Geometry and topology: Applications to cosmological datasets

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Outline

- Background Geometry and Topology
  - ➢Minkowski functionals (Euler characteristic/genus)
  - ➢Homology
  - ➢Hierarchical homology (persistent homology)

- Topological characteristics of CMB fluctuations
  - ➢Temperature
    - ➢Full sky
    - ➢Hemispheres
  - ➢Polarization (full sky)

- Conclusion

- (Structure identification through topology)

Minkowski Functionals

• Predominantly Geometric quantities : include the notion of Volume, surface area, contour length of a manifold M.

• (D+1) quantifiers for D-dimensional sets

• Go by various names and orderings: quermassintegrals, Dehn and Steiner functionals, curvature integrals, intrinsic volumes, Minkowski functionals, and Lipschitz-Killing curvatures.

•We need only MFs and LKCs for our purpose, which when properly defined are related by

$$Q_j(M) = j!\,\omega_j \mathcal{L}_{D-j}(M), \qquad j = 0, \dots, D,$$

$$\omega_k = \pi^{k/2}\,\Gamma\left(\frac{1+k}{2}\right) : \text{volume of k} - \text{dim. unit ball}$$

Pranav et al, MNRAS, 485 (3), 4167-4208

Minkowski Functionals

- A useful way of defining these quantities is via the *Steiner formula* or *Weyl's tube formula :*

$$V_D(\{x \in R^D : \min \| x - y \|\} \leq \rho, y \in M) = \sum_{j=0}^{D} \frac{\rho^j}{j!} Q_j(M) = \sum_{j=0}^{D} \omega_{D-j} \rho^{D-j} \mathcal{L}_j(M)$$

- The set in the LHS is known as the "tube around M", and ρ is small.
- Trivial to check Q_0 and $\mathcal{L}_D$ measure D-dimensional volume (set $\rho = 0$ above).
- $Q_1$and $2\mathcal{L}_2$ measure surface area.
- Other functionals are harder to define, but always a true and deep result that:

$$\chi(M) = \mathcal{L}_0(M) = \frac{1}{D!\, \omega_D} Q_D(M).$$

- In 3D, this only leaves Q_2 and $\mathcal{L}_1$. If the manifold M is convex, $\mathcal{L}_1(M) = Q_2(M)/2\pi$ is twice the *caliper diameter* of M.

Pranav et al, MNRAS, 485 (3), 4167-4208, 2019

Geometry and Topology

- Theorema Egrerium (latin for remarkable theorem) of Gauss states that the Gaussian curvature of a surface is an *intrinsic invariant*, meaning it is a constant irrespective of how the surface is bent (or twisted) in space

- Leads to the Gauss-Bonnet theorem

$$\int_M K\, dA + \int_{\partial M} k_g\, ds = 2\pi\chi(M)$$

- K : Gaussian curvature of M, k_g: geodesic curvature of the boundary of M

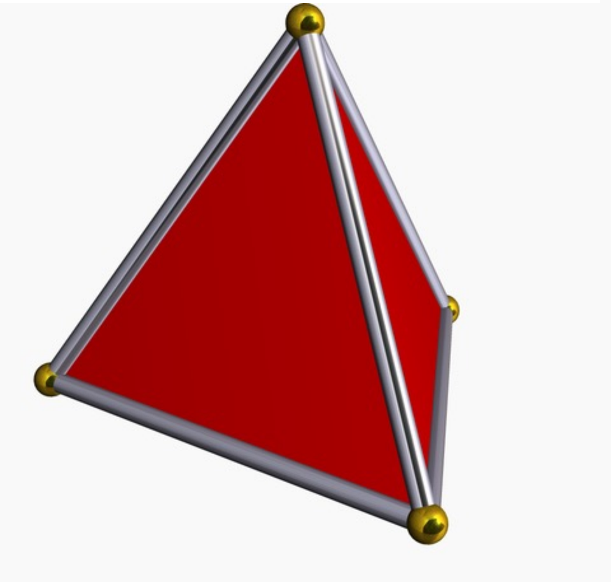
- The theorem is remarkable because it links and proves that a topological invariant (EC) can be computed purely from geometrical properties.

Pranav et al, MNRAS, 485 (3), 4167-4208, 2019

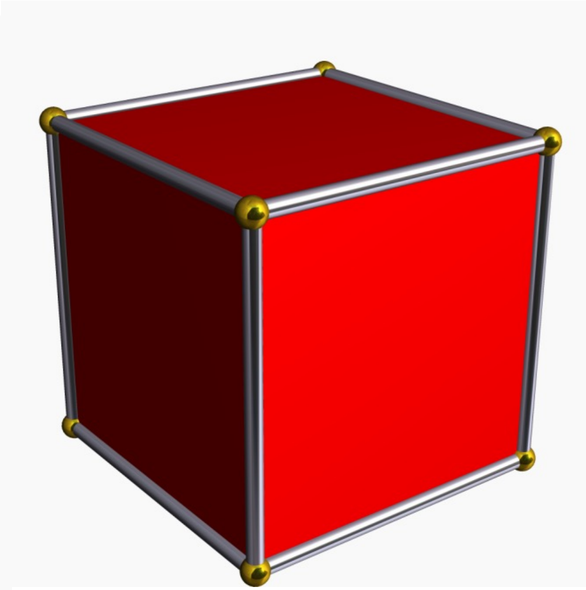
Euler characteristic

- Originally defined for polyhedra

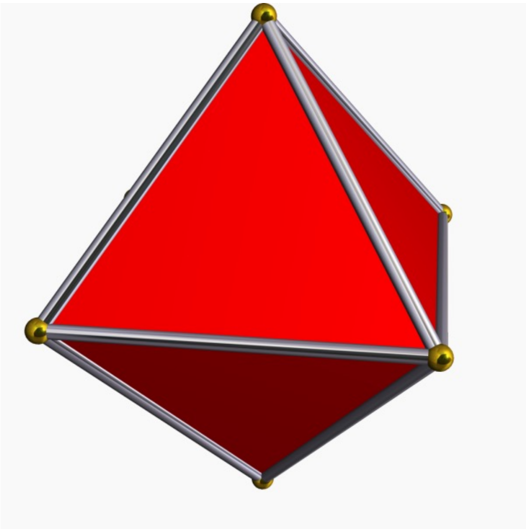
$$\chi = V - E + F$$



Tetrahedron
V = 4, E = 6, F = 4



Cube
V = 8, E = 12, F = 6



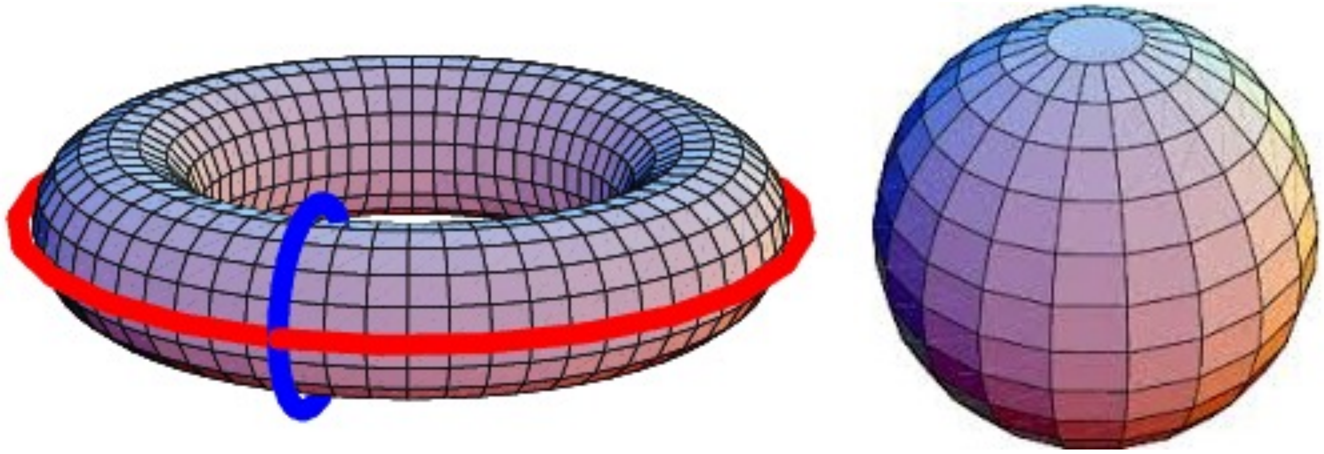
octahedron
V = 6, E = 12, F = 8

- Modern definition through algebraic topology, specifically *Homology*

Genus

- **For a connected, orientable surface, the Genus has a linear relationship with the maximal number of independent simple closed curves that can be drawn on the surface without rendering it disconnected**

- **Number of handles attached to a surface**



Why the Euler characteristic?

SIMPLICIAL TOPOLOGY

Simplices, complexes,
cycles, numbers of simplices,
Betti numbers

$$\sum_k (-1)^k \#\{k\text{-dimensional simplices}\}$$

INTEGRAL GEOMETRY

Convexity, convex ring
kinematic formulae
Minkowski functionals

$$\mathcal{M}_k(M) = c_{dk} \int_{\mathrm{Graff}(d,d-k)} \chi(M \cap V)\, d\mu^d_{d-k}(V)$$

$$\chi$$

$$\sum_k (-1)^k \beta_k$$

ALGEBRAIC TOPOLOGY

Homology, homotopy,
dimensions of groups,
Betti numbers, persistence

$$\sum_k (-1)^k \#\{\text{critical points of index } k\}$$

$$\int_M \mathrm{Tr}(R^{m/2}) \mathrm{Vol}_g$$

DIFFERENTIAL TOPOLOGY

Curvature, forms, Betti numbers,
Morse theory, integration,
Lipschitz-Killing curvatures

Beyond the Genus, EC and MFs

- **The genus is defined only for connected and closed 2-dimensional surfaces, and has no generalizations in higher or lower dimensions.**

- **The Euler-Poincare formula states that the Euler characteristic is an alternating sum of another topological invariant called the *Betti numbers***

- **A formalism capable of expressing topology in a hierarchical fashion would present an interesting and powerful extension for describing the hierarchical structures in the cosmos.**

Genus, Euler & Betti

- Euler – Poincare formula

  Relationship between Betti Numbers & Euler Characteristic χ:

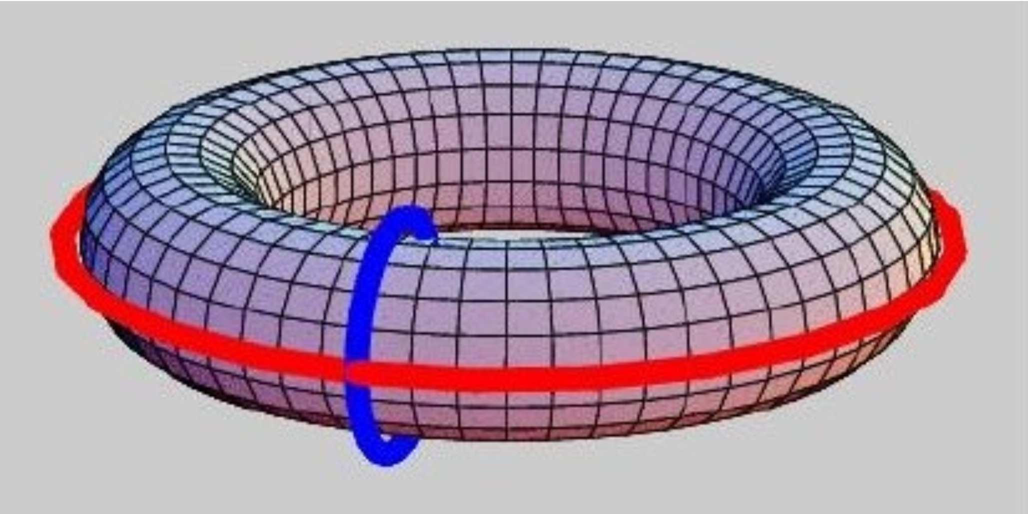
$$\chi = \sum_{k=0}^{d} (-1)^k \beta_k$$

Pranav et al, MNRAS, 465 (4), 4281-4310, 2017

Homology

Topology:
Study of connectivity and spatial relations that remain invariant under homeomorphisms (= continuous mapping between two topological objects)

Homology:
- Description of topology of a space in terms of cycles/boundaries.
- Fundamental lemma : Boundary of a boundary is necessarily empty.



Torus:	one 0-cycle:	rank group H_0:	1
	two 1-cycles:	rank group H_1:	2
	one 2-cycle	rank group H_2:	1

p-chain: sum of p-simplices
p-cycle: boundary of (p+1) chain
0-cycle: closed component
1-cycle: closed loop of edges, or finite union
2-cycle: closed surface, or finite union

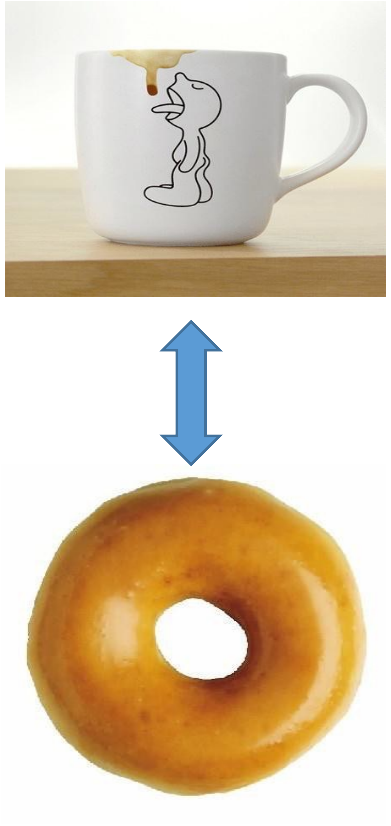
Mutually homologous p-cycles → p-class

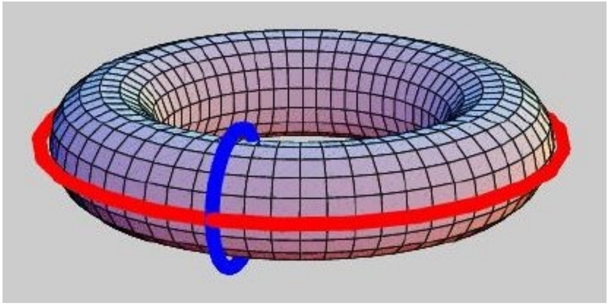
collection of independent p-classes :
Homology group H_p

