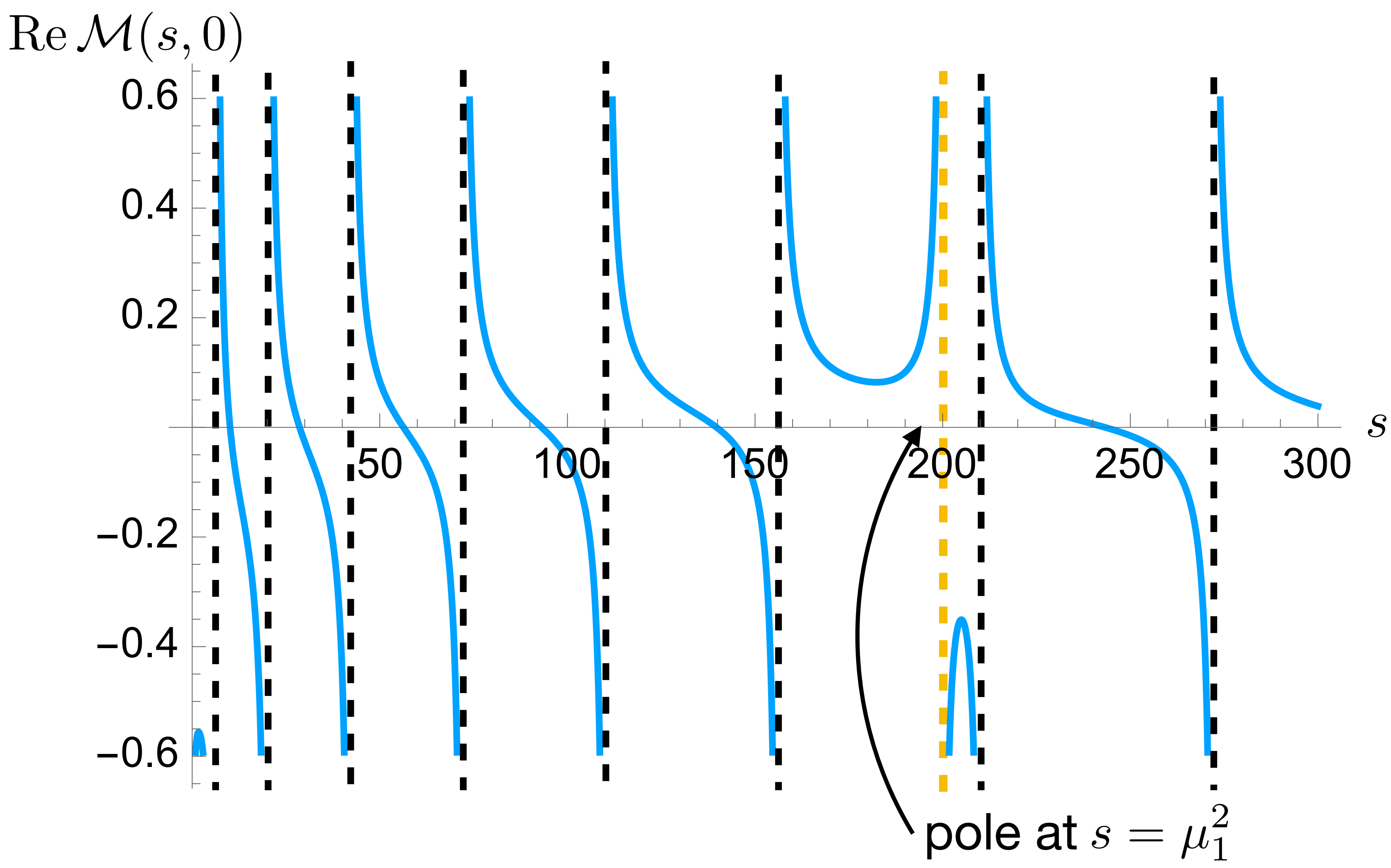
Bottom-up approach

- What about $\mathcal{A}(s) = \dfrac{\zeta'\left(\frac{1}{2} + i\sqrt{s}\right)}{\zeta\left(\frac{1}{2} + i\sqrt{s}\right)}$?
  - ✘ Only poles in the wrong places: $s = \frac{(4n+1)^2}{4}$



Bottom-up approach

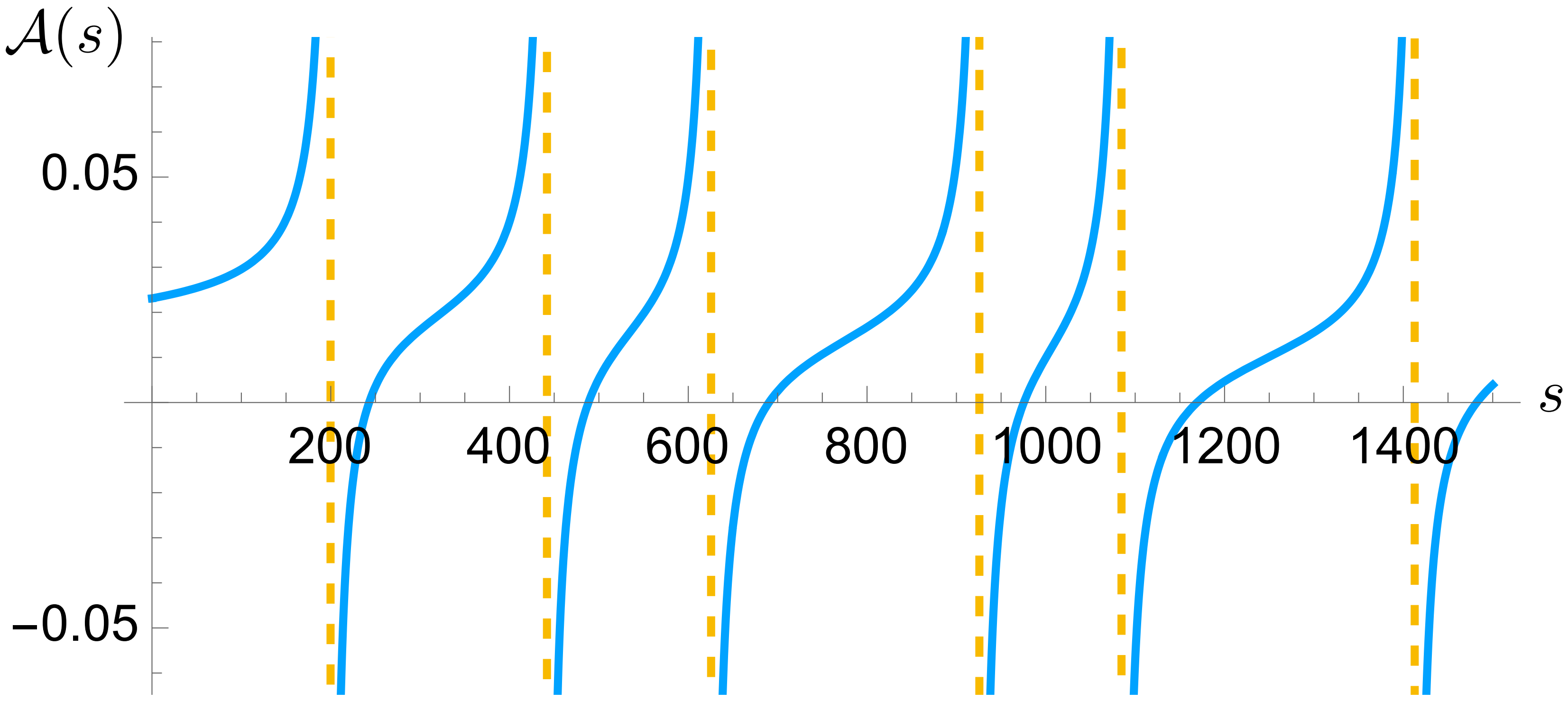
- What about $\mathcal{A}(s) = -\frac{i}{2\sqrt{s}} \frac{\zeta'\left(\frac{1}{2}+i\sqrt{s}\right)}{\zeta\left(\frac{1}{2}+i\sqrt{s}\right)}$?
  - ✘ Still have extra poles in the wrong places: $s = \frac{(4n+1)^2}{4}$



A Riemann zeta amplitude

- To cancel all the wrong poles, we compute their residues and add terms to remove them. Also adding a term to make the forward amplitude real, we find:

$$\mathcal{A}(s) = -\frac{i}{4\sqrt{s}}\left[\psi\left(\tfrac{1}{4}+\tfrac{i}{2}\sqrt{s}\right)+\frac{2\zeta'\left(\frac{1}{2}+i\sqrt{s}\right)}{\zeta\left(\frac{1}{2}+i\sqrt{s}\right)}\right]+\frac{i\log\pi}{4\sqrt{s}}-\frac{1}{s+\frac{1}{4}}$$



- Poles at $s=\mu_n^2$

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Digamma function: $\psi(z) = \Gamma'(z)/\Gamma(z)$
Poles at $\psi(-n)$ cancel trivial zeros at $\zeta(-2n)$ for integer $n > 0$
Pole at $\psi(0)$ canceled by $1/\left(s+\frac{1}{4}\right)$ term

No branch cuts: $\lim\limits_{\epsilon\to 0}\mathcal{A}(s+i\epsilon)-\mathcal{A}(s-i\epsilon)=0$

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- In terms of the Landau-Riemann xi functions,

$$\Xi(z) = \xi\left(\tfrac{1}{2}+iz\right)$$

$$\xi(z) = \tfrac{1}{2}z(z-1)\pi^{-z/2}\Gamma\left(\tfrac{z}{2}\right)\zeta(z)$$

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$\mathcal{A}(s)$ can be written very compactly as:

$$\boxed{\mathcal{A}(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\Xi(\sqrt{s})}$$

$$\mathcal{M}(s,t) = \mathcal{A}(s)+\mathcal{A}(u)$$

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3. The forward amplitude satisfies

$$\lim_{s \to 0} \frac{\mathrm{d}^2}{\mathrm{d}s^2} \mathcal{M}(s, 0) \neq 0$$

A Riemann zeta amplitude

- Were the square roots necessary?

A Riemann zeta amplitude

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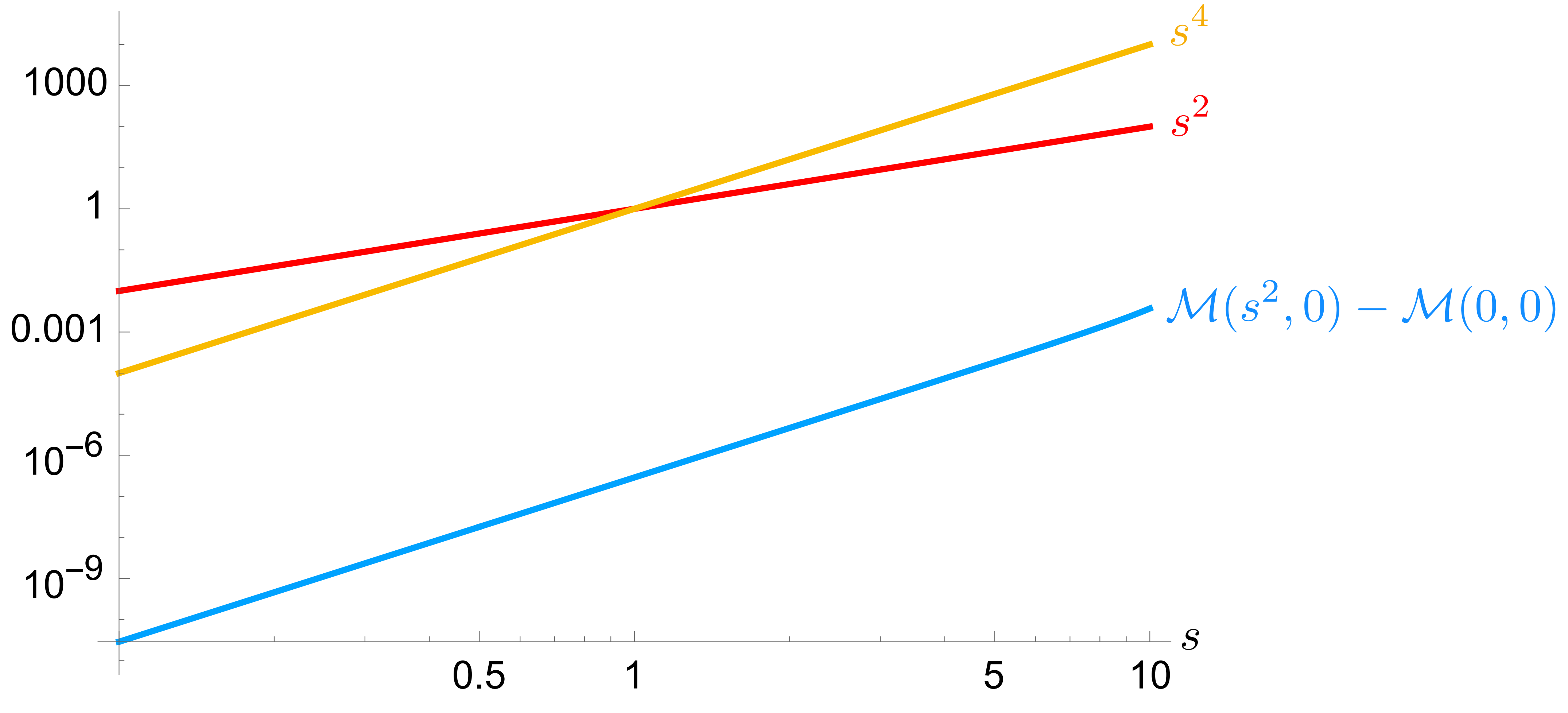
  Yes: If we send $s \to s^2$ in $\mathcal{M}(s,0)$ to eliminate the square roots, then the forward amplitude scales with s^4 at small momentum.

  This violates the s^2 scaling required by dispersion relations. Adams et al. [hep-th/0602178]



Properties of $\mathcal{A}(s)$

- Connection between low-momentum behavior and the zeros of zeta:

$$\frac{c_0}{2} = \lim_{s\to 0} \mathcal{A}(s) = -4 + \frac{\pi^2}{8} + G + \frac{\zeta''\left(\frac{1}{2}\right)}{2\zeta\left(\frac{1}{2}\right)} - \frac{1}{8}\left(\gamma + \frac{\pi}{2} + \log 8\pi\right)^2$$

Catalan's constant $G = \sum_{k=0}^{\infty} (-1)^k/(2k+1)^2$

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using the Hadamard product form of the zeta function (more on this later).

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- Poles corresponding to the nontrivial zeros: $\zeta\left(\frac{1}{2} \pm i\mu_n\right) = 0$

If the Riemann hypothesis holds, these poles are all at real, positive masses.

$$m_n = \mu_n$$

The poles have the correct (positive) residue required by unitarity:

$$\oint_{s=\mu_n^2} i\mathcal{A}(s + i\epsilon)\mathrm{d}s > 0$$

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All simple zeros $\Longrightarrow$ Universal coupling of massive states

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- Can parameterize any $g_n \neq 1$ by allowing degeneracies among the μ_n

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- Locality: All poles are simple ones.

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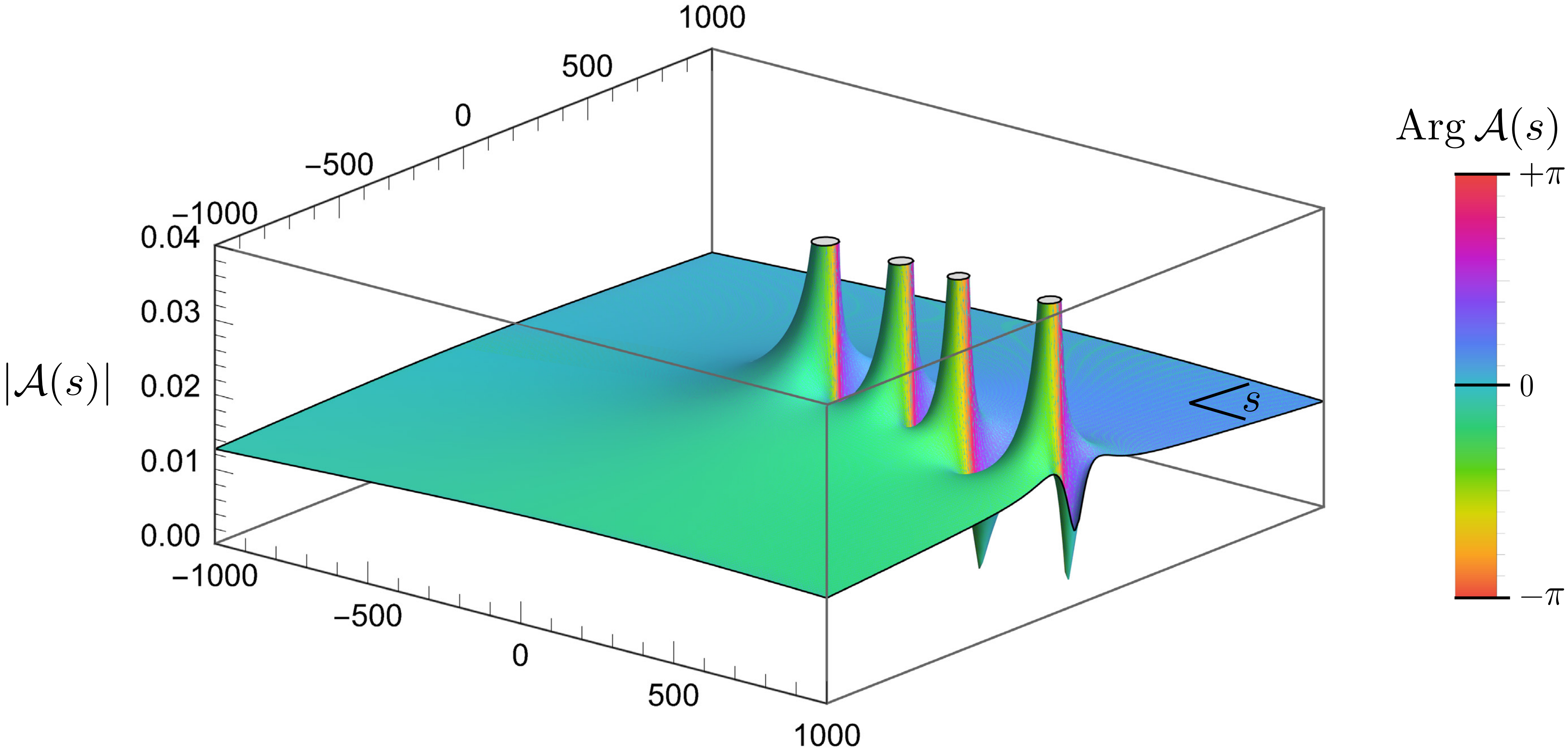
- Nonlocality in $\mathcal{A}(s) \sim 1/(-s+\mu_n^2)^k$ for $k > 1$ would correspond to an essential singularity in the Riemann zeta function,

$$\zeta(z) \sim e^{\frac{\alpha}{(z-z_n)^{k-1}}}$$

Locality in $\mathcal{A}$ $\longleftrightarrow$ Meromorphicity in ζ

Properties of $\mathcal{A}(s)$

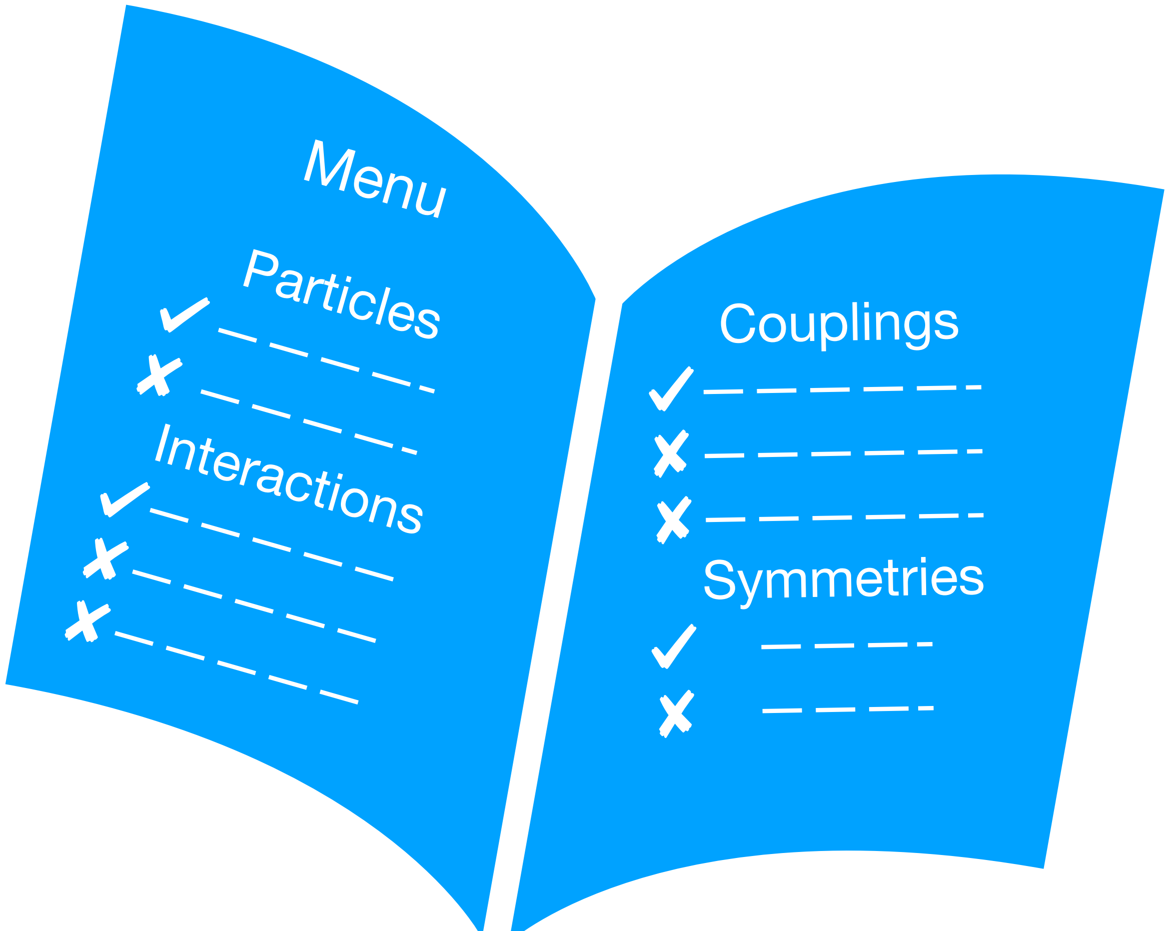


Analytic dispersion relations

Which theories are possible?