Betti numbers denote the rank of the groups

Topological cycles and holes

- intuitive interpretation





0 dimensional holes : gaps between connected objects

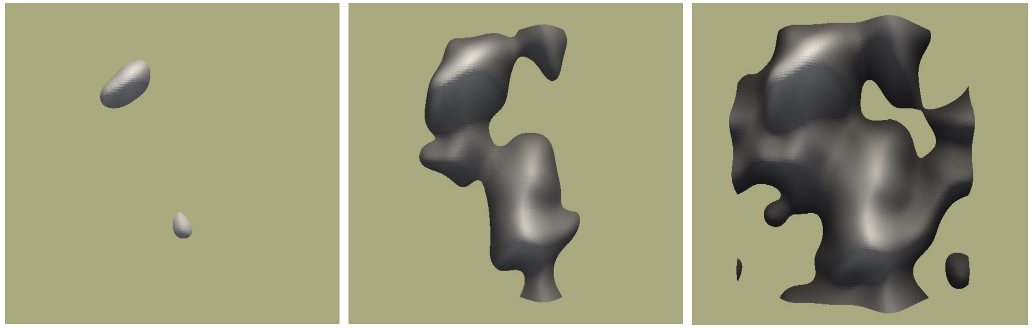
1 dimensional holes : loops/tunnels

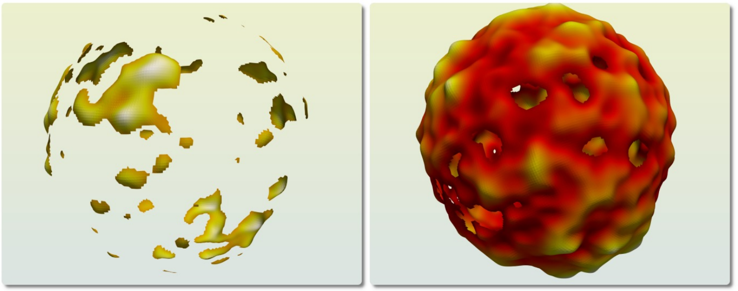
2 dimensional holes : voids

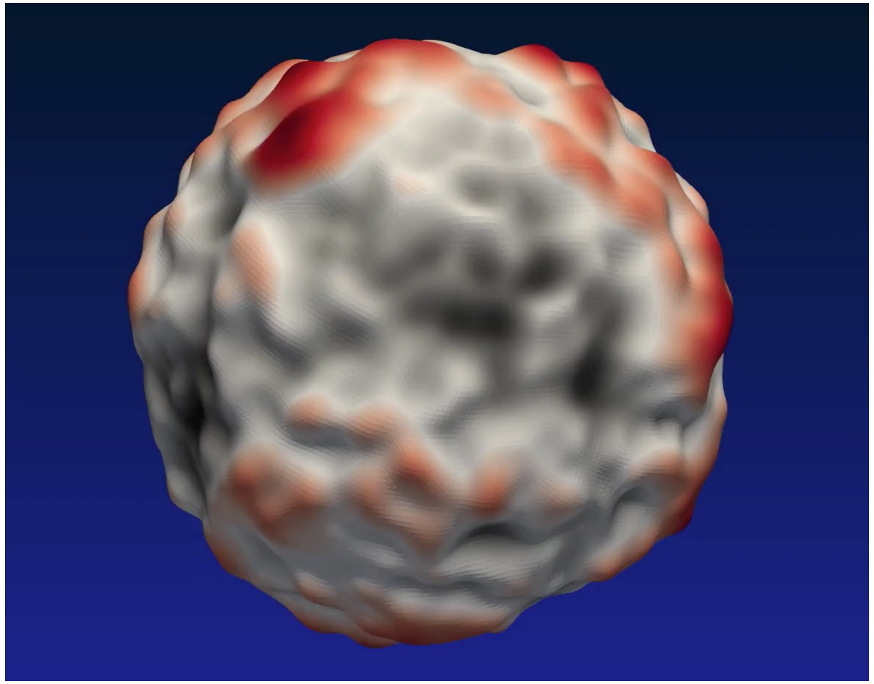
Critical points and filtration

birth and death of topological cycles

- Study the change in topology of a manifold w.r.t. the growing excursion sets of the function f
- *Topology only changes at critical points of the function*
- Addition of a critical point with index k, either creates a k-dimensional hole, or it destroys a (k-1)-dimensional hole







$$\emptyset = M_0 \subseteq M_1 \subseteq \cdots \subseteq M$$

Filtrations

- Let K be a simplicial complex,
  $f : K \to \mathbb{R}$ a monotonic function, i.e. $f(\sigma) \leq f(\tau)$ when σ is a face of τ,
  Let $a_1 < \ldots < a_n$ be the values of f on the simplices of K.
- We get $n+1$ different subcomplexes $K_i = f^{-1}(-\infty, a_i]$ of K which give a filtration of f:

$$\emptyset = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_n = K.$$

- For $i \leq j$, the inclusion map $K_i \to K_j$ induces the homomorphism $f_p^{i,j} : H_p(K_i) \to H_p(K_j)$ for each dimension p.
- Sequence of homology groups, for each dimension p:

$$0 = H_p(K_0) \to H_p(K_1) \to \ldots \to H_p(K_n) = H_p(K)$$

p-th persistent homology groups

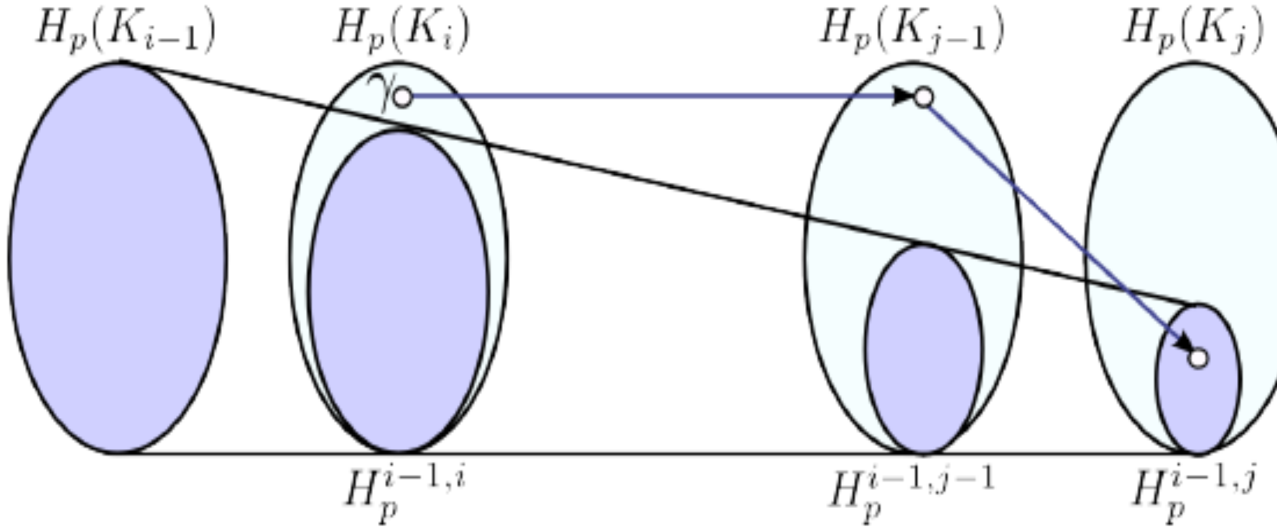
The p-th persistent homology groups are

$$H_p^{i,j} = \mathrm{im}(f_p^{i,j}) \qquad \text{for } 0 \leq i \leq j \leq n.$$

$H_p^{i,j}$ consists of the homology classes of K_i that are still alive at K_j:

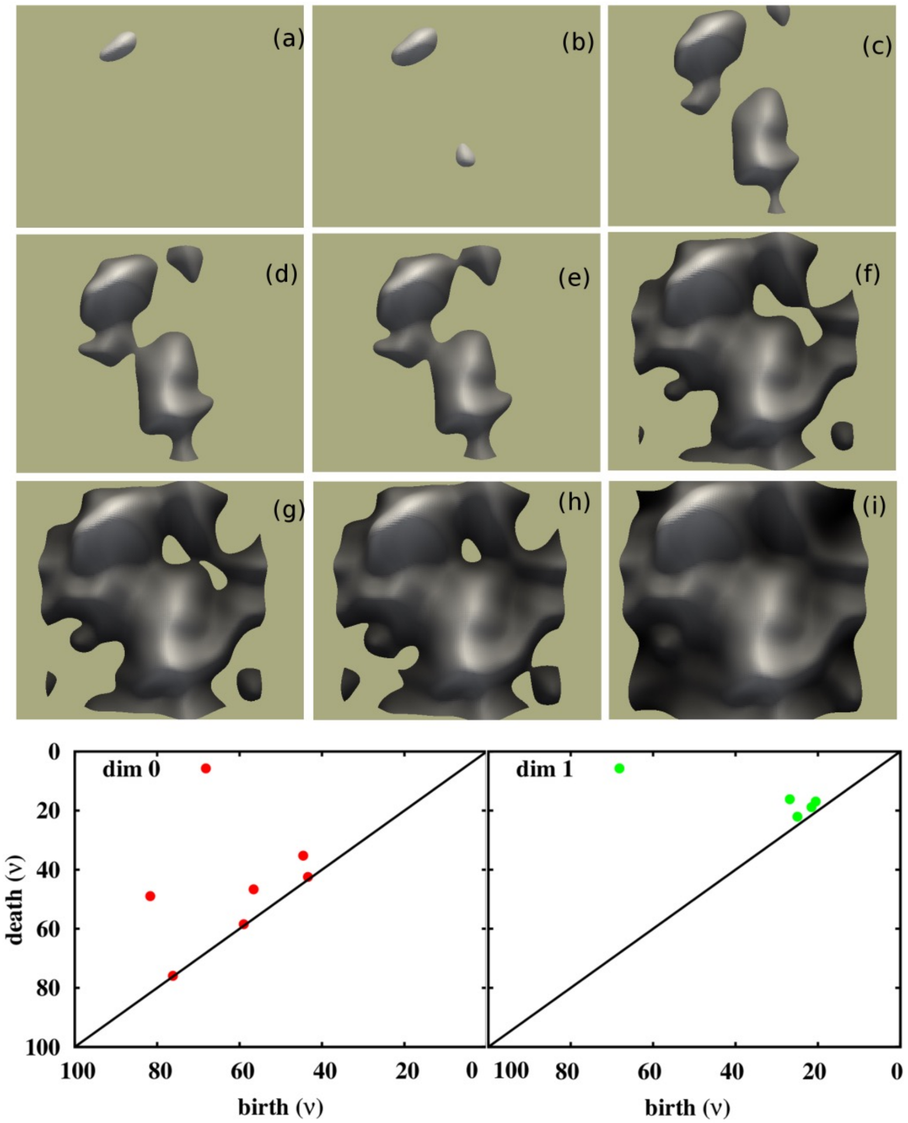
$$H_p^{i,j} = Z_p(K_i)/(B_p(K_j) \cap Z_p(K_i)).$$

The p-th persistent Betti numbers are $\beta_p^{i,j} = \text{rank } H_p^{i,j}$.



Persistence: $\text{pers}(\gamma) = a_j - a_i$.

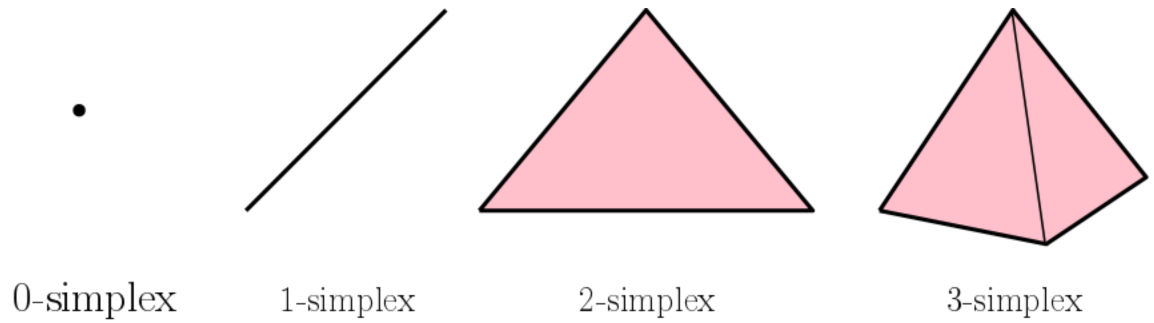
Birth, death and life-time(persistence): hierarchical topology

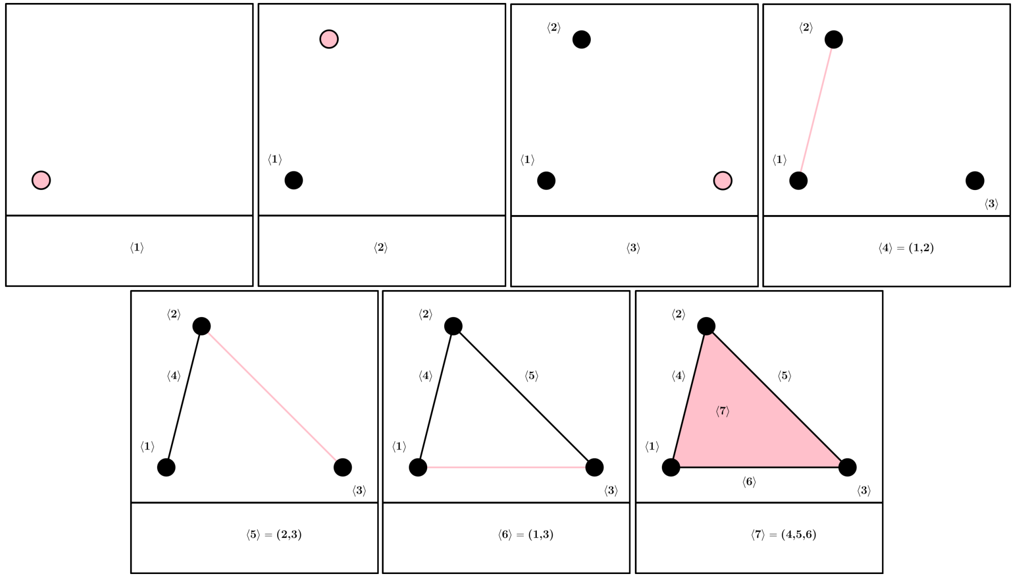


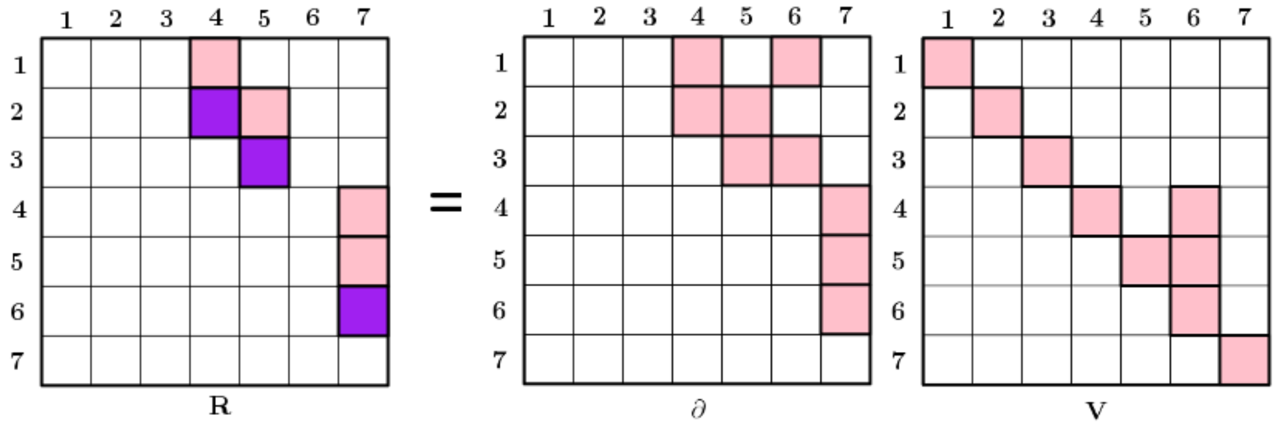
- **Representation of multi-scale topology**
- **Dots in the diagram record birth and death**
- **A diagram for each ambient dimension of the manifold**
  - **0-dimensional diagram: representation of merger of isolated objects (merger trees)**
  - **1-dimensional diagrams: formation and filling up of loops (percolation)**
  - **2-dimensional diagrams: formation and destruction of topological voids (voids)**

Pranav et al, MNRAS, 465 (4), 4281-4310, 2017

Computation







Topology and Geometry of 3D Gaussian fields

Pranav et. al. MNRAS, 485(3), 4167 (2019)

Gaussian Random fields

•Given a spatial location s, a Gaussian random field is a random function X(s) on R^3 such that, when restricted to any finite set, one has a multivariate normal distribution.