Can any Lagrangian be a consistent EFT?

- Certain signs or magnitudes of couplings violate fundamental physics principles:
  - Unitarity
  - Causality
  - Analyticity
  - Thermodynamics



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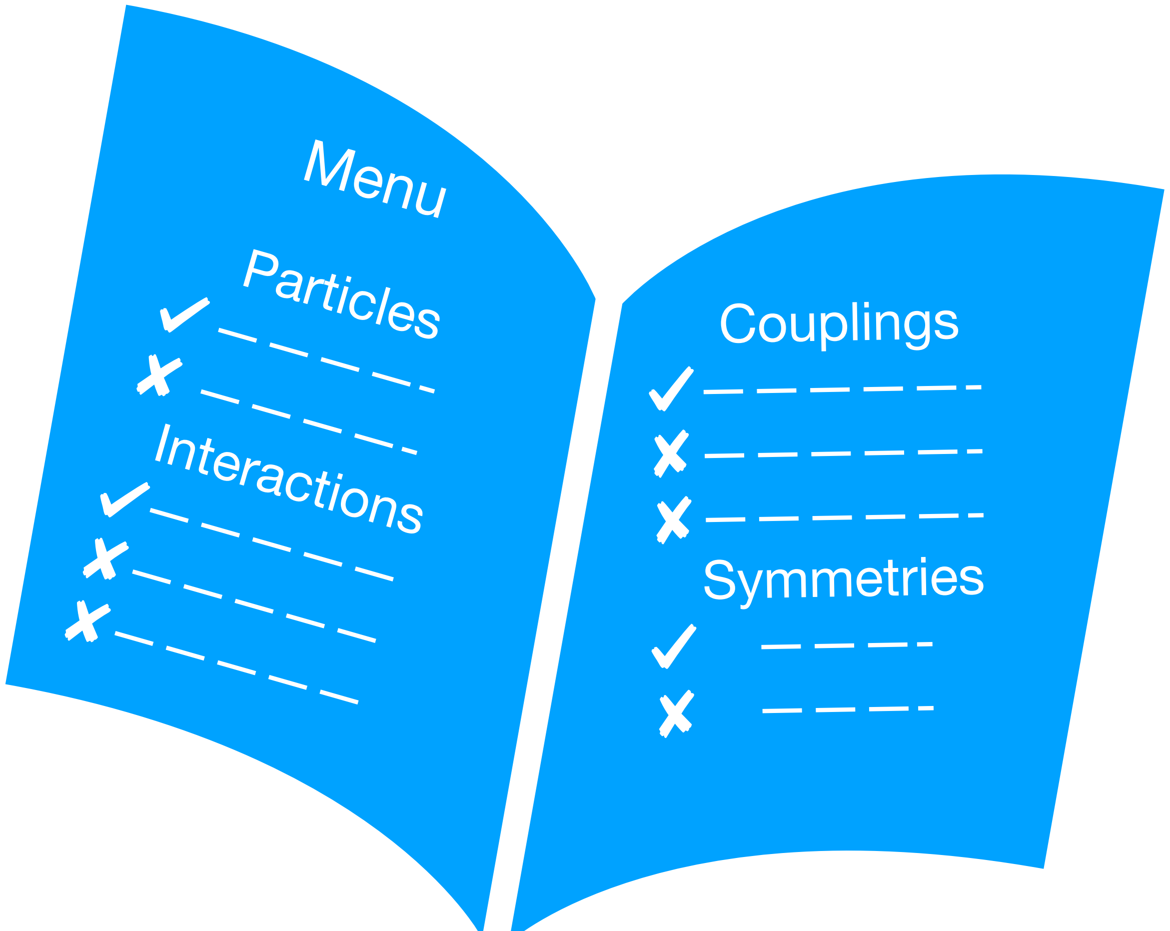
"infrared consistency"



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- Examples:
    - Standard Model EFT — **GR**, Rodd [1908.09845] & (2022, forthcoming)
    - Flavor physics — **GR**, Rodd [2004.02885, 2010.04723]
    - Higher-curvature terms — Bellazzini, Cheung, **GR** [1509.00851]; Cheung, **GR** [1608.02942]; Gruzinov, Kleban (2006)
    - Massive gravity — Cheung, **GR** [1601.04068]
    - Einstein-Maxwell theory — Cheung, **GR** [1407.7865]; Cheung, Liu, **GR** [1801.08546, 1903.09156]; Arkani-Hamed, Huang, Liu, **GR** [2109.13937]
    - Scalar theories — Adams et al. (2006); Chandrasekaran, **GR**, Shahbazi-Moghaddam [1804.03153]
    - a-theorem — Komargodski, Schwimmer (2011); Elvang et al. (2012)

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- Our $\mathcal{M}(s,t)$ built from the zeta function will by definition satisfy the requirements of analyticity and unitarity for scattering amplitudes.

- **Question:** What happens if we run $\mathcal{M}(s,t)$ through the mechanics of analytic dispersion relations?

Example theory

We'll first briefly review how infrared consistency bounds the coefficients of an EFT, based on analyticity, unitarity, and causality. Adams et al. [hep-th/0602178]

Example EFT: massless scalar with shift symmetry

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \frac{c}{M^4}(\partial\phi)^4$$

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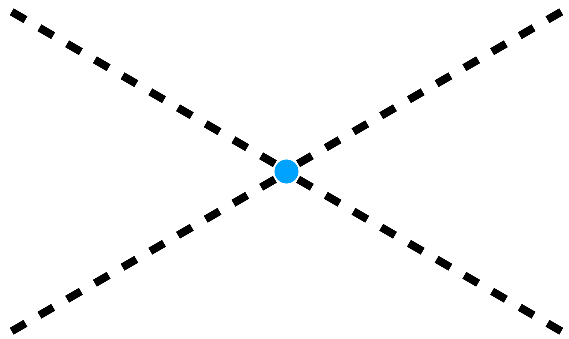
Two-to-two scattering amplitude:



$$\mathcal{M}(s,t) = \frac{2c}{M^4}(s^2 + t^2 + u^2)$$

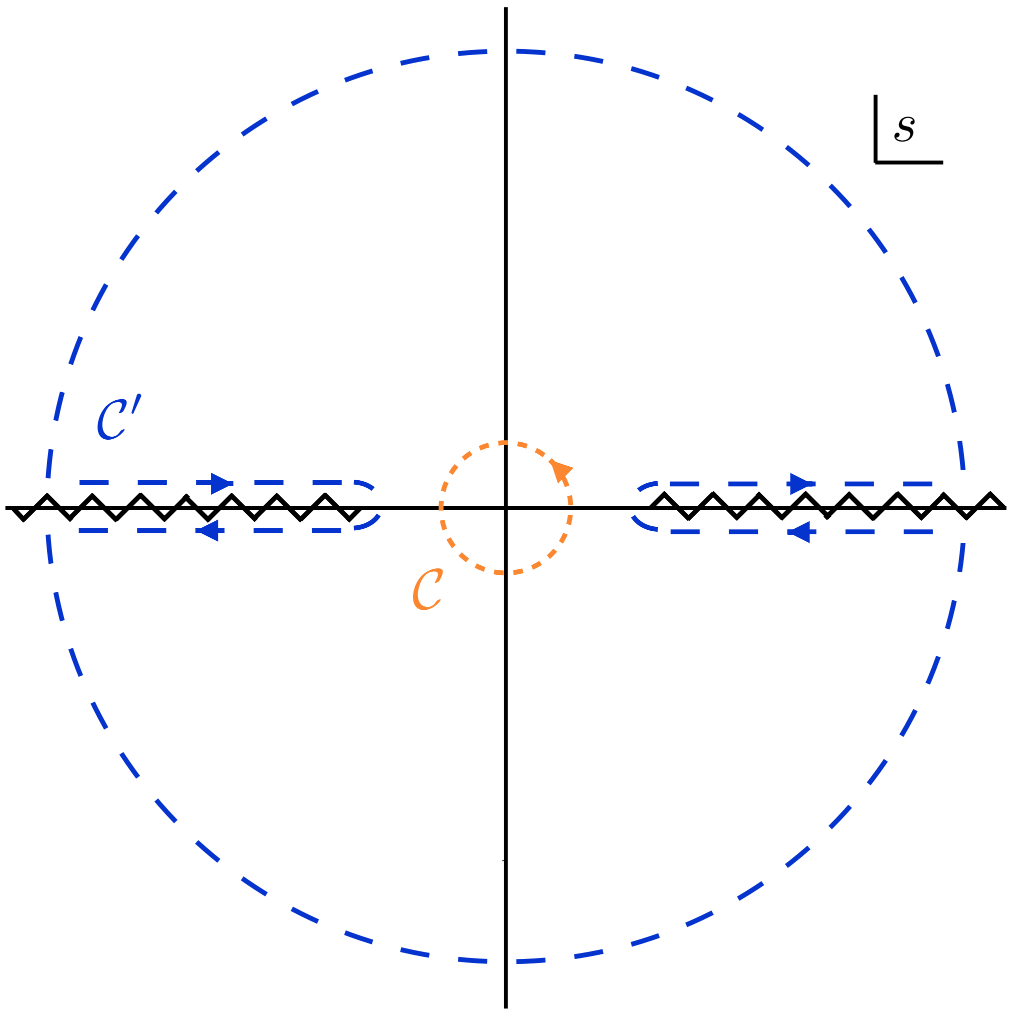
Forward amplitude (in state = out state):

$$\mathcal{M}(s,0) = \frac{4c}{M^4}s^2$$

$$s = -(p_1 + p_2)^2$$
$$t = -(p_1 + p_3)^2$$
$$u = -(p_1 + p_4)^2$$

Analyticity and unitarity

The Wilson coefficient of interest can be extracted via a contour integral of the forward amplitude:



$$\frac{4c}{M^4} = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{\mathrm{d}s}{s^3}\mathcal{M}(s,0)$$