• m-point joint distribution function:

$$P[f_1,...,f_m]df_1...df_m = \frac{1}{(2\pi)^N(\det M)^{1/2}}.\exp(-\frac{\sum \Delta f_i (M^{-1})_{ij}\Delta f_j}{2})df_1...df_m$$

- Mean: $\Delta f_i = f_i - \langle f_i \rangle$

- Covariance matrix:

$$M_{ij} = \langle \Delta f_i \Delta f_j \rangle$$

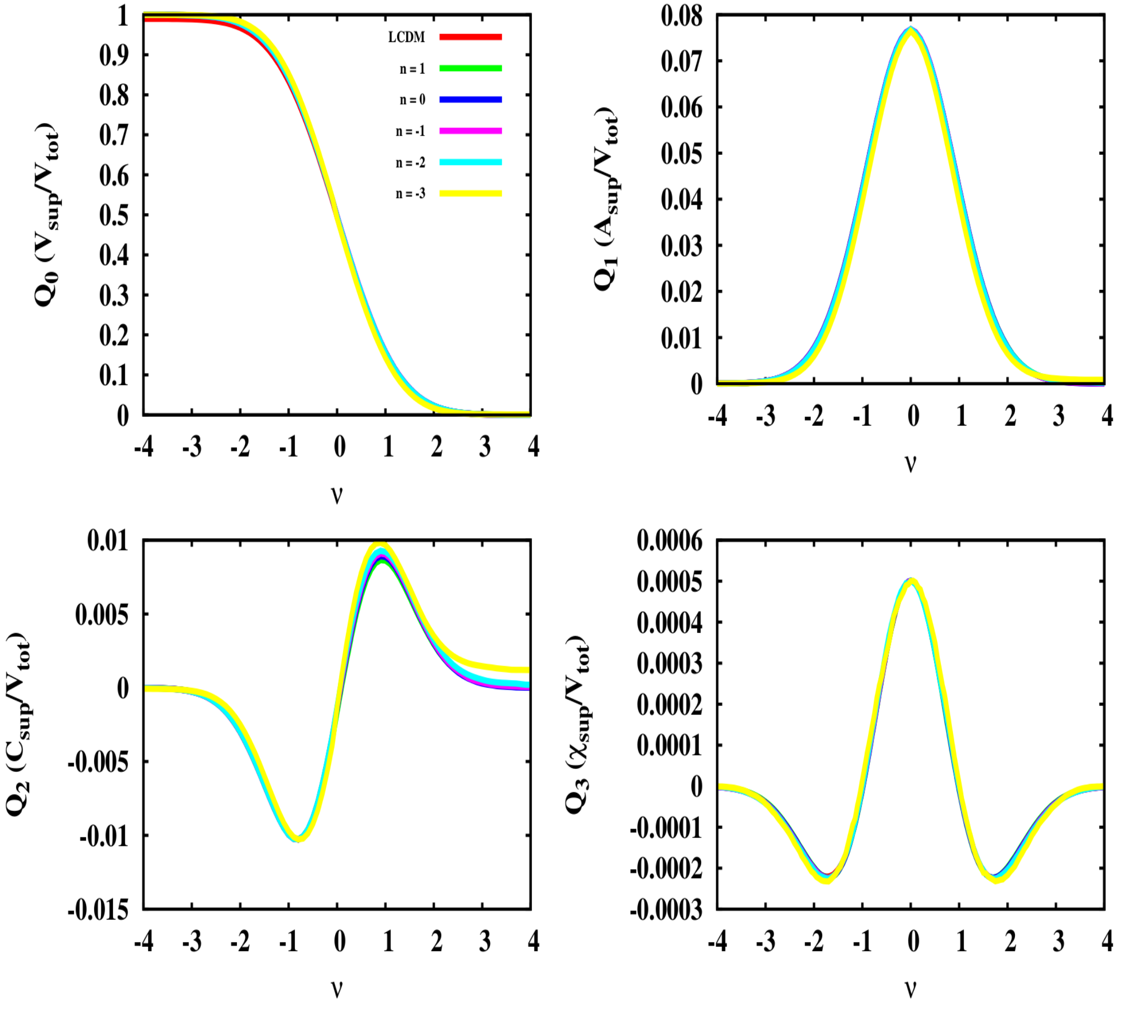
Gaussian fields: Minkowski Functionals

$$Q_0(\upsilon) = \frac{1}{2} - \frac{1}{\sqrt{2}}\Phi(\upsilon)$$

$$Q_1(\upsilon) = \frac{2}{3}\frac{\lambda}{\sqrt{2\pi}}\exp(-\frac{1}{2}\upsilon^2)$$

$$Q_2(\upsilon) = \frac{2}{3}\frac{\lambda^2}{\sqrt{2\pi}}\upsilon\exp(-\frac{1}{2}\upsilon^2)$$

$$Q_3(\upsilon) = \frac{\lambda^3}{\sqrt{2\pi}}(\upsilon^2 - 1)\exp(-\frac{1}{2}\upsilon^2)$$



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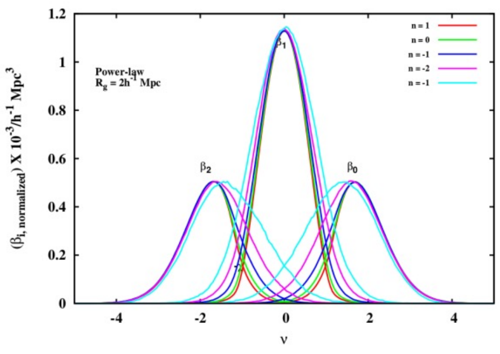
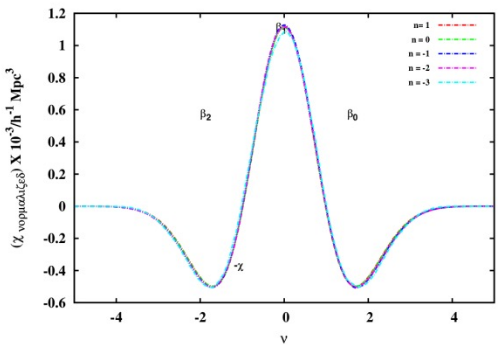
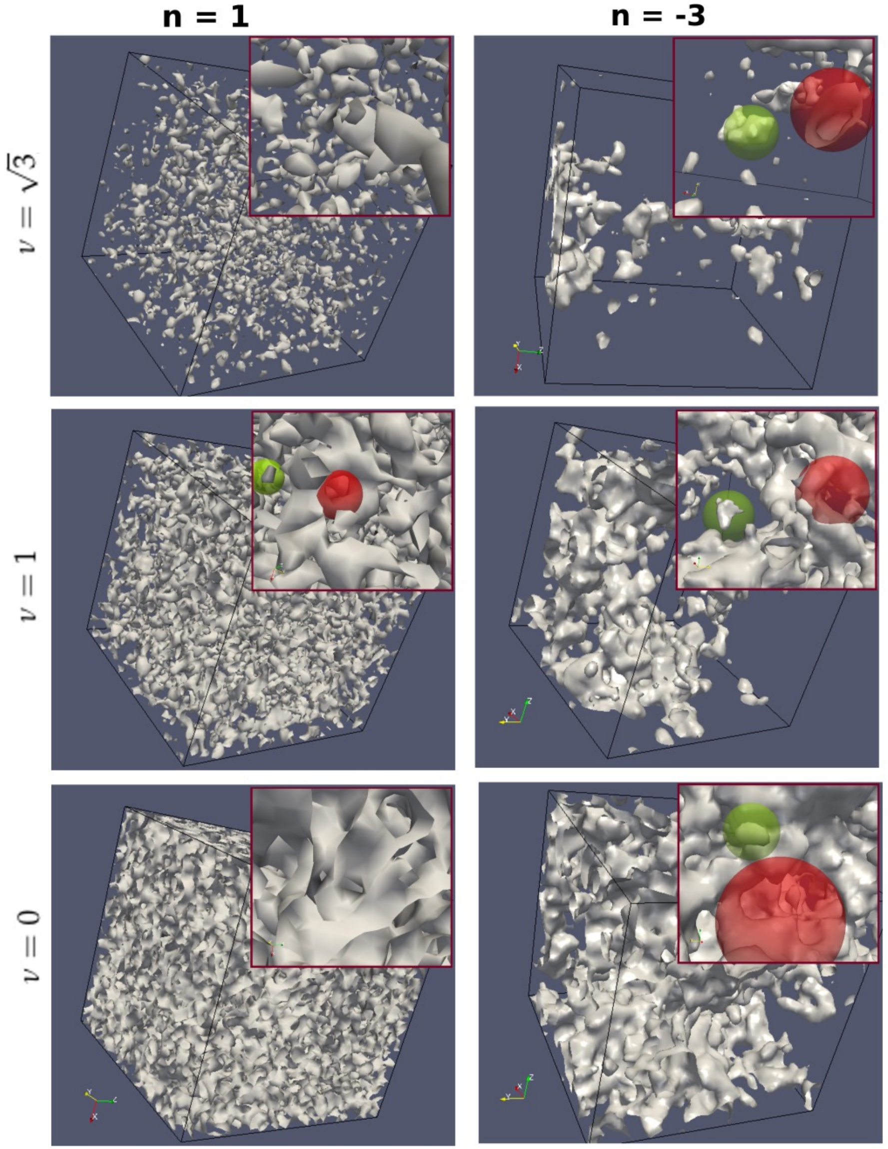
Gaussian fields: Betti Topology







- *Shape of Betti numbers dependent on power spectrum; EC and MF are not*

- *Gott et. al. (1986): Sponge-like topology of the LSS*

Pranav et. al. MNRAS, 485(3), 4167 (2019)

Topology of the CMB

Relative Homology

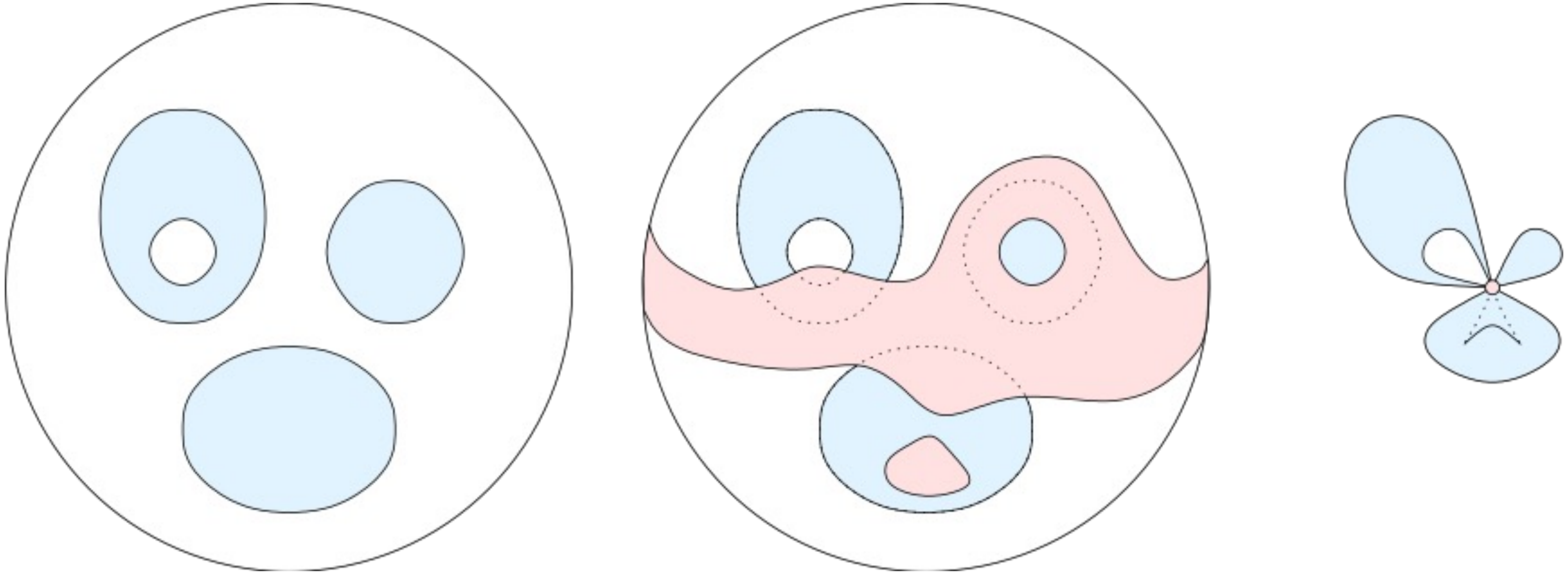


Figure 1: *Left*: A blue excursion set on the sphere consisting of an upper left component with a hole, an upper right component, and a lower component. Its Betti numbers are $\beta_0 = 3, \beta_1 = 1, \beta_2 = 0$, and its Euler characteristic is $\mathsf{EC} = 3 - 1 + 0 = 2$. *Middle*: A pink mask in which the data is not reliable. It covers part of the upper left component and hole, its hole is fully contained in the upper right component, and it overlaps the lower component in two disconnected pieces. *Right*: A visualization of the relative homology groups obtained by shrinking the mask to a point and pulling the excursion set with it. We have $b_0 = 0$ because all three components connect to the shrunken mask, $b_1 = 2$ because the loop in the upper left component is preserved and a new loop in the lower component is formed, and $b_2 = 1$ because the upper right component takes on the shape of sphere. The (relative) Euler characteristic is therefore $\mathsf{EC}_{\text{rel}} = 0 - 2 + 1 = -1$.

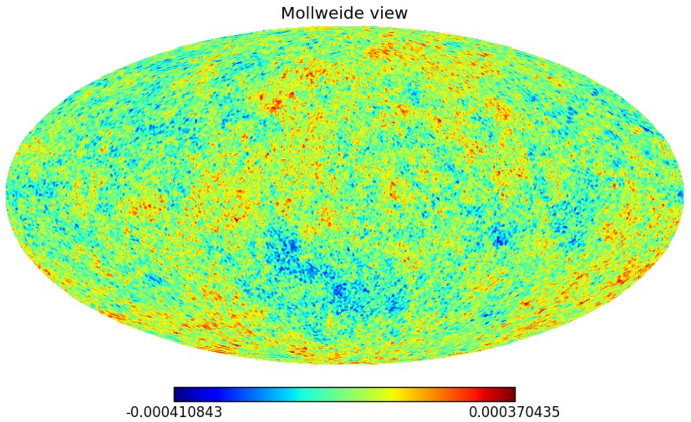
Planck Data

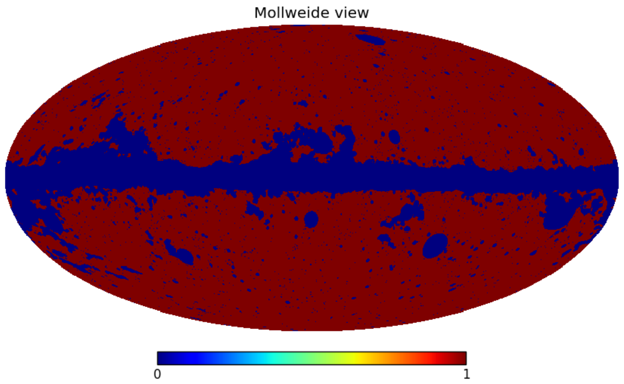
- Specified on S2, as the deviation from the background average (HEALPIX format)

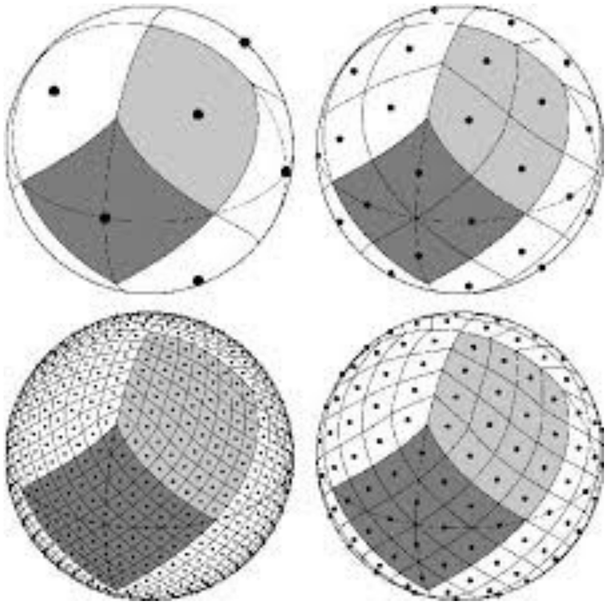
- Measurement unreliable in some parts: foregrounds

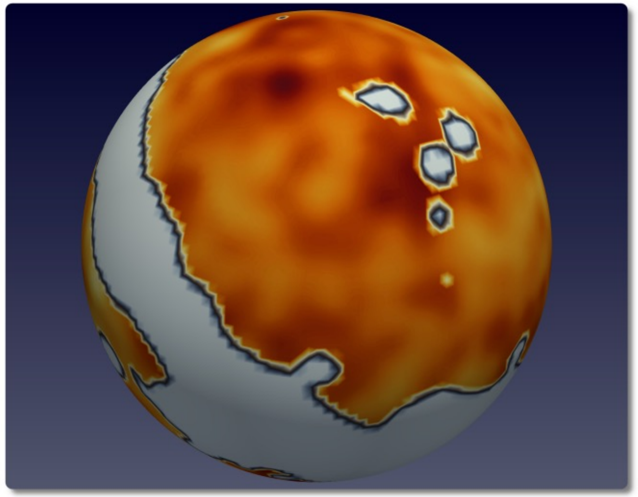
- Unreliable parts masked

- Field converted to N (0,1) using unmasked pixels only



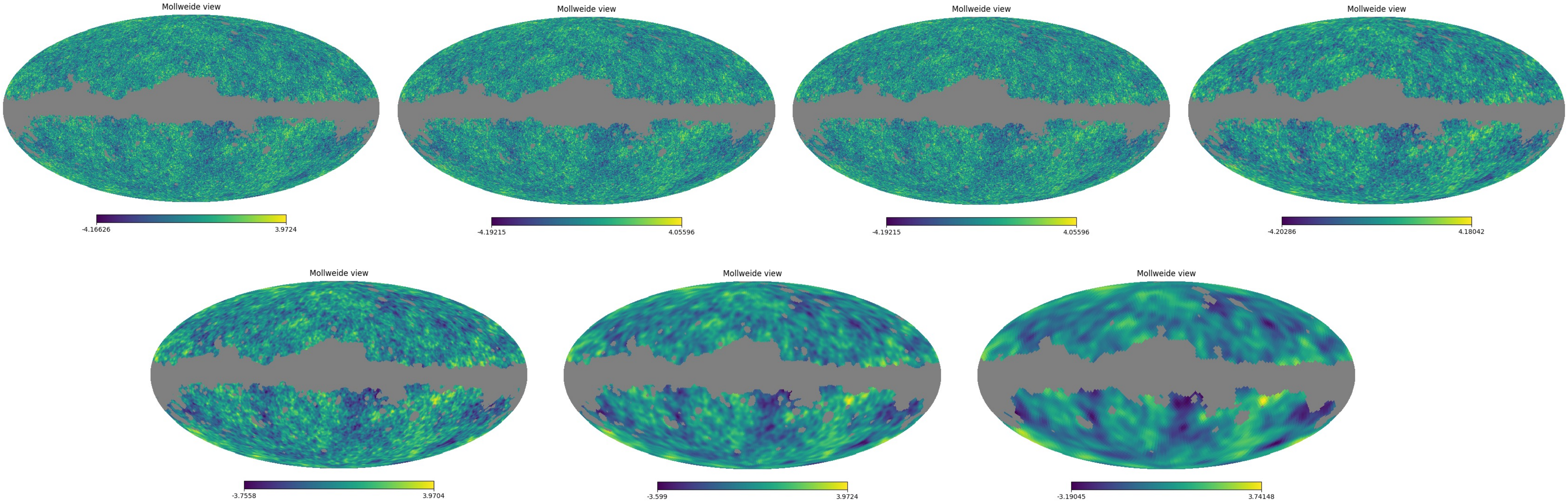




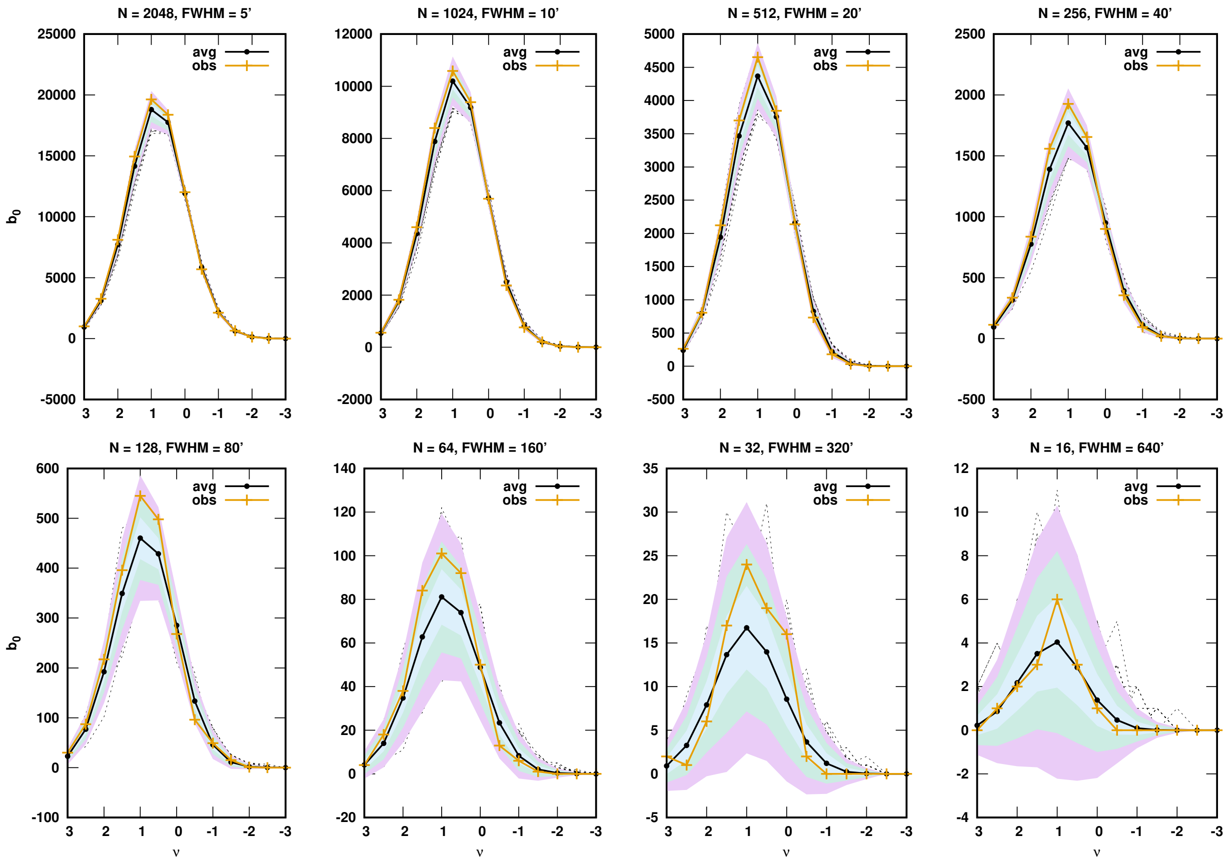


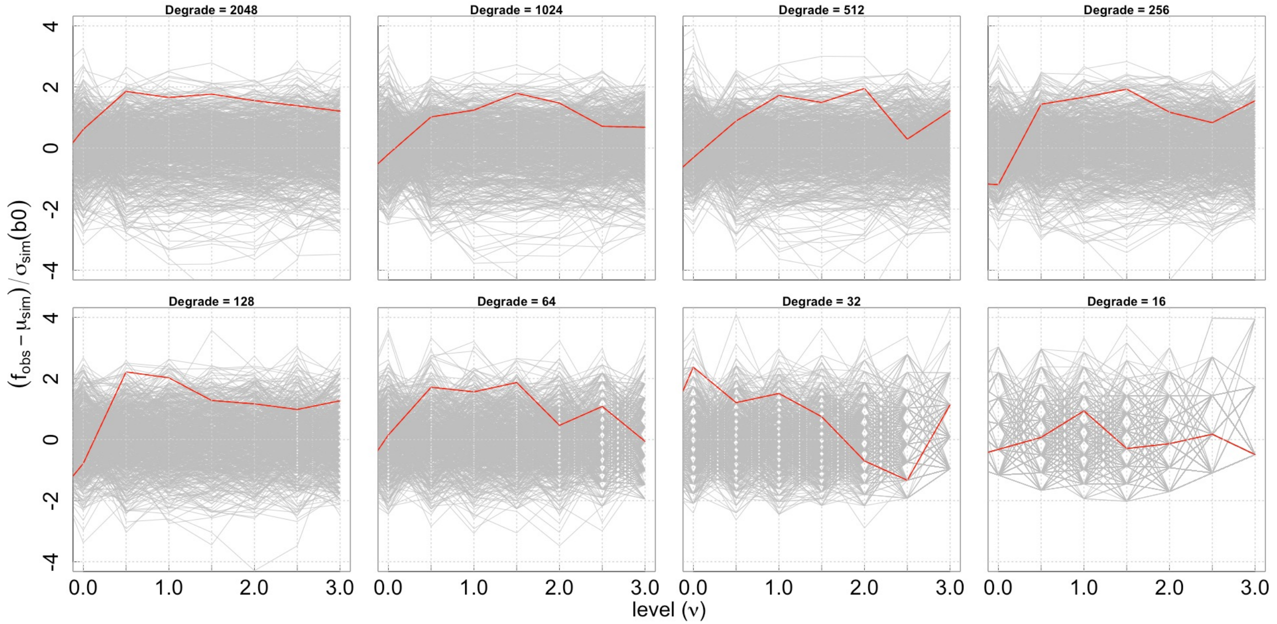
Temperature
Full Sky

Masked degraded maps (multi-scale analysis)



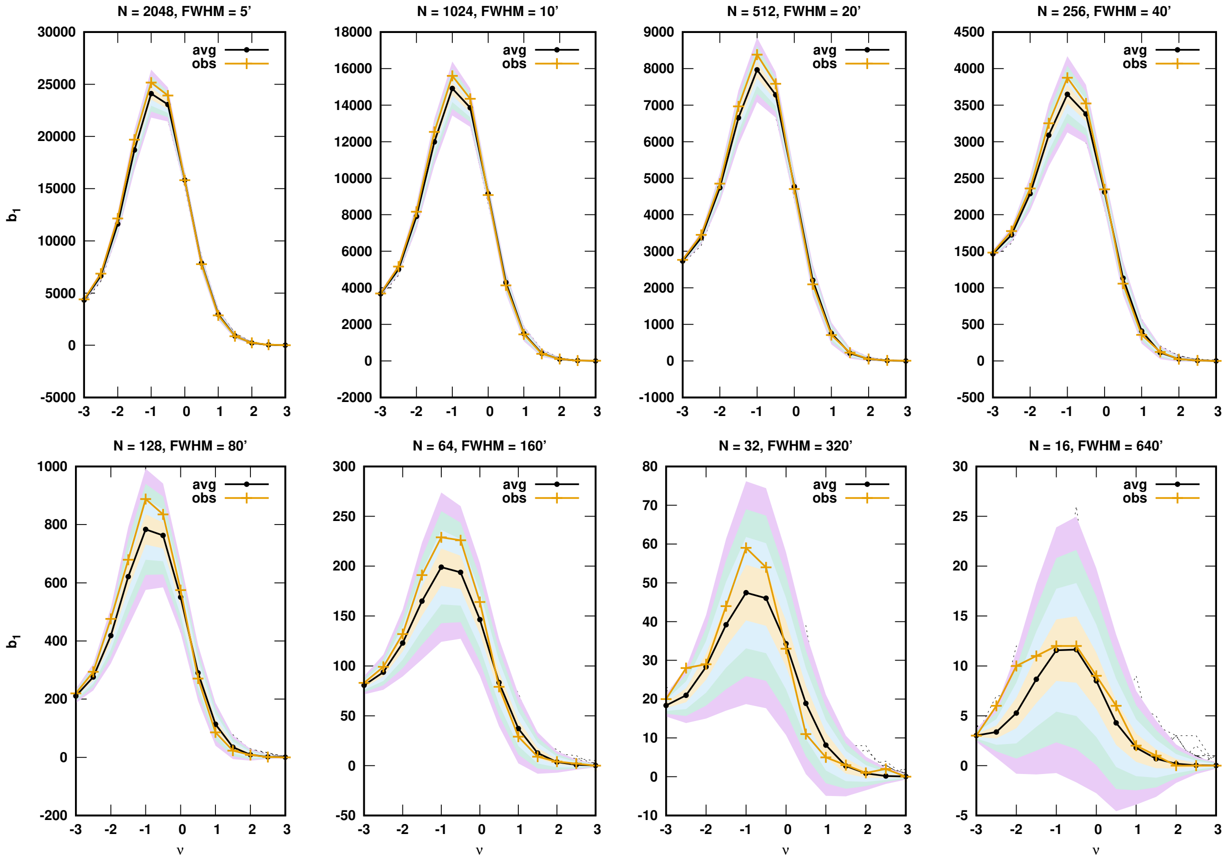
- Maps degraded to N_side = 2048, 1024, 512, 256, 128, 64, 32 and 16 (not shown), corresponding to FWHM = 5’, 10’, 20’, 40’, 80’, 160’, 320’, 640'
- Binary Mask degraded similarly (converts it to non-binary) – reconverted to binary by setting the threshold 0.9 (as done by Planck coll.)

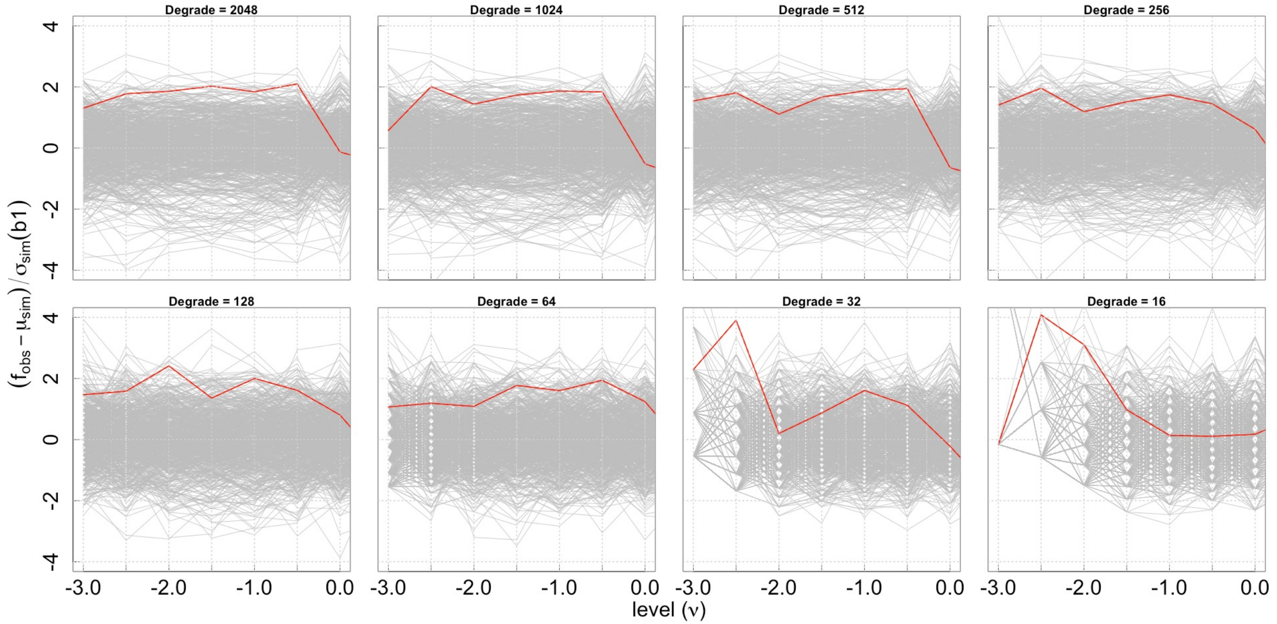




Isolated objects (betti 0)

P.Pranav, A&A 659, A115 (2022)





Loops (betti 1)