residue theorem

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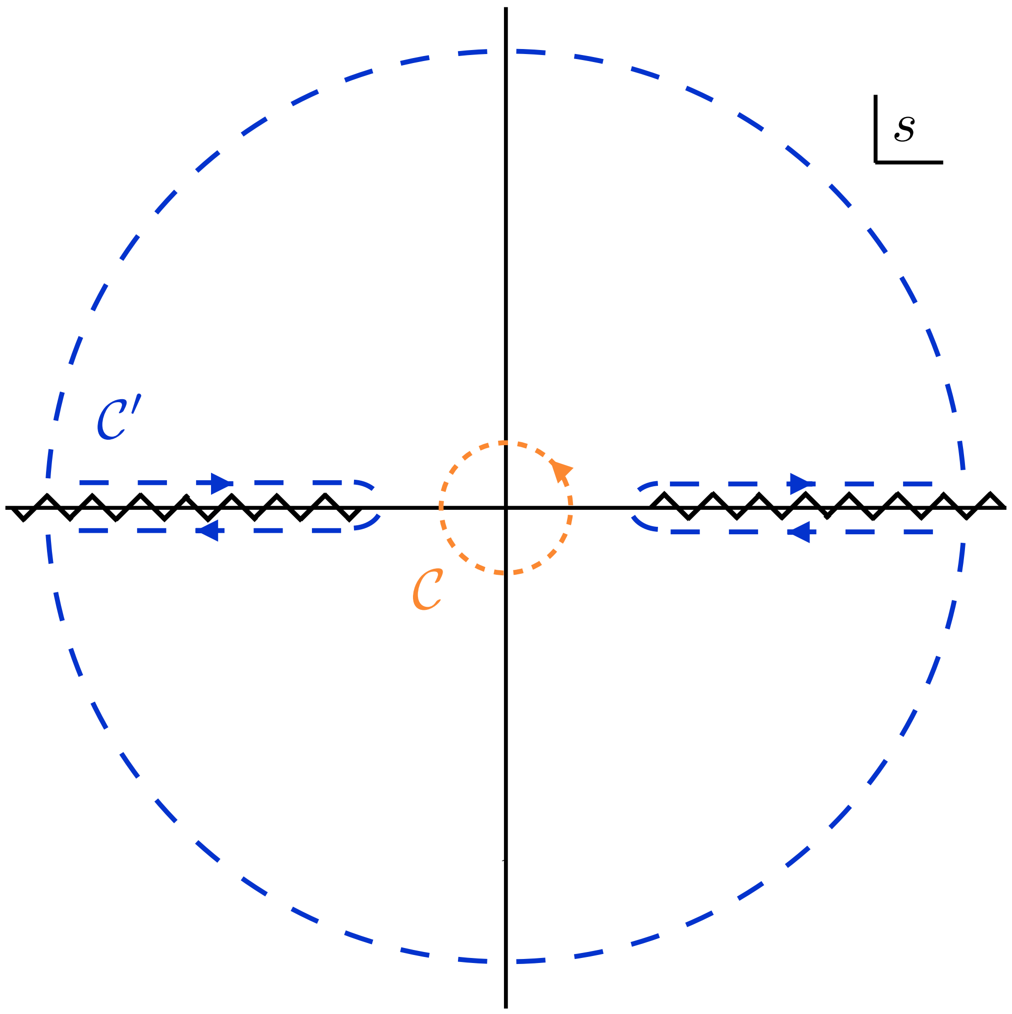

$$\frac{4c}{M^4} = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{\mathrm{d}s}{s^3}\mathcal{M}(s,0)$$
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use analyticity to deform the contour

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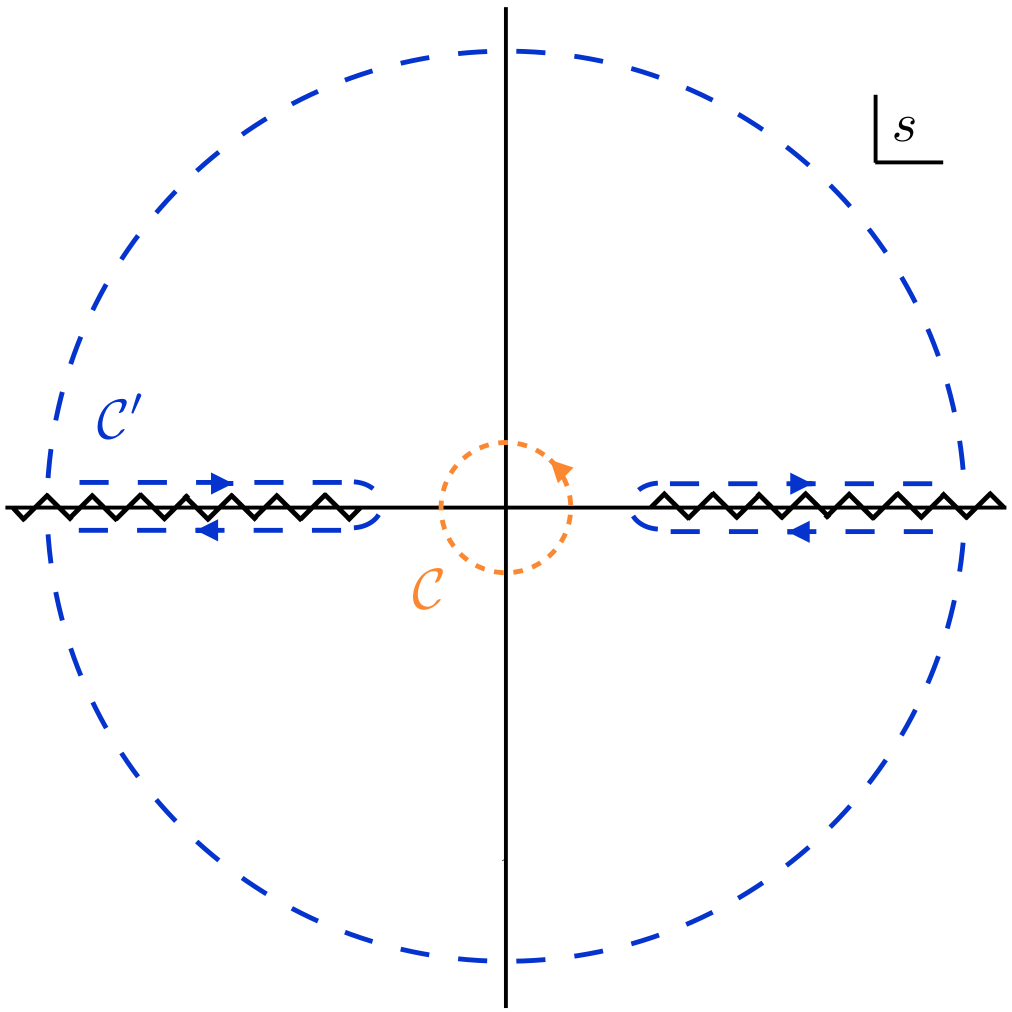

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boundary term at infinity vanishes

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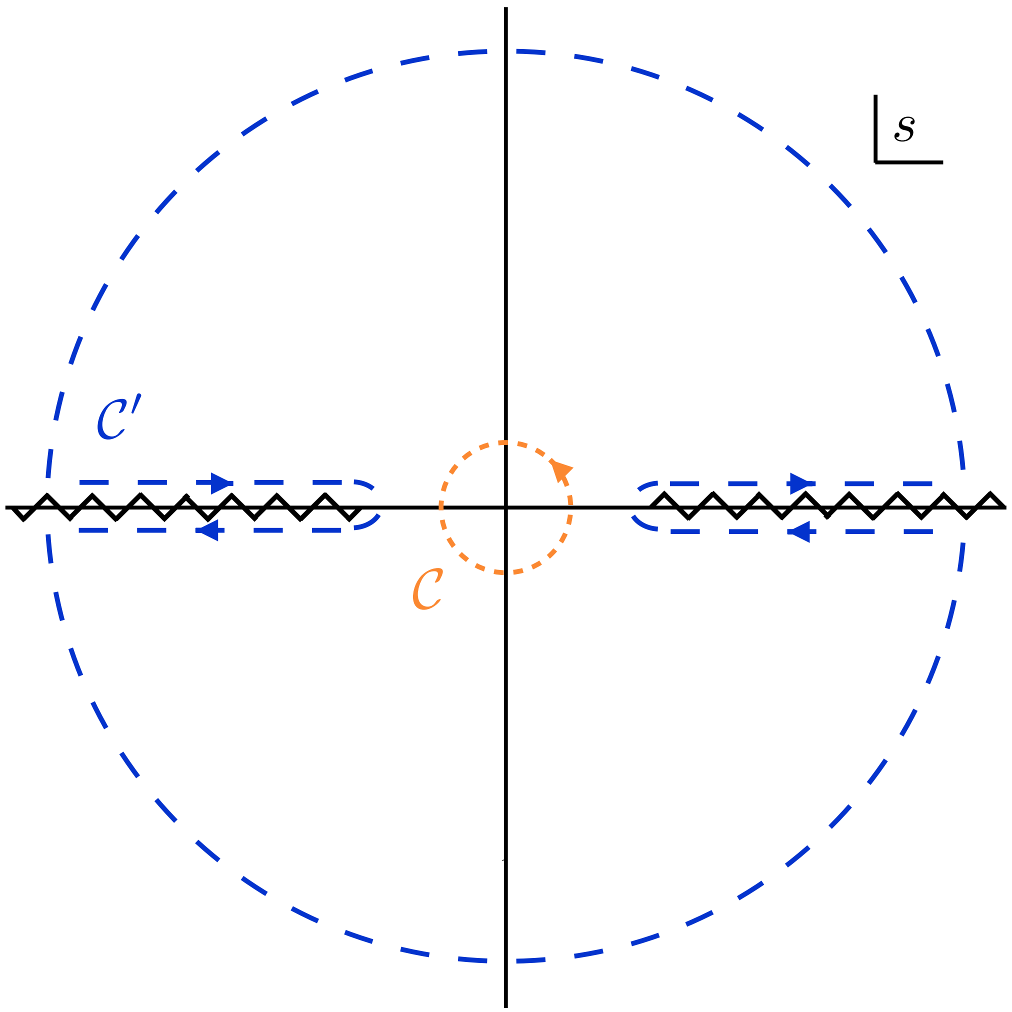

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crossing symmetry: $\mathcal{M}(s,0) = \mathcal{M}(-s,0)$

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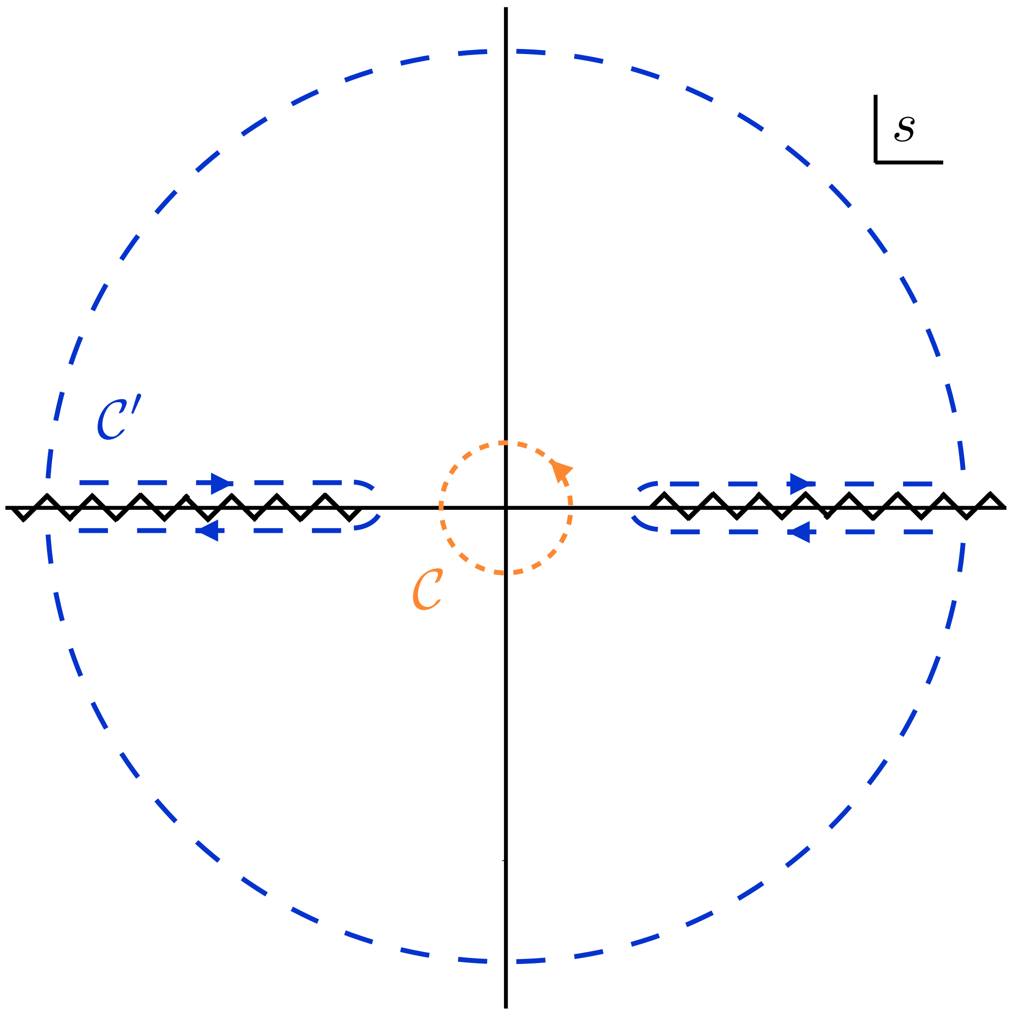