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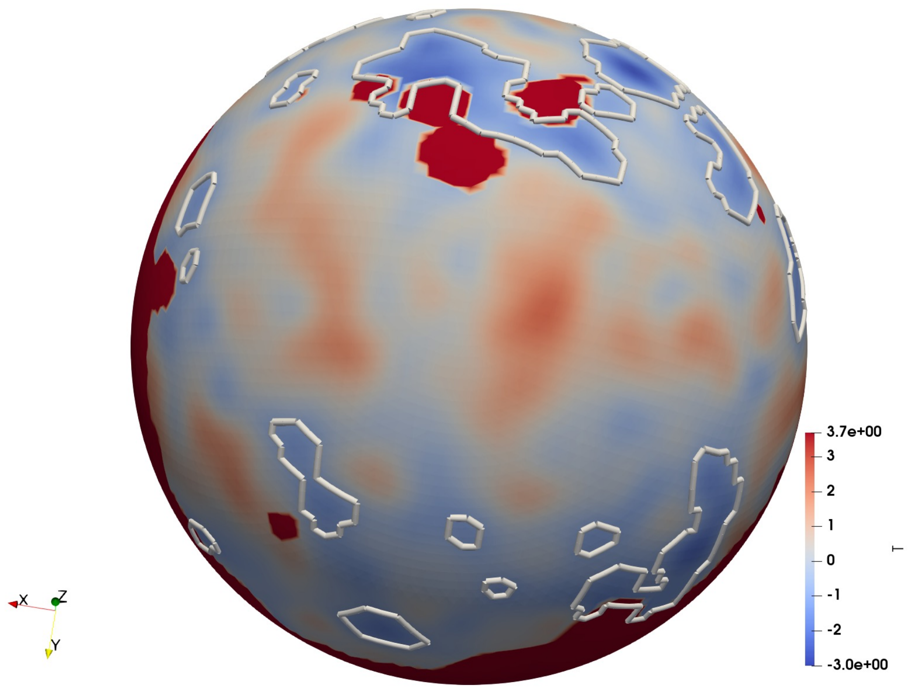
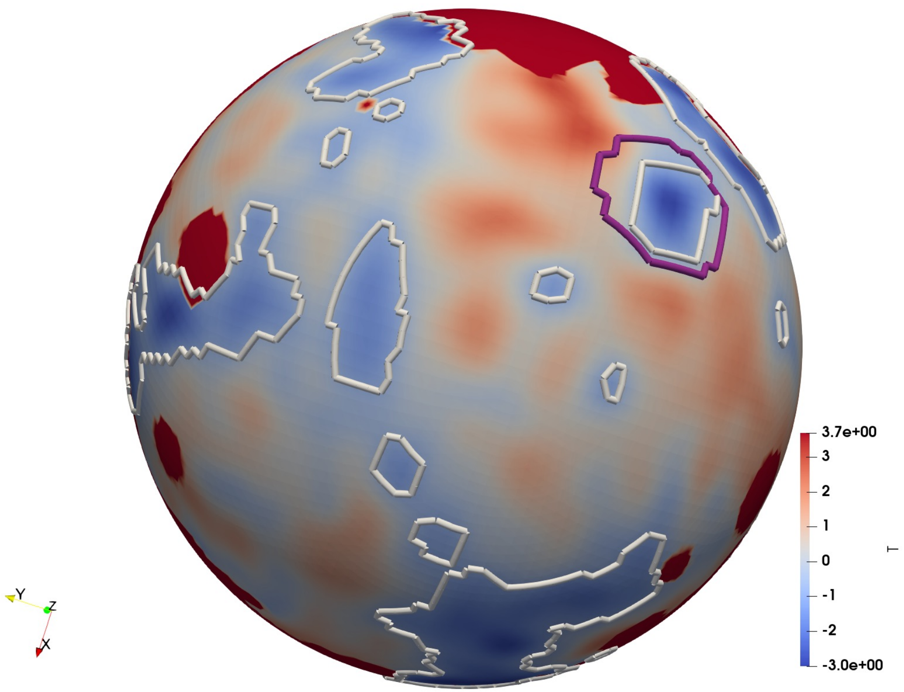
CMB Loops





Statistical tests

- The data consists of topological summaries (b0, b1, EC) obtained from simulations, and observed CMB field
- Goal: estimate the probability that the physical model that produced the simulations would produce quantities consistent with those from the observed CMB field
- Let $\mathbf{x}_i \in \mathbb{R}^m, i = 1, \ldots, n,$ be a sample of i.i.d. m-dimensional vectors, drawn from a distribution F. Let $\mathbf{y} \in \mathbb{R}^m$ be another sample point, assumed to be drawn from a distribution G.
- Test the (null) hypothesis that F=G
- p-values compute the probability that $\mathbf{y}$ is `consistent' with this hypothesis.
- Two methods :
    - Mahalanobis Distance or chi^2 test : parametric (Prasanta Chandra Mahalanobis, 1936)
    - Tukey depth : non-parametric (John Tukey, 1974)

Mahalanobis distance or χ2-test:

- Mean : $\bar{\mathbf{x}} = \sum_{i=1}^{n} \mathbf{x}_i / n$
- Variance: $\mathbf{S} = \sum_{i=1}^{n} (\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})^{\mathrm{T}} / (n-1)$
- Mahalanobis distance: $d^2_{\mathrm{Mahal}}(\mathbf{y}) = (\mathbf{y} - \bar{\mathbf{x}})^{\mathrm{T}} \mathbf{S}^{-1} (\mathbf{y} - \bar{\mathbf{x}}).$

- If **F** is assumed to be Gaussian and **n** is large, then under the hypothesis that **G=F** the squared Mahalanobis distance is approximately distributed as a **chi^2** distribution with **m** degrees of freedom. Thus the corresponding p-value is

$$p_{\mathrm{Mahal}}(\mathbf{y}) = P[\chi^2_m > d^2_{\mathrm{Mahal}}(\mathbf{y})].$$

Tukey depth

- **F** may not always conform to elliptical contours and therefore may not be Gaussian. In such a setting, p-values computed using the Mahalanobis distance may not be reliable.

- The Tukey half-space depth provides a general metric for identifying outliers in a flexible manner and in a non-parametric setting.

- Take **x_i, i=1,...,n** and **y** as before, making no assumptions on the structure of **F** and **G**, and let $\mathbf{z}$ be any point in $\mathbf{R}^m$.

- Half-space depth $\mathbf{d}_{\mathbf{dep}}(\mathbf{x}_j; \mathbf{x}_1,\ldots, \mathbf{x}_n)$ of $\mathbf{z}$ within the sample of the $\mathbf{x}_\mathbf{i}$ is the smallest fraction of the $\mathbf{n}$ points \$$\mathbf{x}_1,\ldots,\mathbf{x}_n$\$ to either side of any hyperplane passing through $\mathbf{z}$

- Points that have the same depth constitute a non-parametric estimate of the isolevel contour of the distribution **F**.

- To evaluate a p-value for **y**, compute $\mathbf{d}_\mathbf{j} = \mathbf{d}_{\mathbf{dep}}(\mathbf{x}_\mathbf{j}; \mathbf{x\_1},\ldots,\mathbf{x}_\mathbf{n})$ for every point $\mathbf{x}_\mathbf{j}$, **(j=1,…,n)**, yielding an empirical distribution of depth. p-value is the proportion of points whose depth is lower than that of **y**:

$$p_{\text{dep}}(\mathbf{y}) = \#\{j \mid d_j > d_{dep}(y)\}/n$$

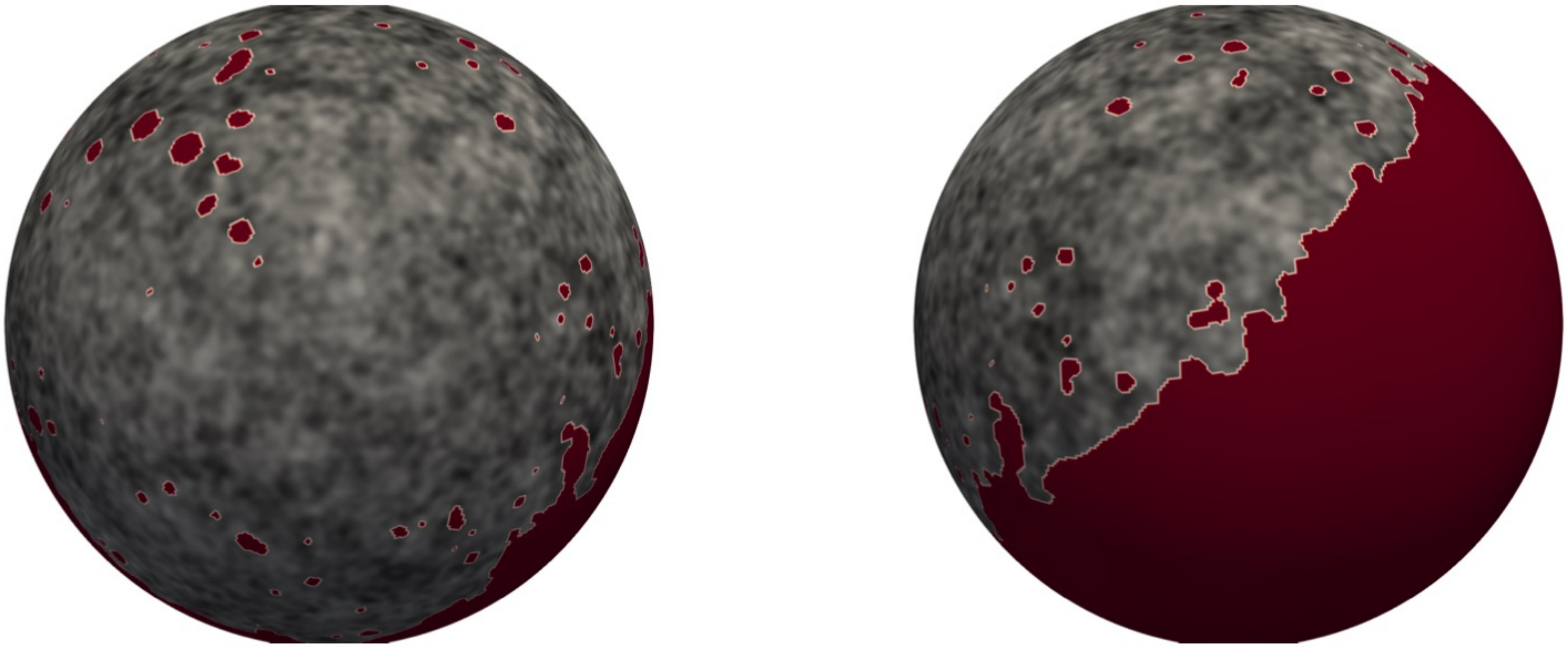
Relative homology										
		χ^2 (theoretical)			χ^2 (empirical)			Tukey Depth		
Res	FWHM	b_0	b_1	$\mathsf{EC}_{\mathrm{rel}}$	b_0	b_1	$\mathsf{EC}_{\mathrm{rel}}$	b_0	b_1	$\mathsf{EC}_{\mathrm{rel}}$
threshold = 0.90										
2048	5	0.708	0.569	0.858	0.676	0.584	0.851	0.630	0.318	0.912
1024	10	0.649	0.475	0.383	0.628	0.476	0.363	0.672	0.352	**0.000**
512	20	0.156	0.417	0.203	0.161	0.408	0.183	0.332	0.308	**0.000**
256	40	0.398	0.596	0.772	0.403	0.600	0.756	0.362	0.560	0.642
128	80	0.356	0.204	0.563	0.383	0.186	0.555	**0.000**	0.312	0.730
64	160	0.398	0.494	0.768	0.401	0.468	0.746	0.465	0.718	0.755
32	320	**0.020**	**0.002**	**0.001**	**0.028**	**0.013**	**0.001**	**0.000**	**0.000**	**0.000**
16	640	0.974	**0.001**	0.203	0.981	**0.030**	0.222	0.983	**0.000**	0.440
summary	NA	0.478	**0.023**	**0.045**	0.290	0.076	**0.052**	0.803	**0.000**	**0.000**

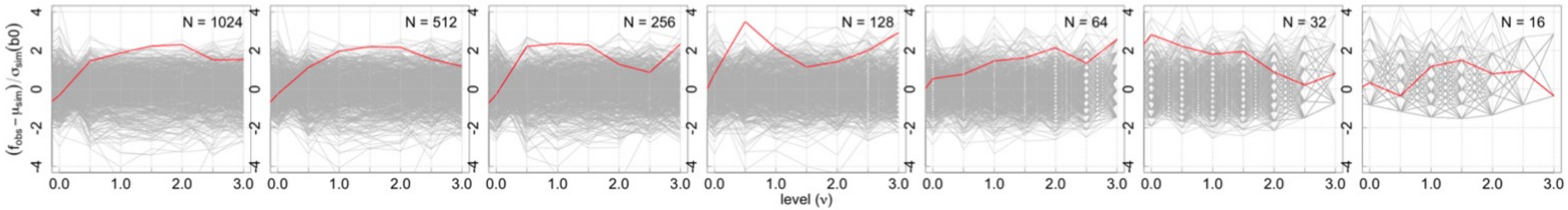
Table 1: Table displaying the two-tailed p-values for relative homology obtained from parametric (Mahalanobis distance) and non-parametric (Tukey depth) tests, for different resolutions and smoothing scales for the `NPIPE` dataset. The last entry is the p-value for the summary statistic computed across all resolutions. Marked in boldface are p-values 0.05 or smaller.

P.Pranav, A&A 659, A115 (2022)

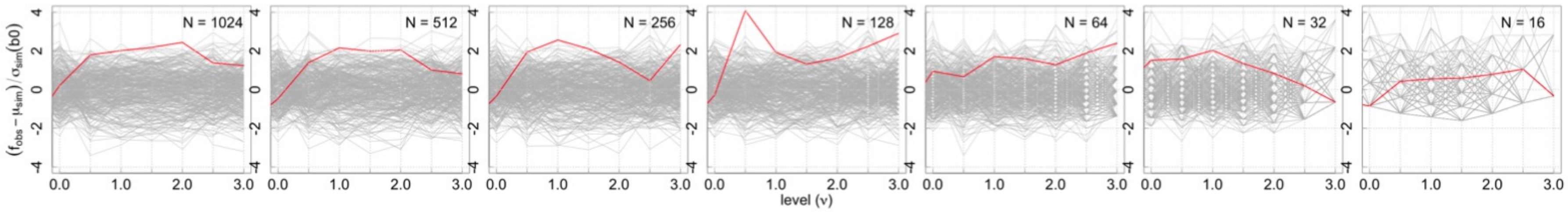
Temperature

Hemispheres

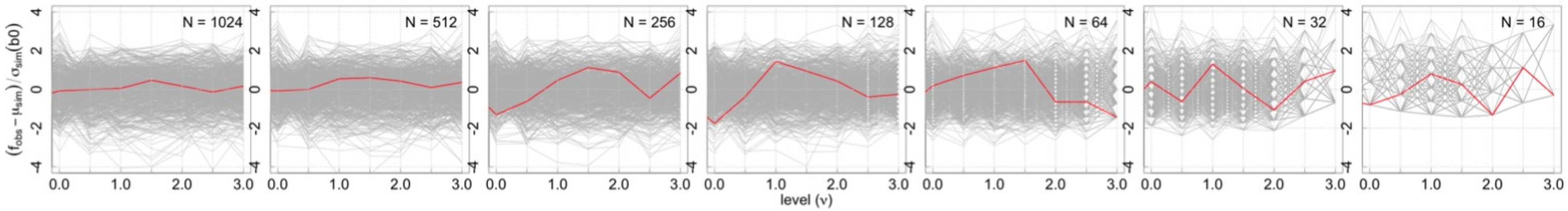




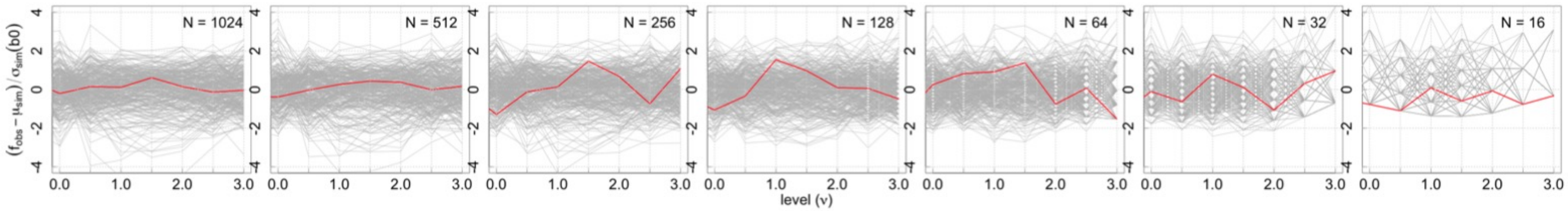
(a)



(b)



(c)

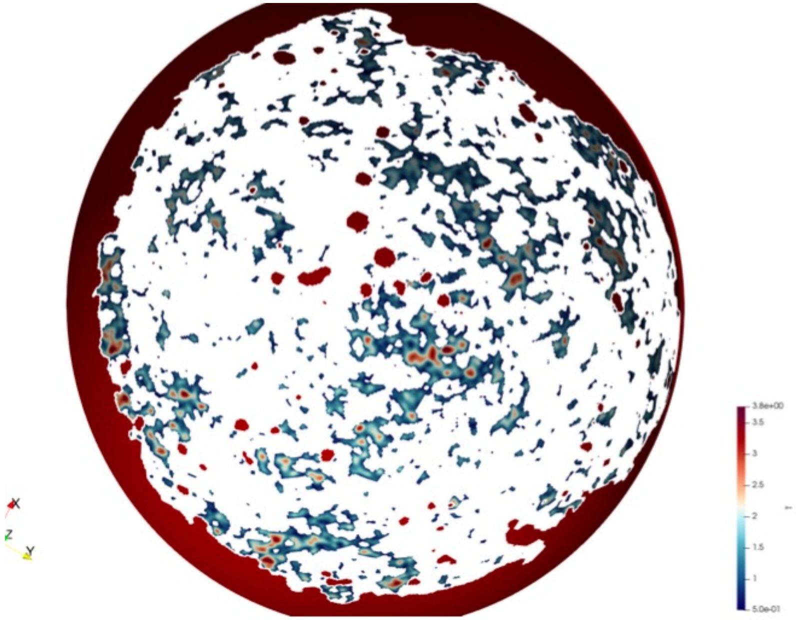


(d)

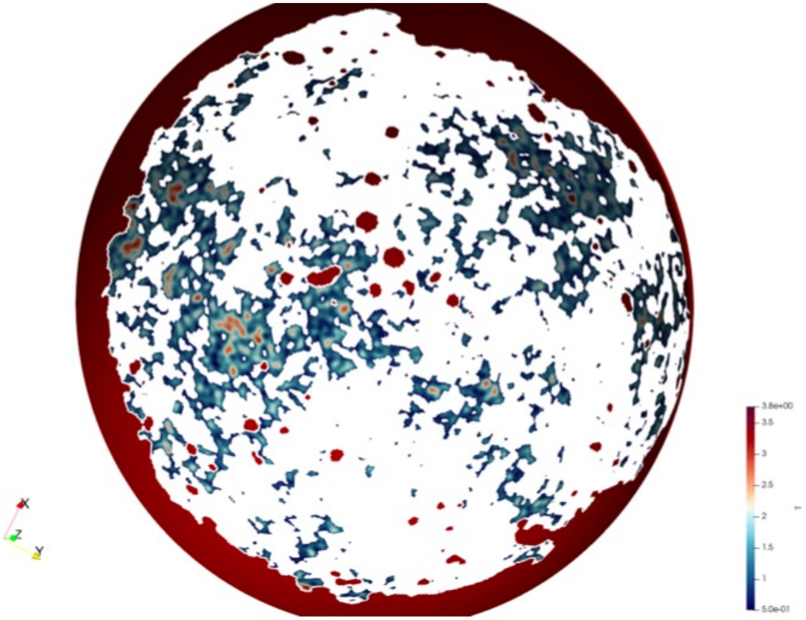
- *b*0 for the temperature maps (NPIPE and FFP10 dataset) Northern (top two rows) and the southern hemisphere (bottom two rows).

- The graphs present the normalized differences, and each panel presents the graphs for a range of degradation and smoothing scales.

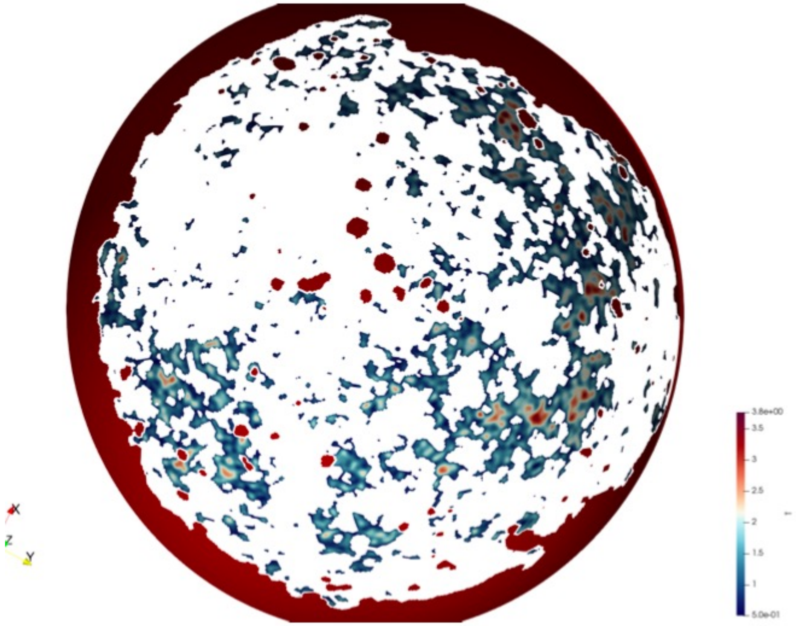
- PR3 temperature common mask.



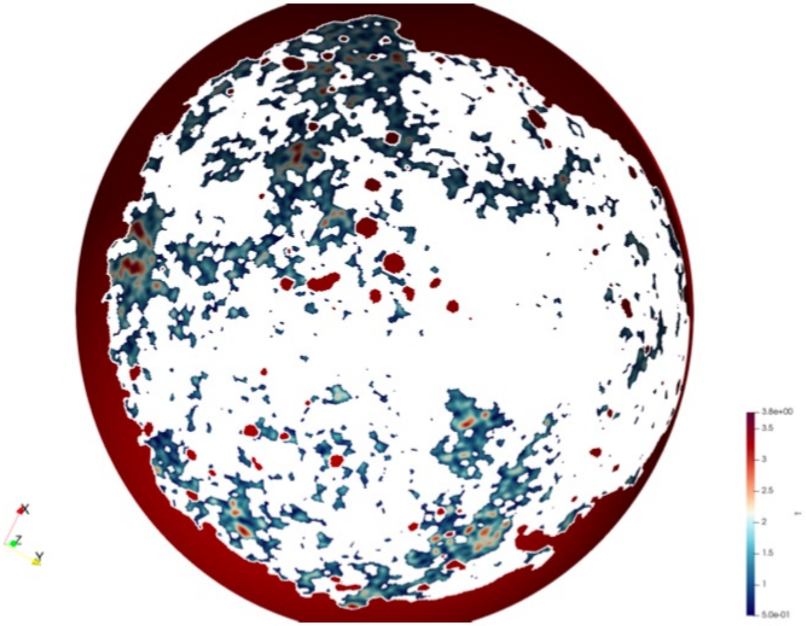
(a)



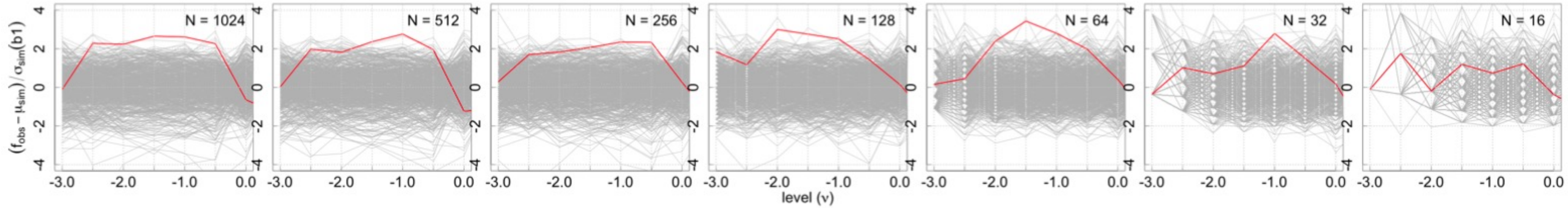
(b)



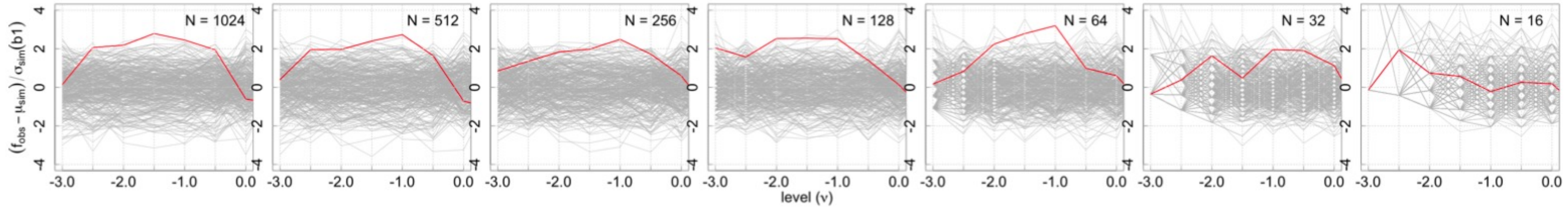
(c)



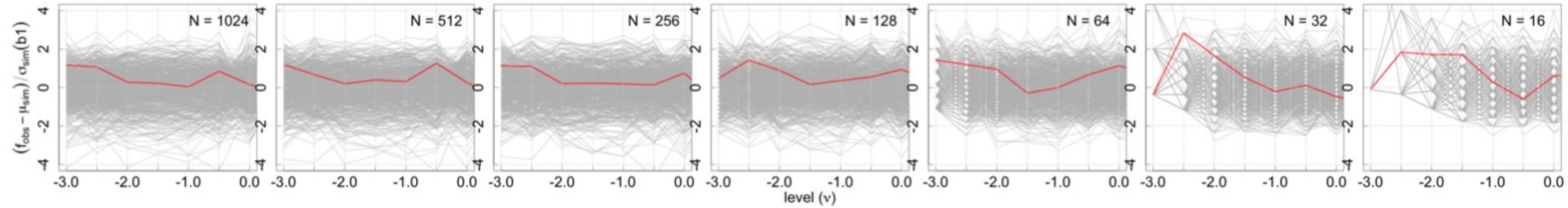
(d)



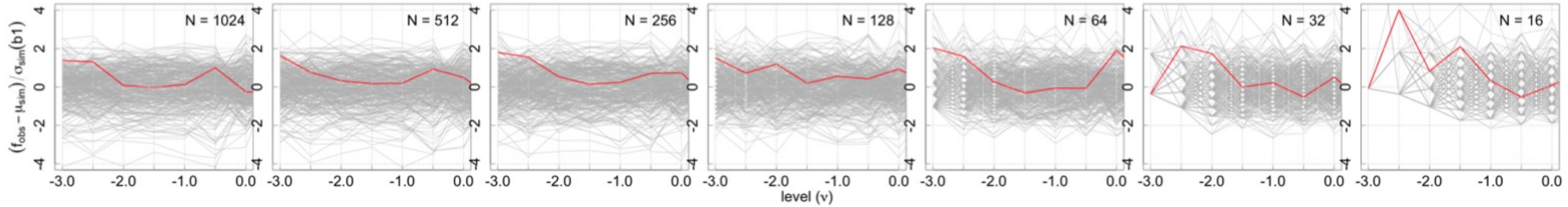
(a)



(b)



(c)



(d)

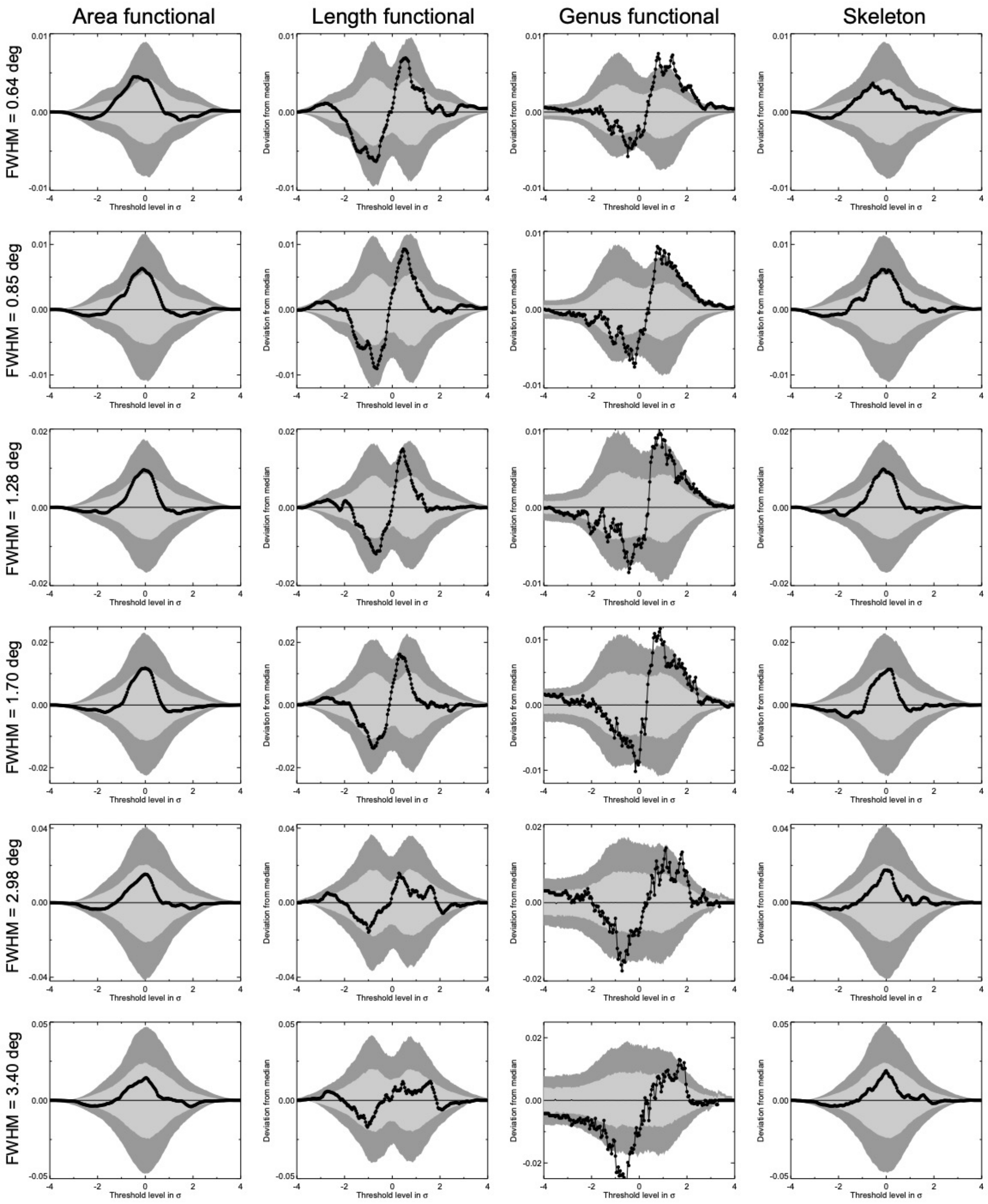
- *b1* for the temperature maps (NPIPE and FFP10 dataset) Northern (top two rows) and the southern hemisphere (bottom two rows).