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by definition

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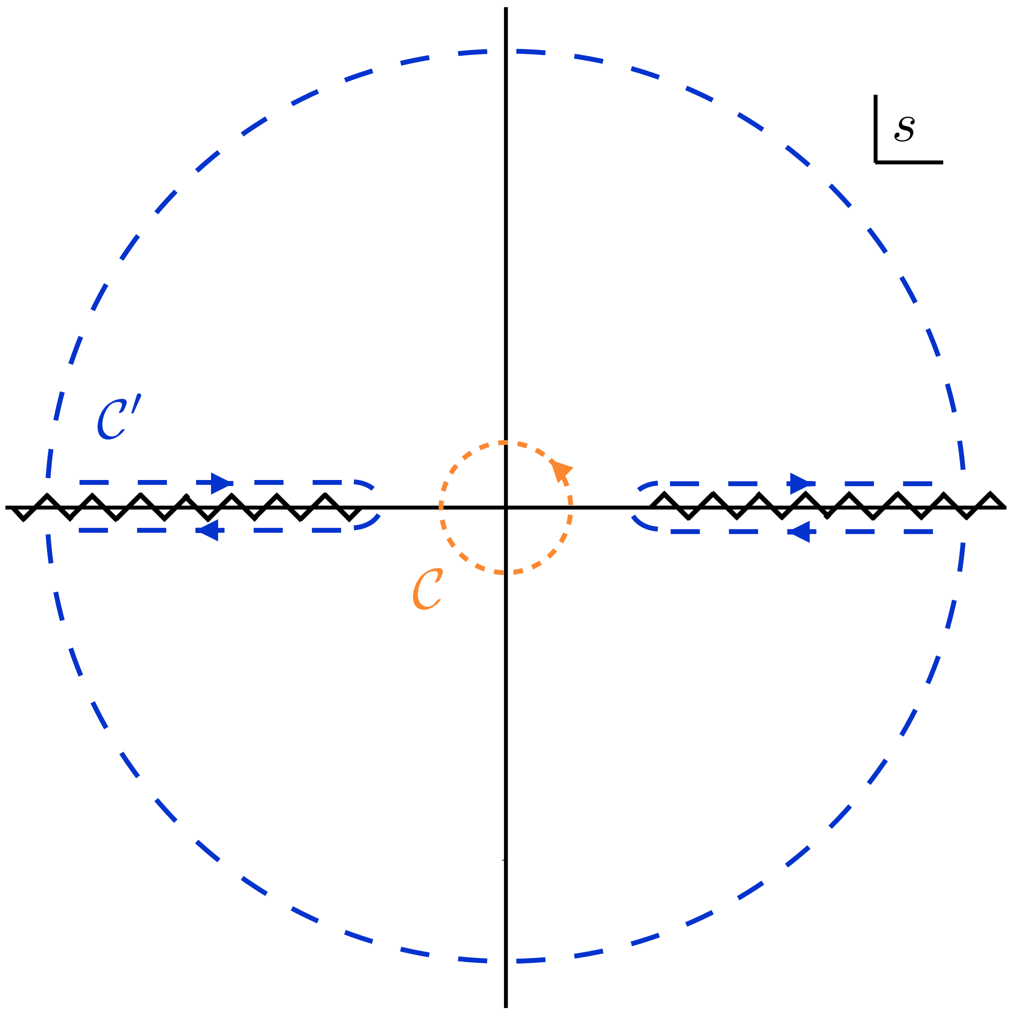

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Schwarz reflection principle:

$$\mathcal{M}(s^*,0) = (\mathcal{M}(s,0))^*$$

Analyticity and unitarity

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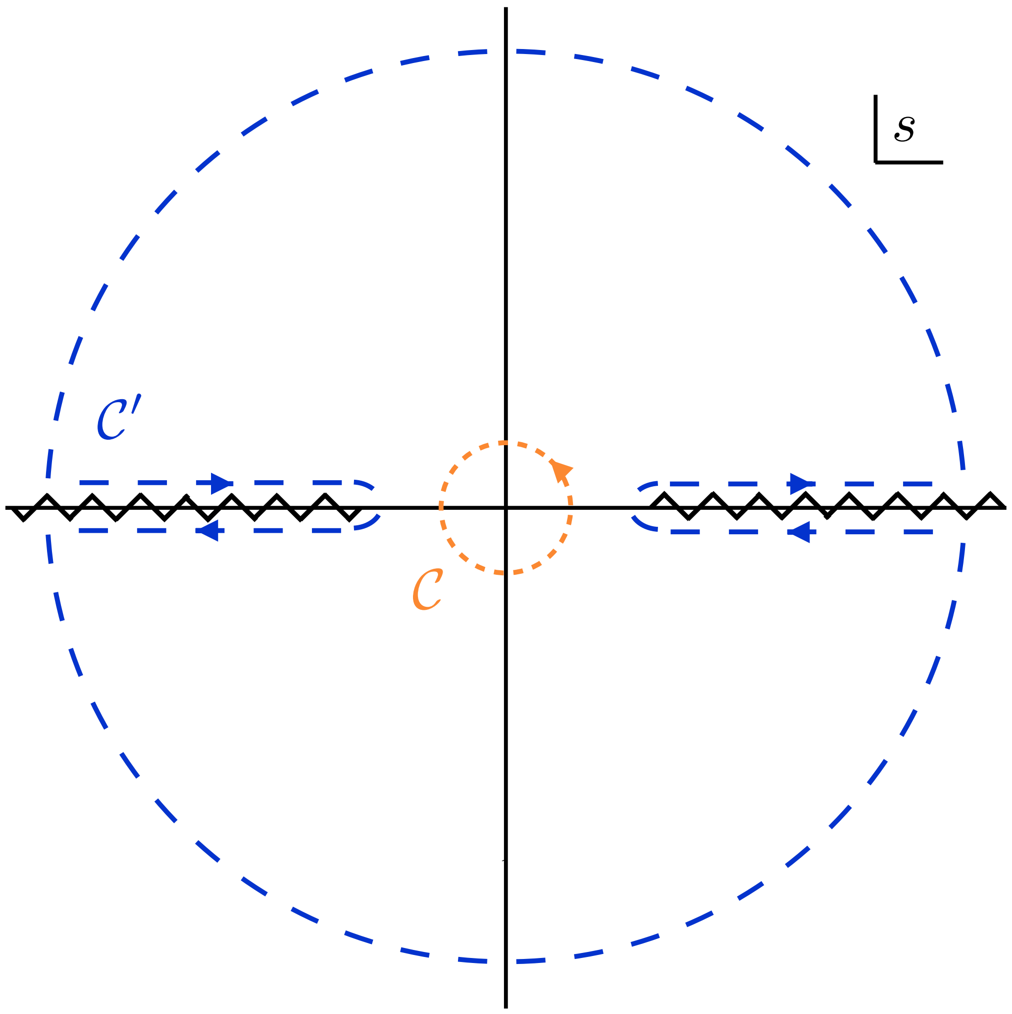

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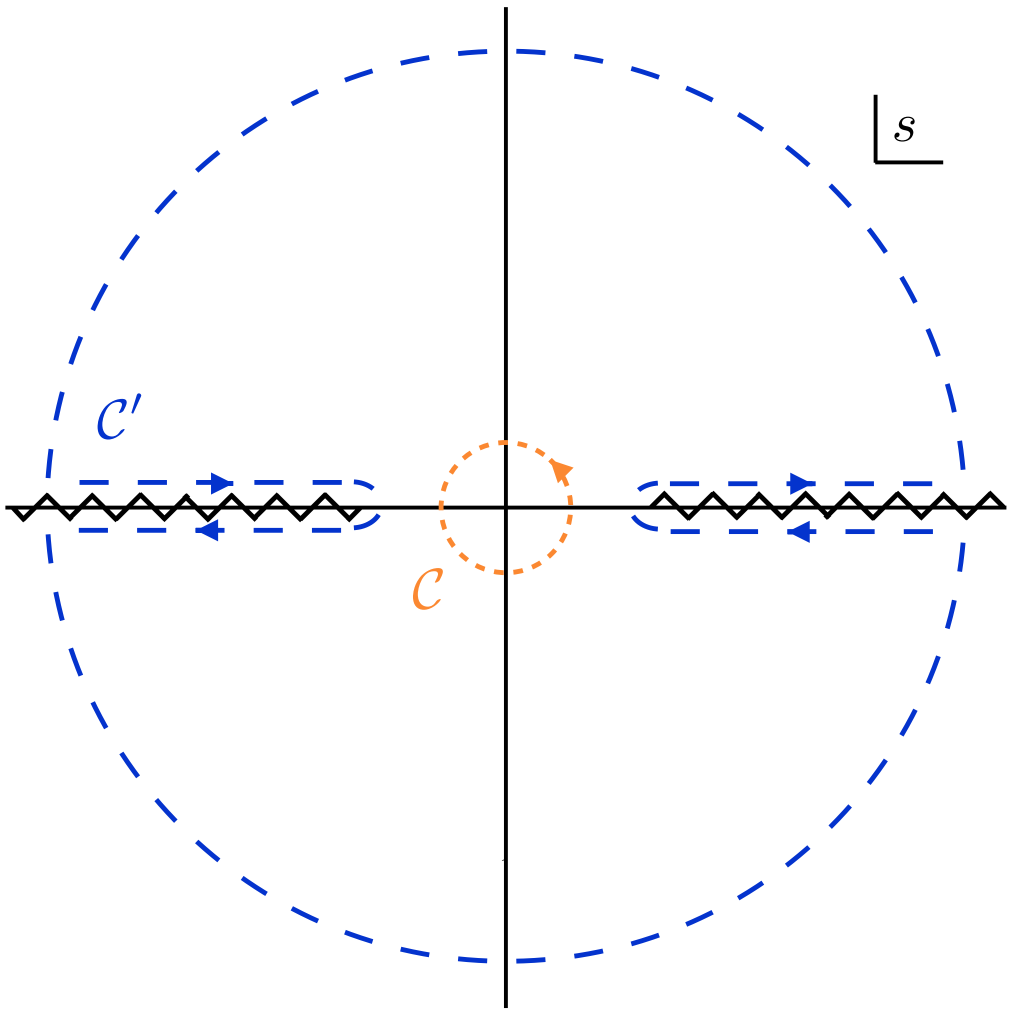
The Wilson coefficient of interest can be extracted via a contour integral of the forward amplitude:



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using the optical theorem (unitarity):

$$\mathrm{Im}\,\mathcal{M}(s,0) = s\,\sigma(s)$$

$$\boxed{\implies c > 0}$$

Adams et al. [hep-th/0602178]

Analyticity and unitarity

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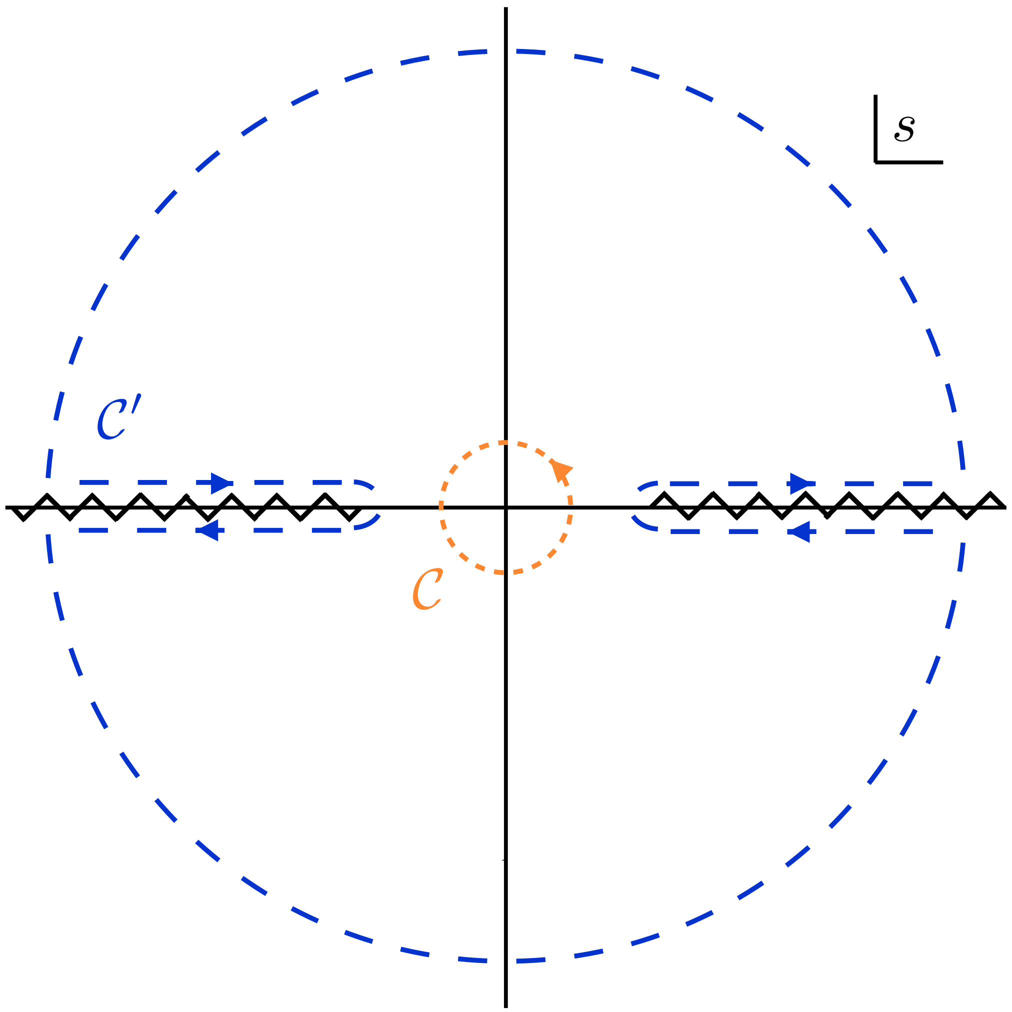

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using the optical theorem (unitarity):

$$\mathrm{Im}\,\mathcal{M}(s,0) = s\,\sigma(s)$$

More generally,

$$\boxed{\lim_{s\to 0}\frac{\mathrm{d}^{2k}}{\mathrm{d}s^{2k}}\mathcal{M}(s,0) > 0}$$

Adams et al. [hep-th/0602178]

Wilson coefficients for the zeta amplitude

- Let's now apply the dispersion relation formalism to our zeta amplitude. Define a power series of the forward amplitude at small momentum:

$$\mathcal{M}(s,0) = \sum_{k=0}^{\infty} c_{2k} s^{2k}$$

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- Boundary term:

$$c_{\infty}^{(2k)} = \frac{1}{2\pi i} \oint_{|s|=\infty} \frac{\mathrm{d}s}{s^{2k+1}} \mathcal{M}(s,0)$$

Nonzero $c_{\infty}^{(2k)}$ would mean that $\Xi(z)$ grows at least as fast as $e^{\alpha z^{4k+2}}$ (i.e., growth order $4k+2$), contradicting known growth order 1. Titchmarsh (1951)

$$\implies c_{\infty}^{(2k)} = 0$$

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- Let's now apply the dispersion relation formalism to our zeta amplitude. Define a power series of the forward amplitude at small momentum:

$$\mathcal{M}(s,0) = \sum_{k=0}^{\infty} c_{2k} s^{2k}$$

- The properties we have proven for $\mathcal{M}(s,t)$ give a beautiful relation between the Wilson coefficients and the nontrivial zeros:

$$\boxed{c_{2k} = \sum_{n=1}^{\infty} \frac{2}{\mu_n^{2(2k+1)}}}$$

e.g.,

$$c_0 = \sum_{n=1}^{\infty} \frac{2}{\mu_n^2}$$

$$c_2 = \sum_{n=1}^{\infty} \frac{2}{\mu_n^6}$$

$$c_4 = \sum_{n=1}^{\infty} \frac{2}{\mu_n^{10}}$$

$$\vdots$$

Riemann hypothesis $\Longrightarrow \; c_{2k} > 0$

Wilson coefficients for the zeta amplitude

For example, the s^2 coefficient gives us the remarkable identity:

$$
\begin{aligned}
c_2 &= \frac{1}{2}\lim_{s\to 0}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{M}(s,0)\\
&= -128 + \frac{1}{7680}\psi^{(5)}\left(\tfrac{1}{4}\right) - \zeta_1^6\left(\tfrac{1}{2}\right)\\
&\quad + 3\zeta_1^4\left(\tfrac{1}{2}\right)\zeta_2\left(\tfrac{1}{2}\right) - \frac{9}{4}\zeta_1^2\left(\tfrac{1}{2}\right)\zeta_2^2\left(\tfrac{1}{2}\right)\\
&\quad + \frac{1}{4}\zeta_2^3\left(\tfrac{1}{2}\right) - \zeta_1^3\left(\tfrac{1}{2}\right)\zeta_3\left(\tfrac{1}{2}\right)\\
&\quad + \zeta_1\left(\tfrac{1}{2}\right)\zeta_2\left(\tfrac{1}{2}\right)\zeta_3\left(\tfrac{1}{2}\right) - \frac{1}{12}\zeta_3^2\left(\tfrac{1}{2}\right)\\
&\quad + \frac{1}{4}\zeta_1^2\left(\tfrac{1}{2}\right)\zeta_4\left(\tfrac{1}{2}\right) - \frac{1}{8}\zeta_2\left(\tfrac{1}{2}\right)\zeta_4\left(\tfrac{1}{2}\right)\\
&\quad - \frac{1}{20}\zeta_1\left(\tfrac{1}{2}\right)\zeta_5\left(\tfrac{1}{2}\right) + \frac{1}{120}\zeta_6\left(\tfrac{1}{2}\right)\\
&= \sum_{n=1}^{\infty}\frac{2}{\mu_n^6}
\end{aligned}
$$

using the shorthand $\zeta_n(z) = \zeta^{(n)}(z)/\zeta(z)$

$$\zeta_n^k(z) = [\zeta_n(z)]^k$$

Wilson coefficients for the zeta amplitude

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Can prove (with great effort!) by computing analytic expressions for derivatives of $\zeta(z)$ at $z = \frac{1}{2}$ using polygamma identities and the product form of the zeta function,

$$\zeta(z) = \frac{1}{2(z-1)} (\pi e^{\gamma})^{z/2} \prod_{k=1}^{\infty} \left(1 + \frac{z}{2k}\right) e^{-z/2k} \prod_{z_n \text{ nontrivial zeros}} \left(1 - \frac{z}{z_n}\right)$$

which comes from the Hadamard expansion of the xi function,

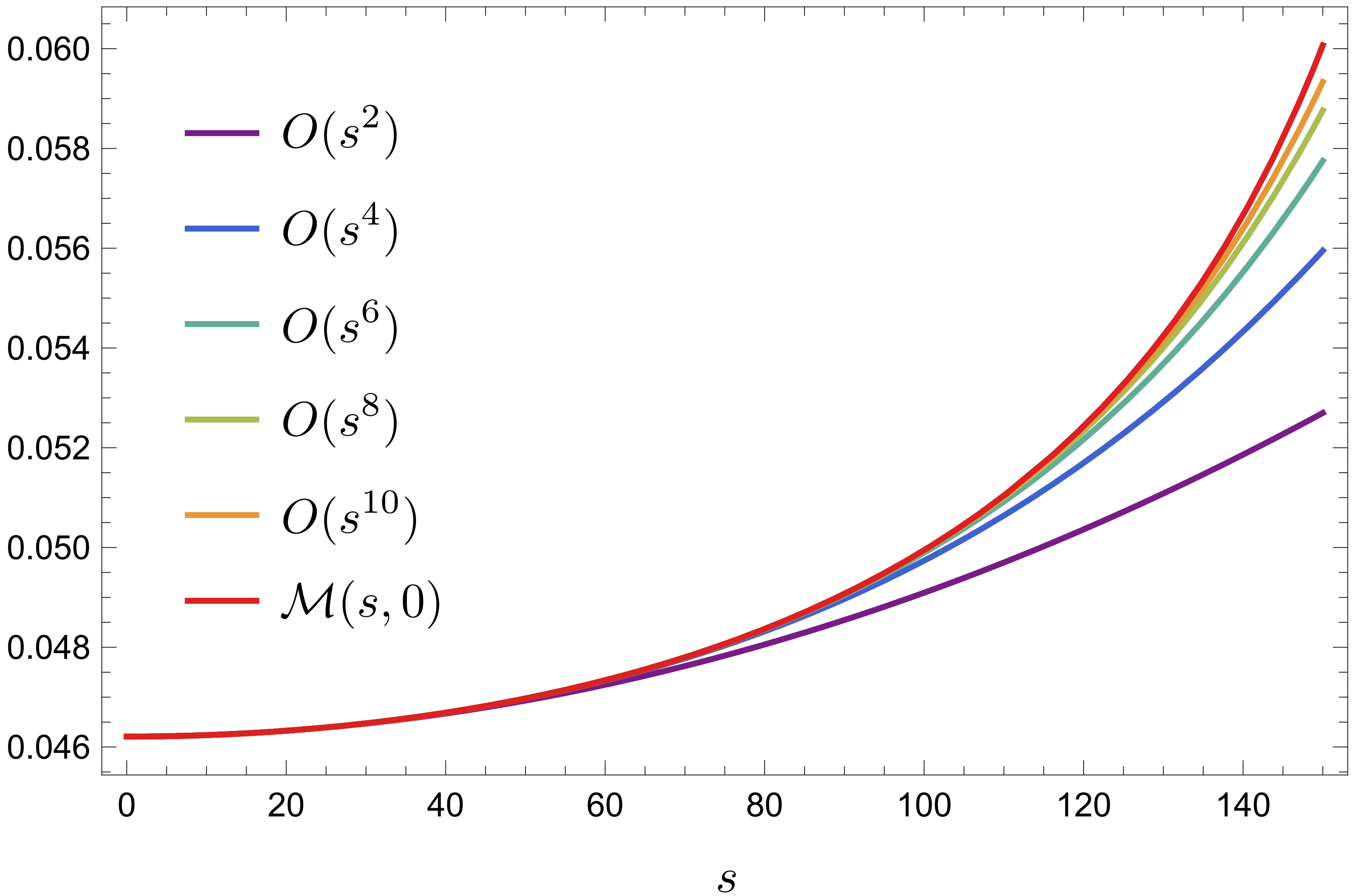
$$\xi(z) = \xi(0) \prod_{z_n \text{ nontrivial zeros}} \left(1 - \frac{z}{z_n}\right)$$