Relative homology – χ^2 (empirical) – Northern hemisphere							
		FFP10			NPIPE		
Res	FWHM	b_0	b_1	$\mathrm{EC}_{\mathrm{rel}}$	b_0	b_1	$\mathrm{EC}_{\mathrm{rel}}$
threshold = 0.90							
1024	10'	0.240	0.110	0.133	0.408	0.073	0.148
512	20'	0.433	0.237	0.287	0.463	0.085	0.165
256	40'	**0.037**	0.327	0.273	0.060	0.335	0.477
128	80'	**0.000**	**0.047**	**0.013**	**0.037**	**0.037**	**0.035**
64	160'	0.063	0.070	0.233	**0.043**	0.058	0.073
32	320'	0.157	0.247	0.343	**0.020**	0.215	0.102
16	640'	0.817	0.540	0.693	0.560	0.290	0.470
summary	N/A	**0.010**	0.087	**0.000**	**0.023**	0.053	**0.017**

(a)

Relative homology – χ^2 (empirical) – Southern hemisphere							
		FFP10			NPIPE		
Res	FWHM	b_0	b_1	$\mathrm{EC}_{\mathrm{rel}}$	b_0	b_1	$\mathrm{EC}_{\mathrm{rel}}$
threshold = 0.90							
1024	10'	0.967	0.457	0.467	0.992	0.762	0.887
512	20'	1.000	0.857	0.967	0.993	0.818	0.993
256	40'	0.090	0.677	0.247	0.350	0.817	0.728
128	80'	0.653	0.623	0.847	0.662	0.902	0.915
64	160'	0.287	0.260	0.507	0.283	0.722	0.850
32	320'	0.853	0.247	0.460	0.597	0.103	0.172
16	640'	0.873	**0.023**	**0.037**	0.428	0.178	0.268
summary	N/A	0.543	0.397	0.013	0.577	0.877	0.925

(b)



- Minkowski functionals and skeleton length (WMAP)

Eriksen et al 2004

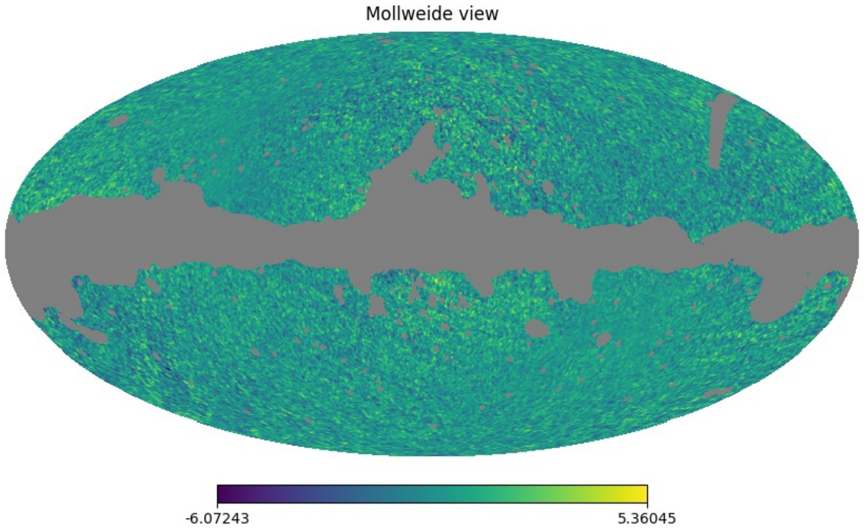
Polarization

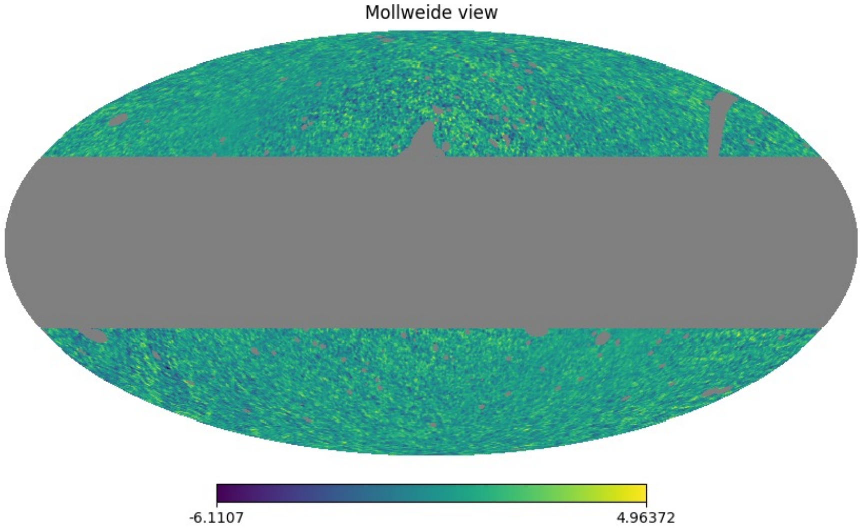
Experimental set up

- Use full sky to generate alms from TQU maps (prevents E/B leakage)

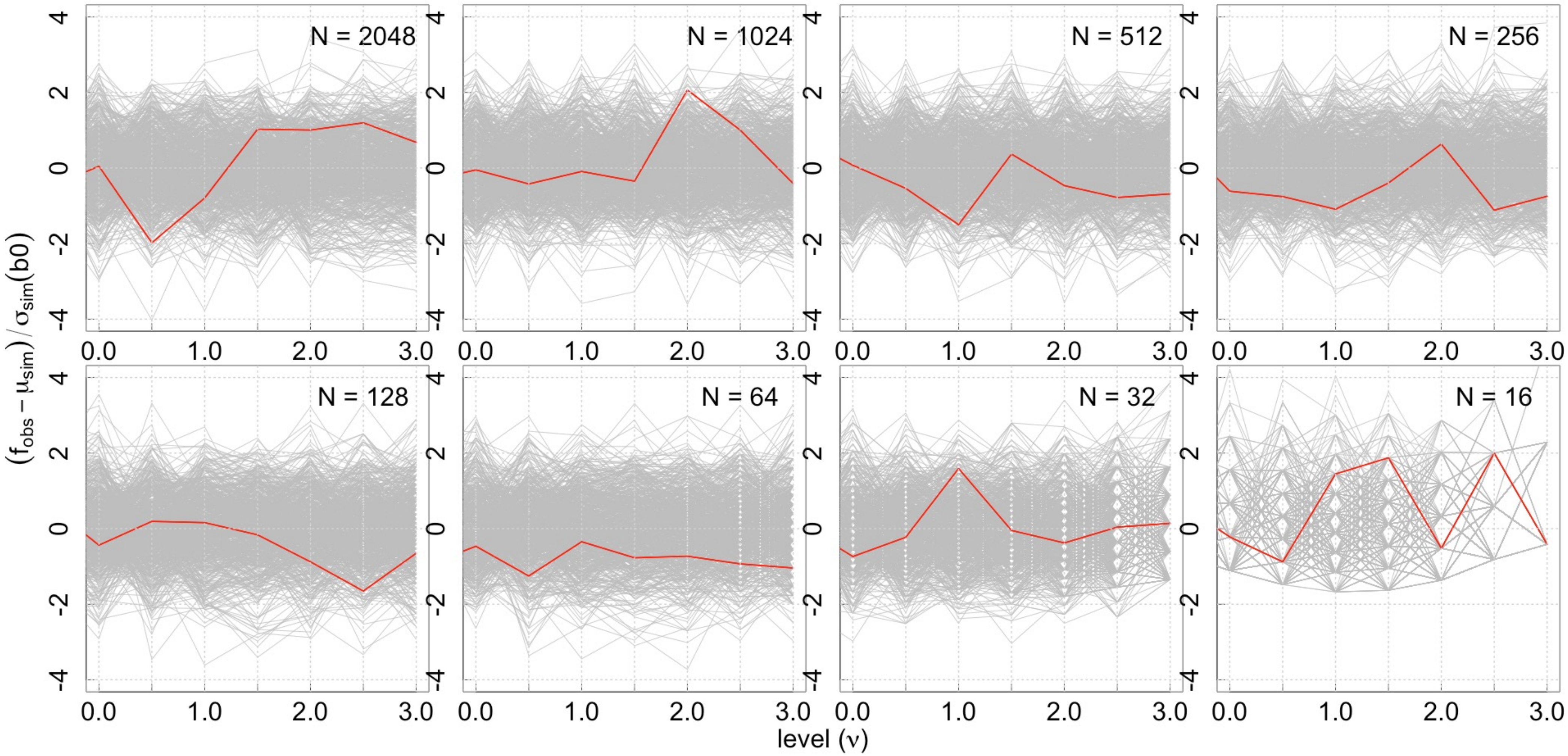
- Use the grad and curl-like like elements to synthesize the E and B maps

- Mask : PR3 common mask (+ b)





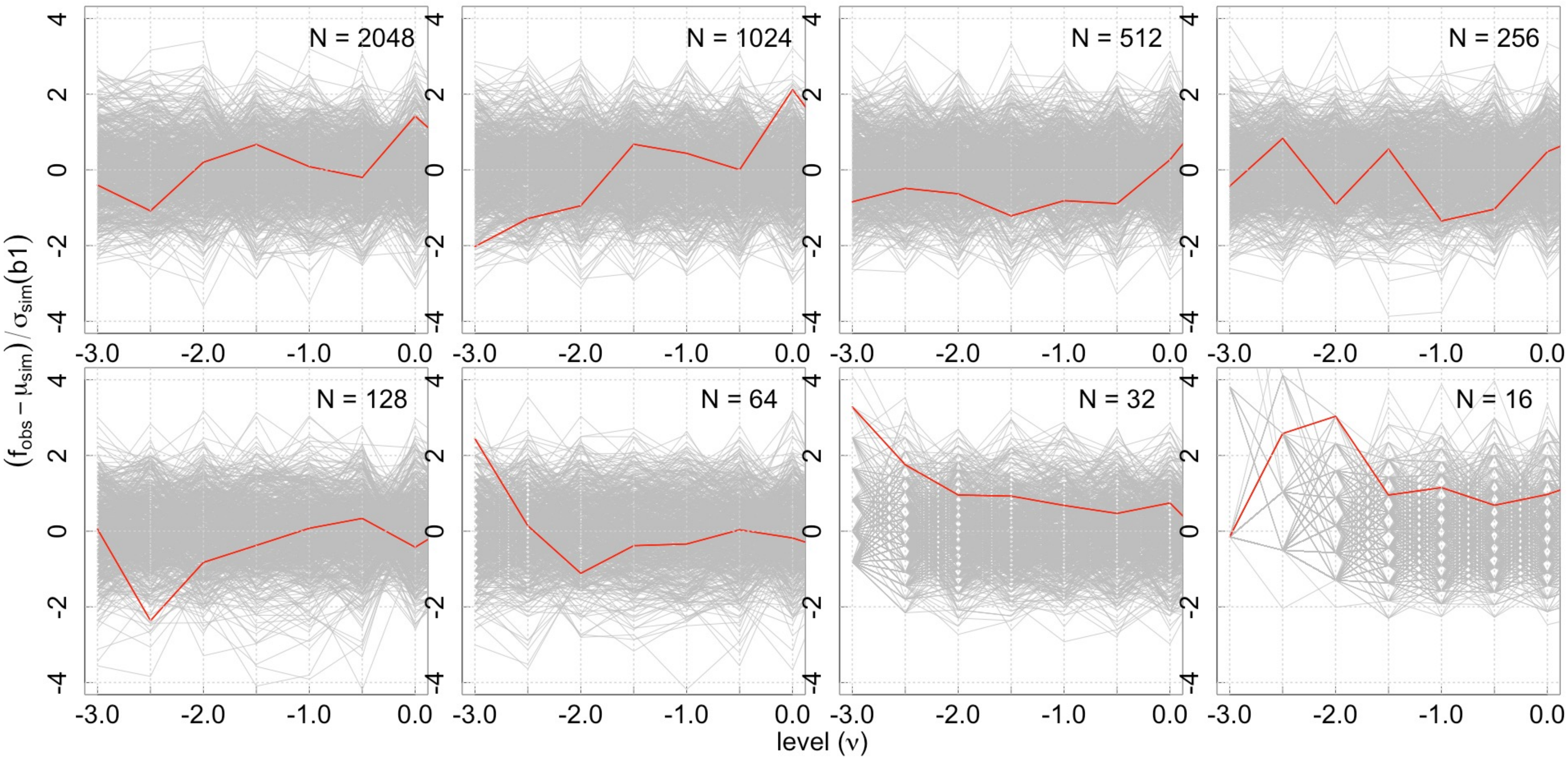
NPIPE : Planck 2018 common Mask



E-mode maps

Isolated objects (betti 0)

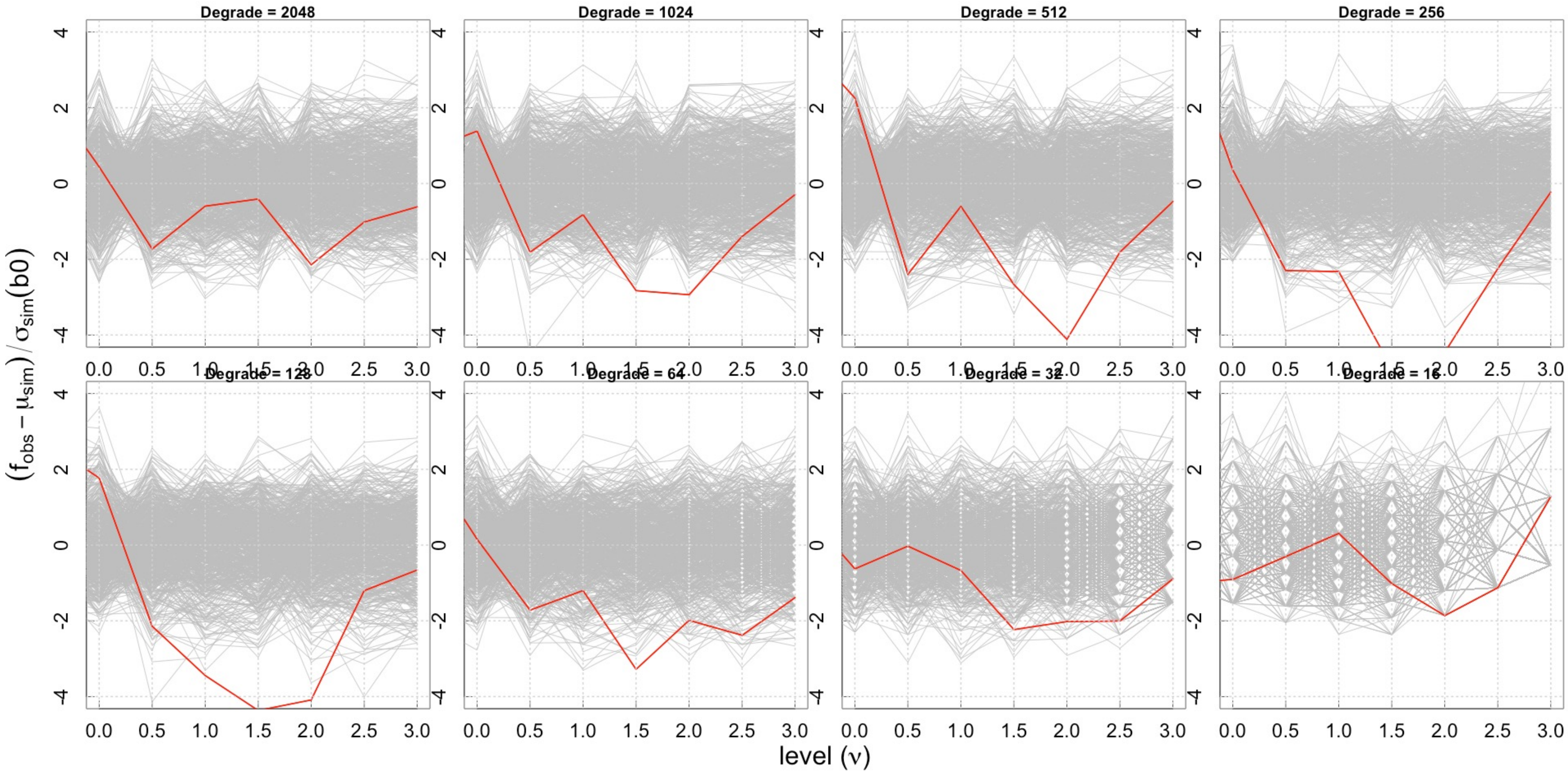
NPIPE : Planck 2018 common Mask



E-mode maps

Loops(betti 1)