What is remarkable is that our amplitude construction allows for much simpler, physical derivations of such identities!

using the shorthand $\zeta_n(z) = \zeta^{(n)}(z)/\zeta(z)$

$$\zeta_n^k(z) = [\zeta_n(z)]^k$$

Wilson coefficients for the zeta amplitude



$$c_0 \simeq 4.6210 \times 10^{-2}$$
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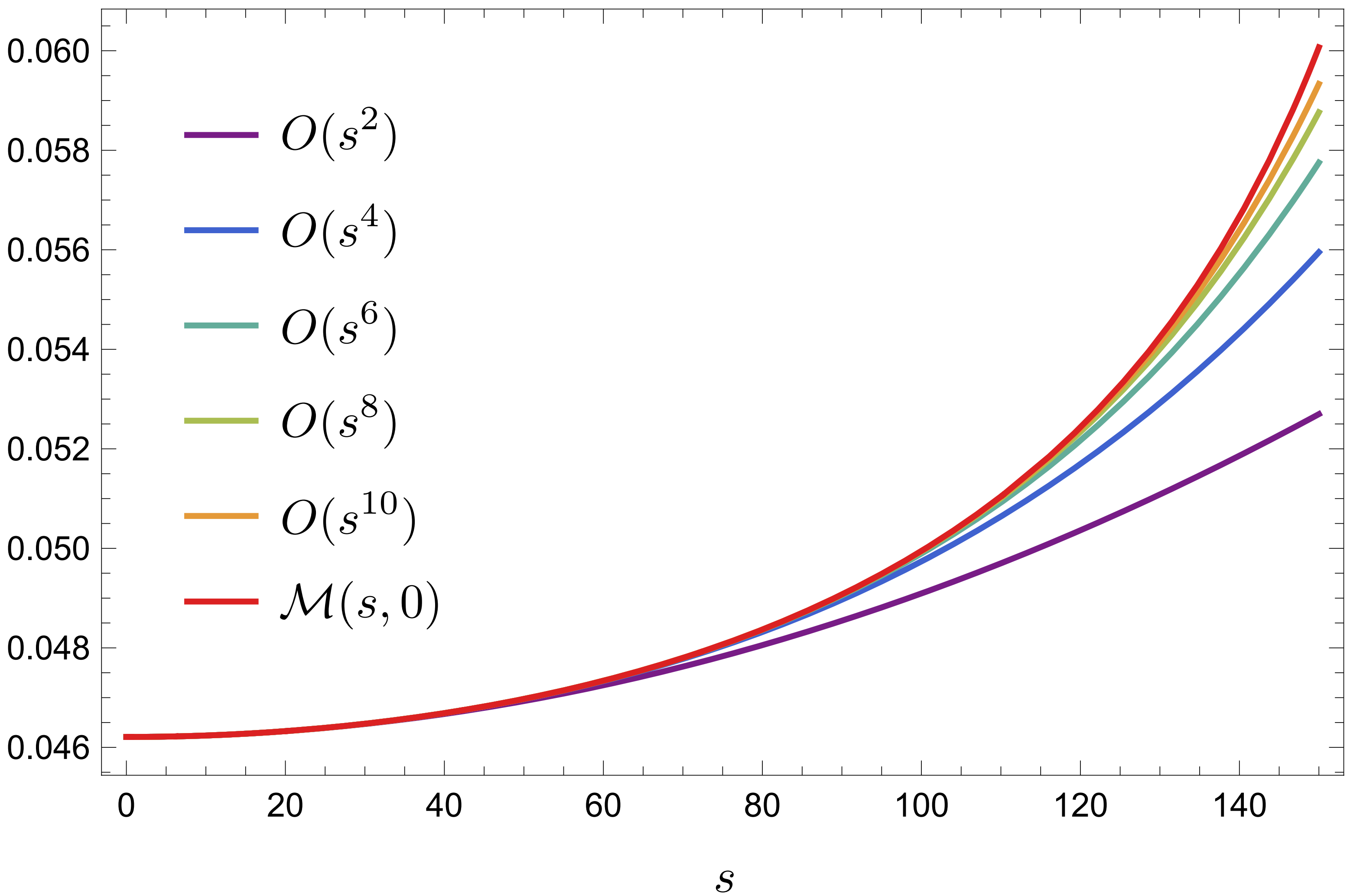

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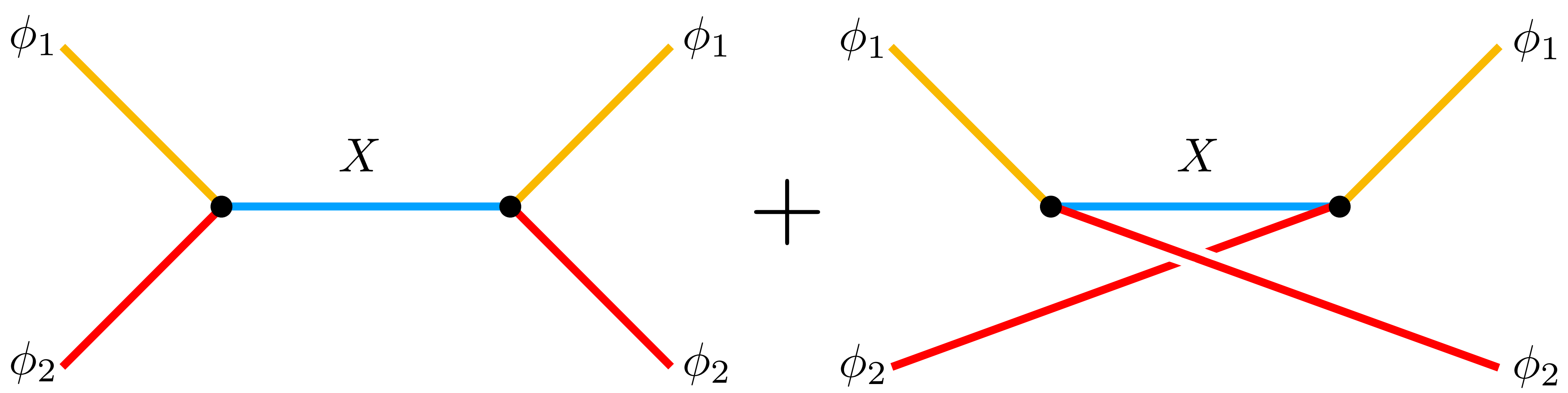
Numerical tests of $c_{4,6,8,10}$ confirm prediction to within relative error of 10^{-30}.

Other properties

What sort of theory is this?

- Our amplitude $\mathcal{M}(s,t)$ describes a theory of two types of massless scalars, ϕ_1 and ϕ_2, exchanging a tower of massive states X in the s and u channels for the process:

$$\phi_1\phi_2 \longrightarrow \phi_1\phi_2$$



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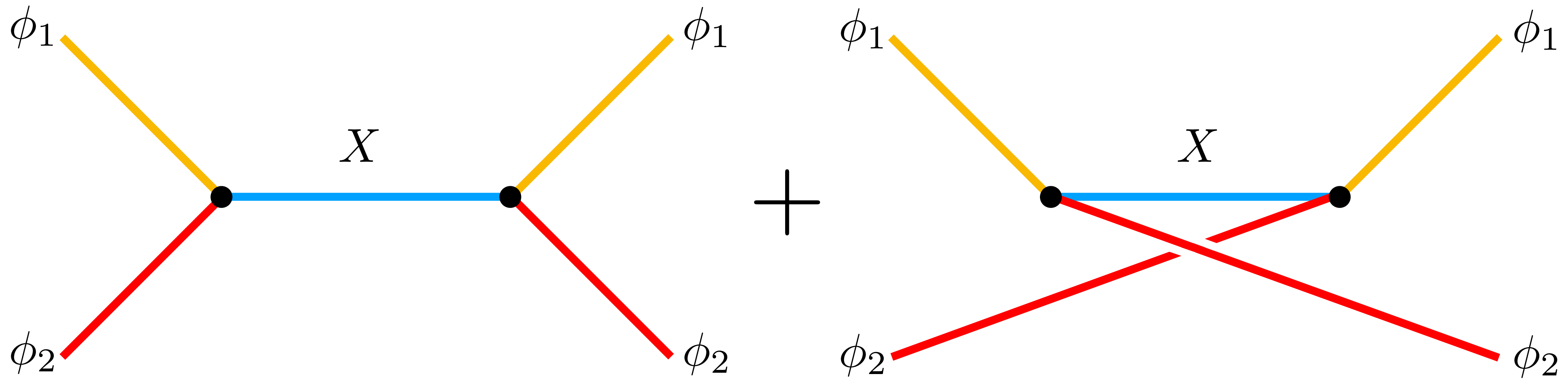
- We alternatively could have defined $\mathcal{M}(s,t)$ as $\mathcal{A}(s)+\mathcal{A}(t)+\mathcal{A}(u)$ to have full Bose symmetry, in which case our amplitude would describe single-scalar scattering $\phi\phi \to \phi\phi$



What sort of theory is this?

- Our amplitude $\mathcal{M}(s,t)$ describes a theory of two types of massless scalars, ϕ_1 and ϕ_2, exchanging a tower of massive states X in the s and u channels for the process:

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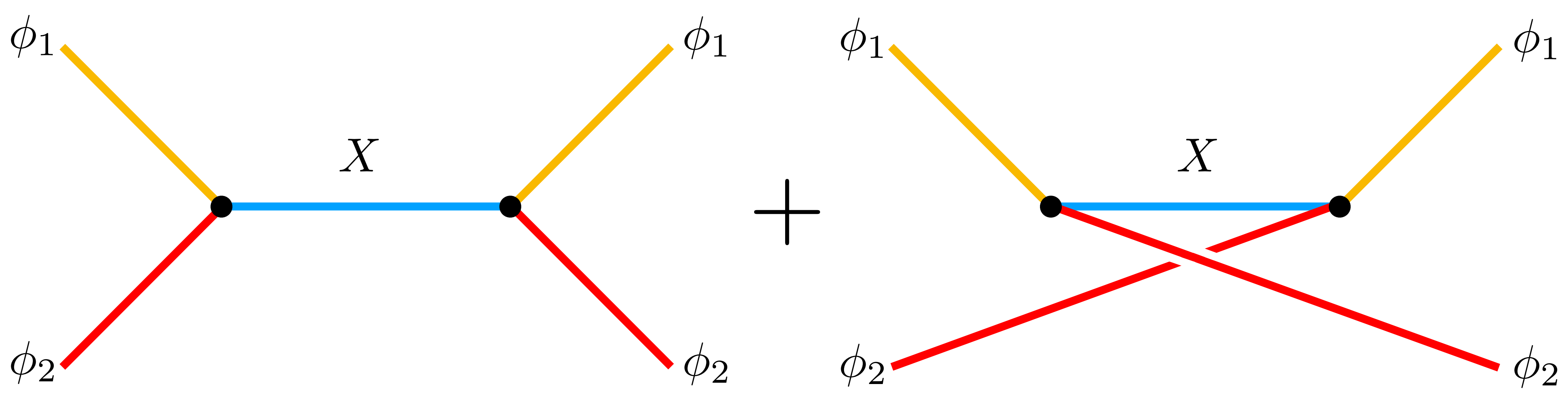
- The spectrum of X is given by $m_n^2 = \mu_n^2$



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- Based on the properties of $\mathcal{A}(s)$, our Riemann zeta amplitude is on-shell constructible from the UV amplitudes $\phi_1\phi_2 \to X$:

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- Proof:

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  By our evaluation of c_0, $\Delta(0) = 0 \quad \implies \quad \Delta(s)$ vanishes everywhere.

Hadamard product

- Integrating our result

$$\mathcal{A}(s) = -\frac{\mathrm{d}\ \log \Xi(\sqrt{s})}{\mathrm{d}s} = \sum_n \frac{1}{-s + \mu_n^2}$$

gives the product form for the Riemann-Landau xi function:

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- Using the Weierstrass product $\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$ along with $\zeta(0) = -\frac{1}{2}$, we have…

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On-shell constructibility of amplitude $\longleftrightarrow$ Hadamard product of zeta function

Reflection symmetry and CPT

- Functional equation $\zeta(z) = 2^z \pi^{z-1} \sin(\pi z/2)\Gamma(1-z)\zeta(1-z)$
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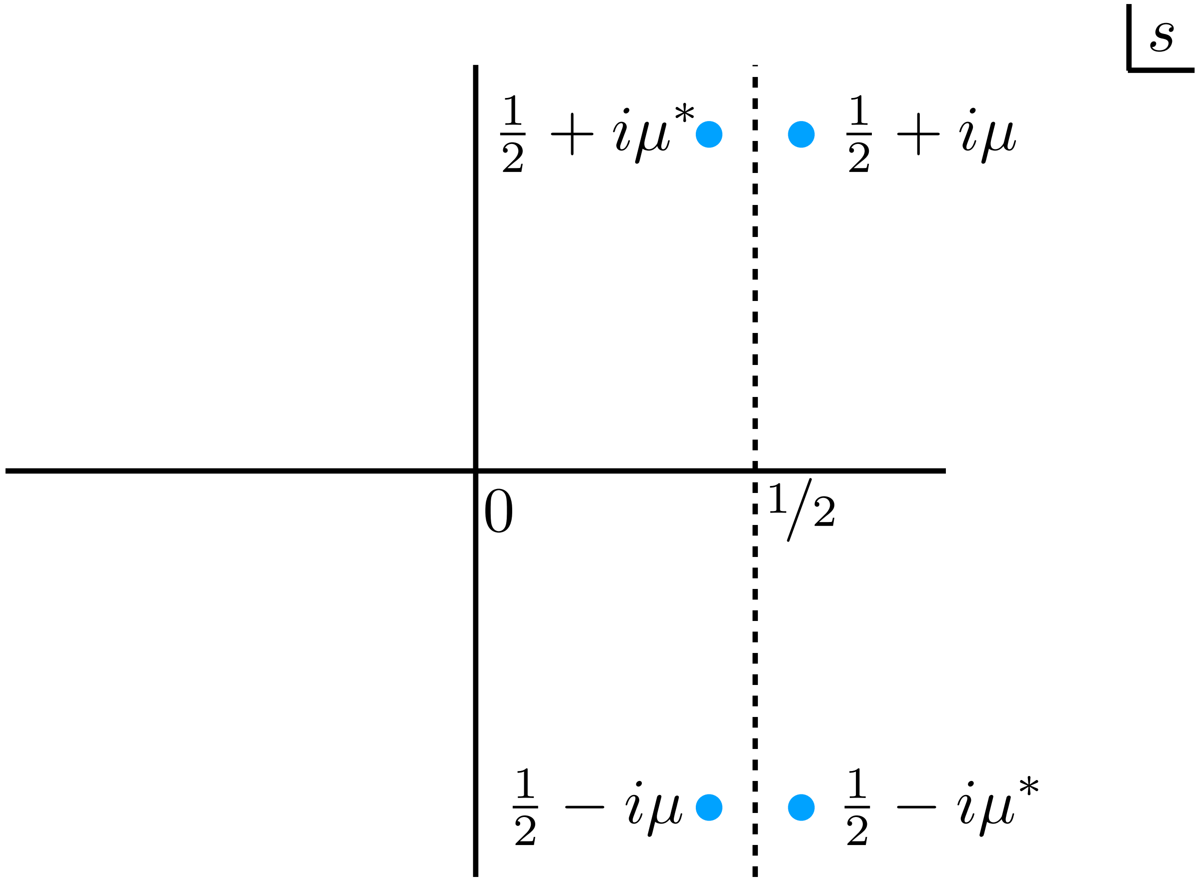
  together imply that the nontrivial zeros enjoy a four-fold symmetry:



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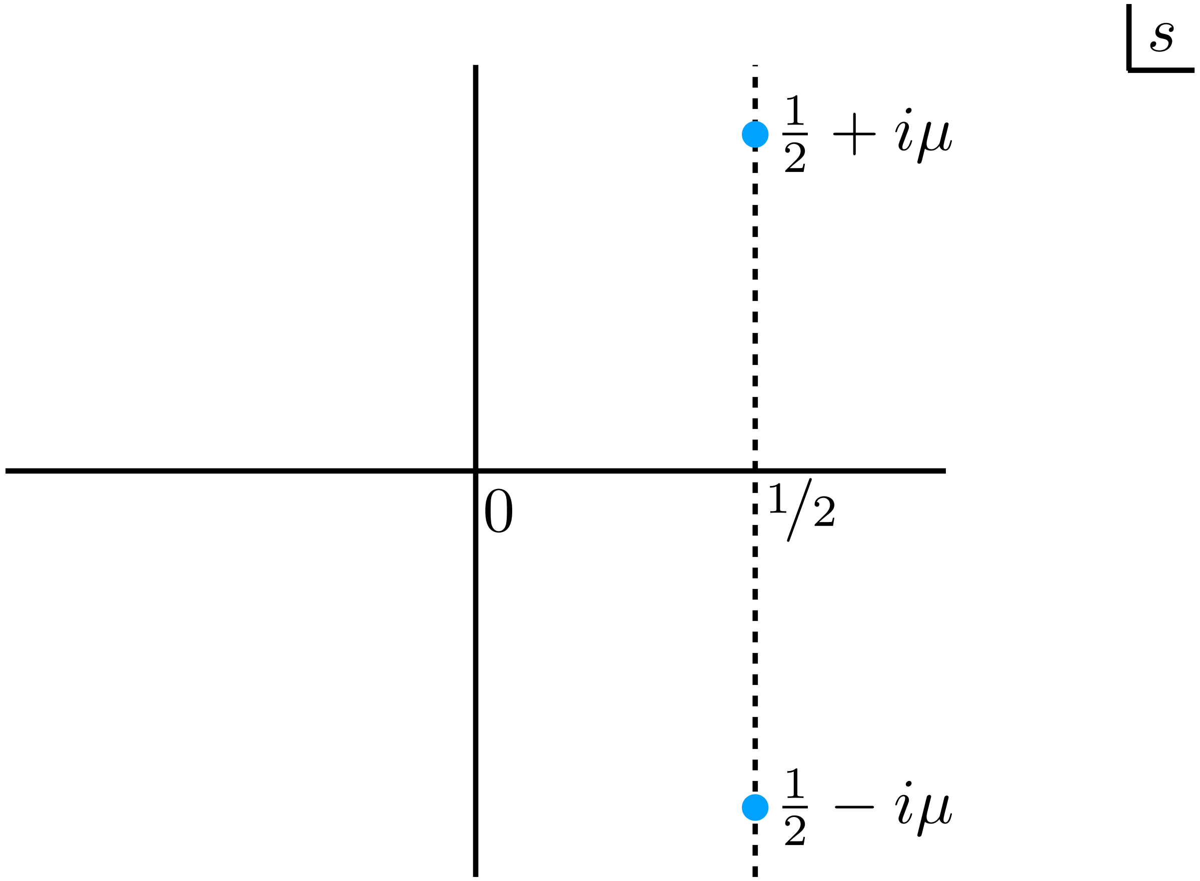

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- Zero-counting: $N(T) = \Big| \{ z | \zeta(z) = 0 \;\&\; 0 < \mathrm{Im}(z) \leq T \} \Big|$

$$= \frac{1}{\pi} \int_0^{\sqrt{T}} \sigma(s)\, \mathrm{d}s$$

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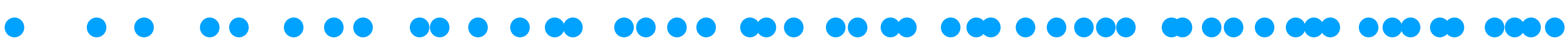
We have constructed an amplitude whose physical attributes correspond to the known or conjectured properties of the nontrivial zeros of zeta.



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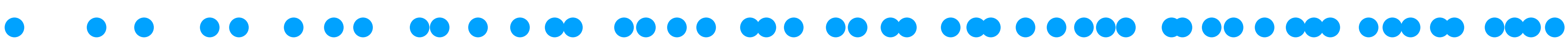

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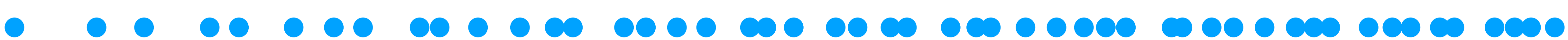
- **Other future directions:**

    - Different couplings or spin for massive states?

    - Zeta function universality

    - Dirichlet *L*-functions




Questions