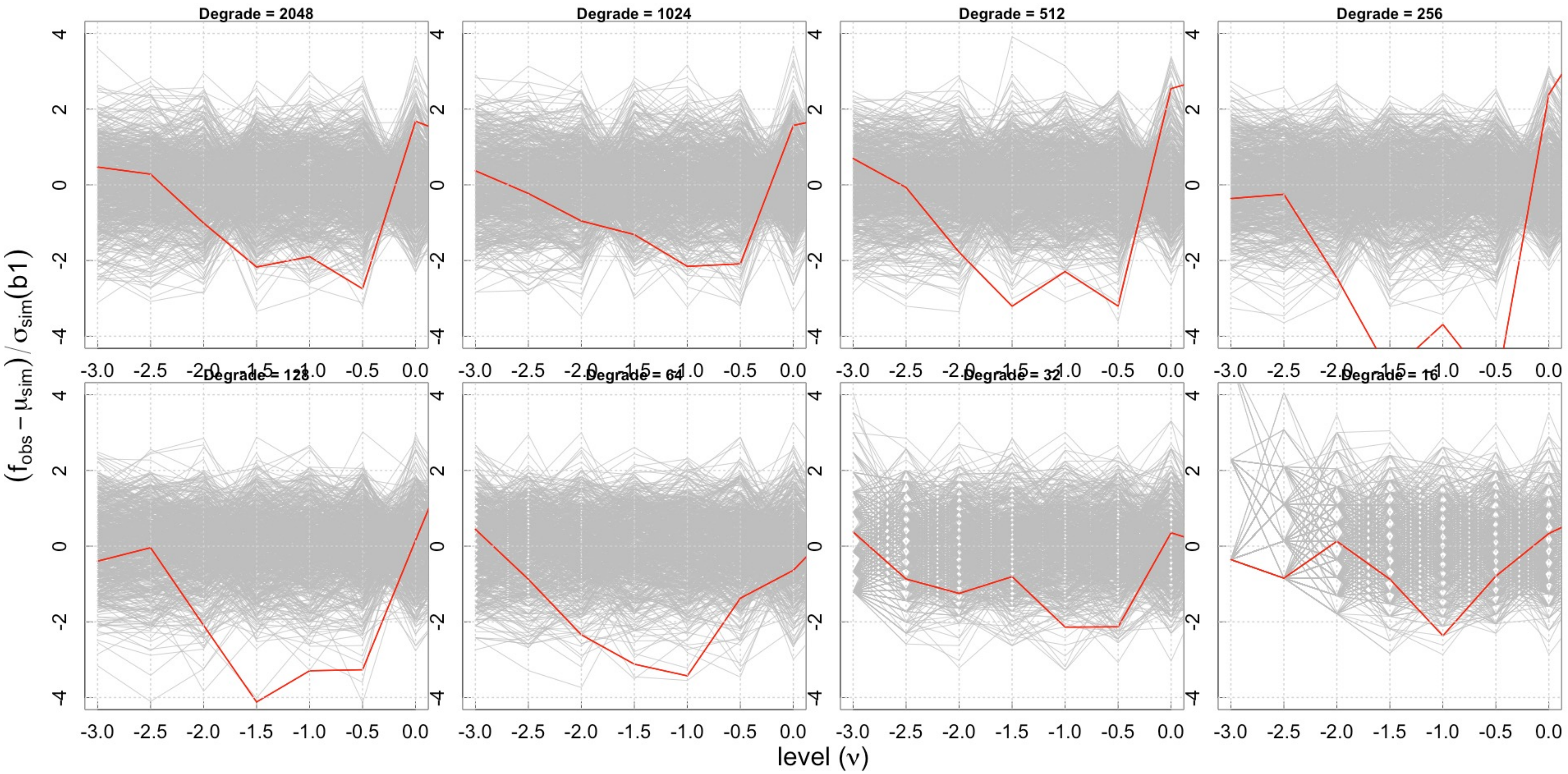
NPIPE : Planck 2018 common Mask



B-mode maps

Isolated objects (betti 0)

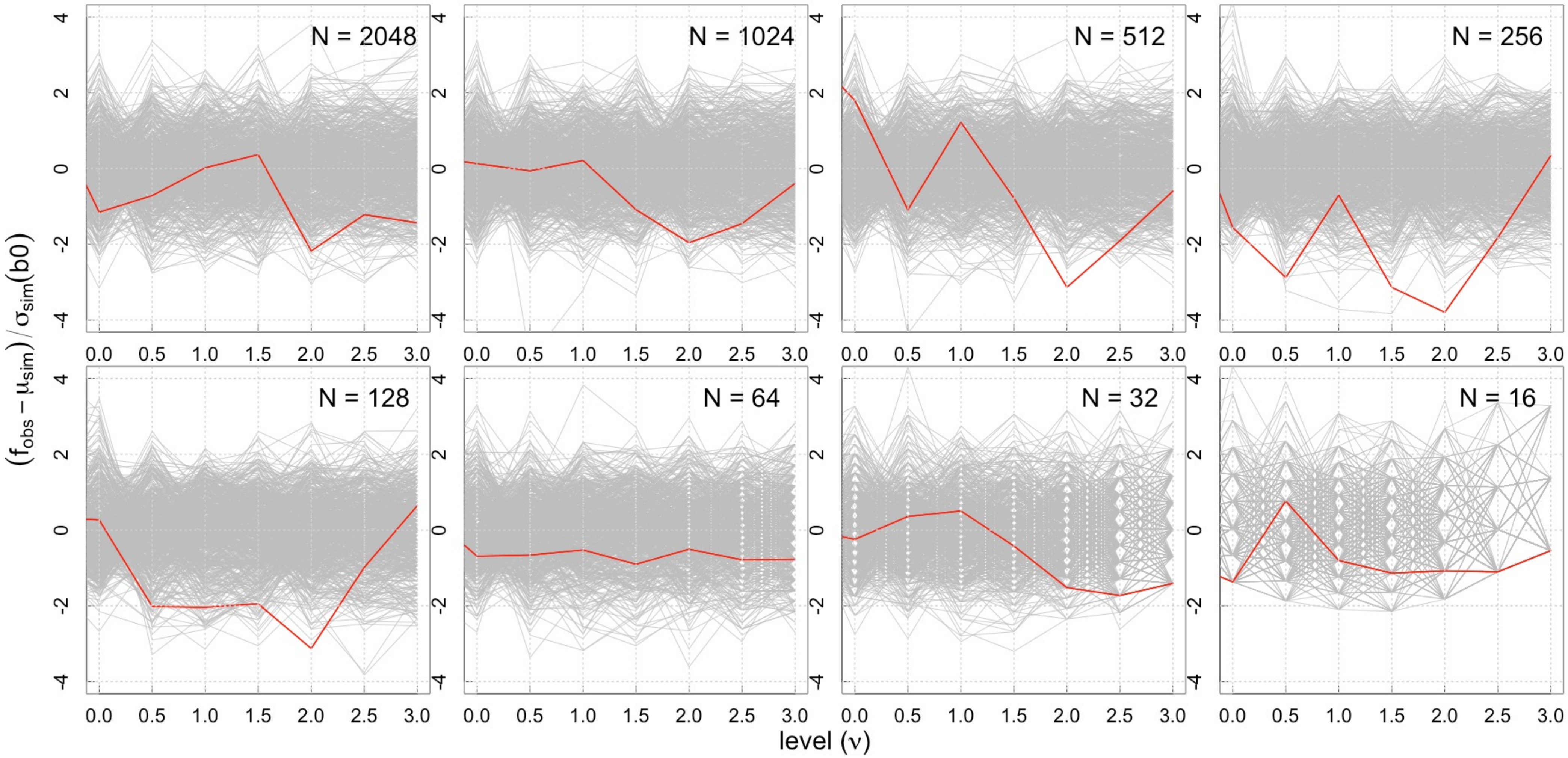
NPIPE : Planck 2018 common Mask



B-mode maps

Loops(betti 1)

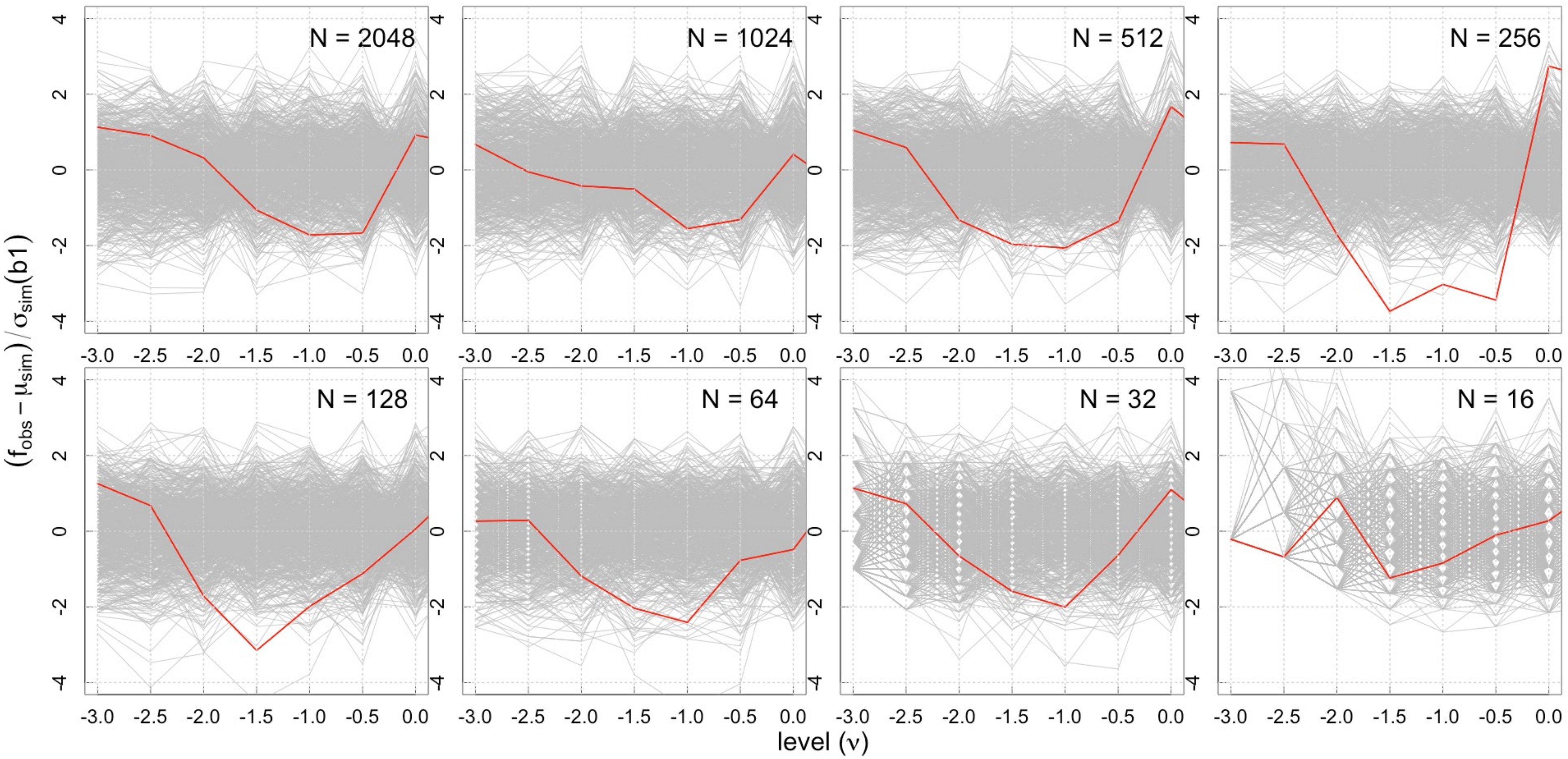
NPIPE : Planck 2018 common Mask + galcut 60



B-mode maps

Isolated objects (betti 0)

NPIPE : Planck 2018 common Mask + galcut 60

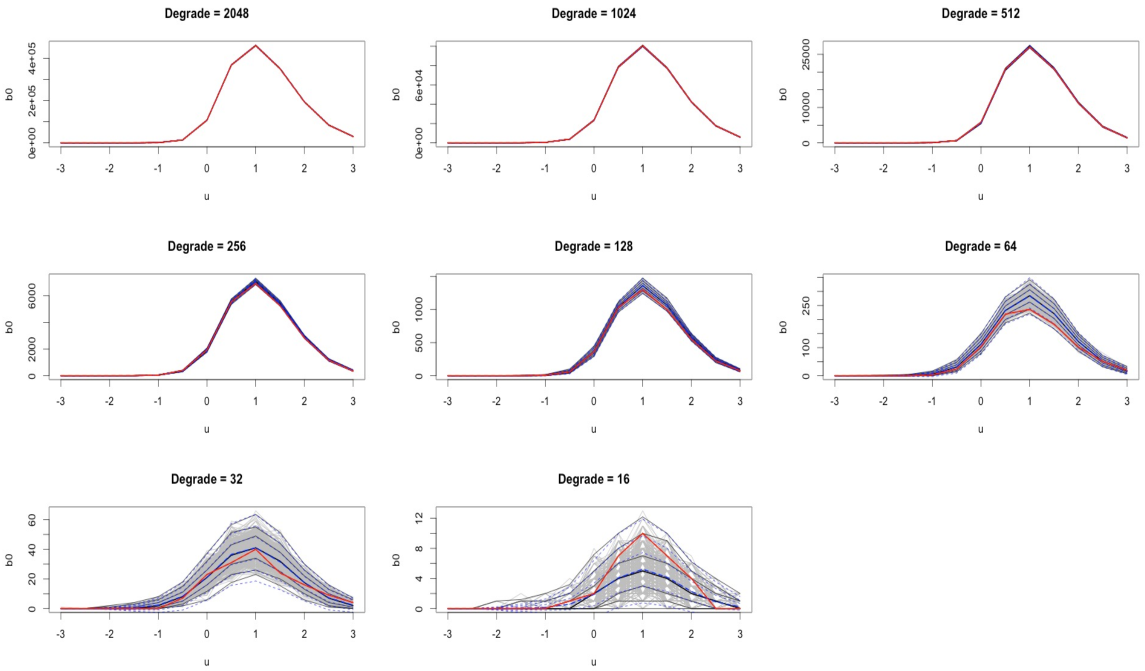


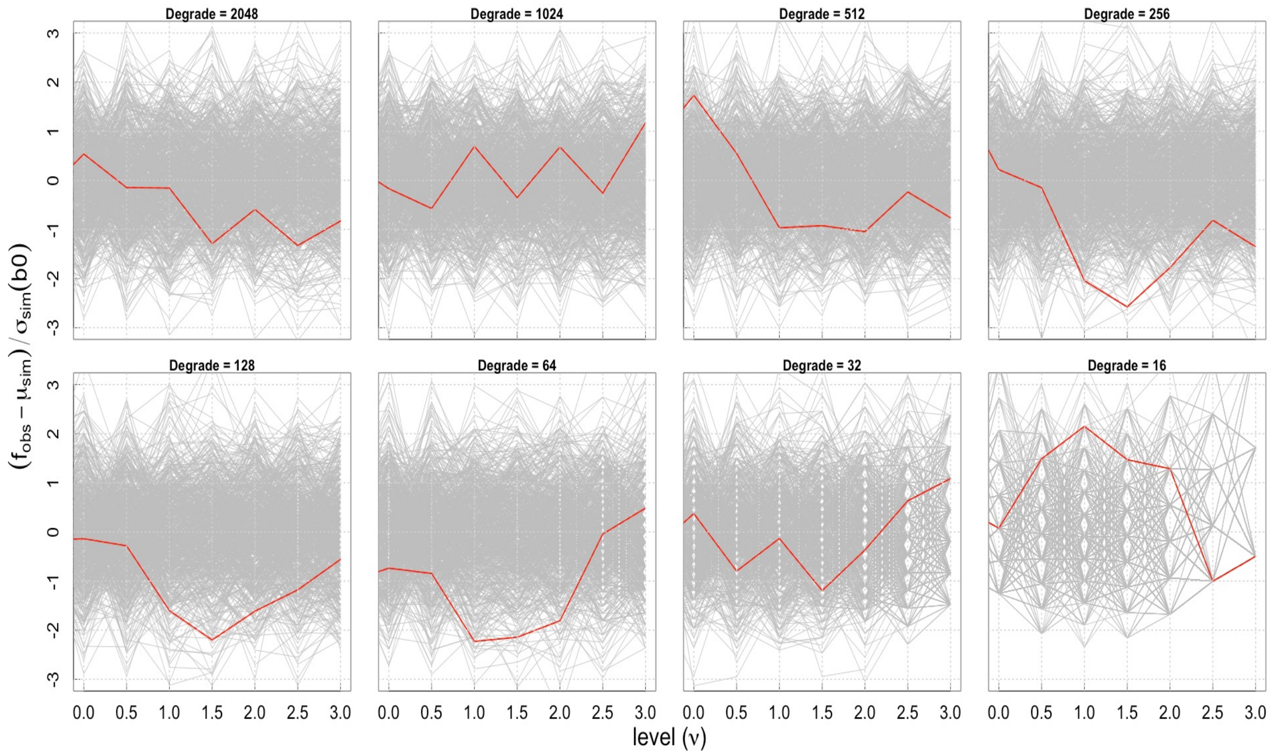
B-mode maps

Loops(betti 1)

Relative homology – `NPIPE` E-mode							
		χ^2 (empirical)			χ^2 (theoretical)		
Res	FWHM	b_0	b_1	$\mathsf{EC}_{\mathrm{rel}}$	b_0	b_1	$\mathsf{EC}_{\mathrm{rel}}$
threshold = 0.90							
2048	5	0.600	0.665	0.457	0.601	0.671	0.470
1024	10	0.498	0.815	0.495	0.504	0.806	0.506
512	20	0.173	0.638	0.215	0.180	0.631	0.214
256	40	0.582	0.187	0.388	0.596	0.190	0.407
128	80	0.447	0.992	0.705	0.476	0.989	0.699
64	160	0.953	0.962	0.993	0.936	0.960	0.991
32	320	0.358	0.027	0.018	0.371	0.016	0.015
16	640	0.670	0.130	0.088	0.764	0.198	0.058

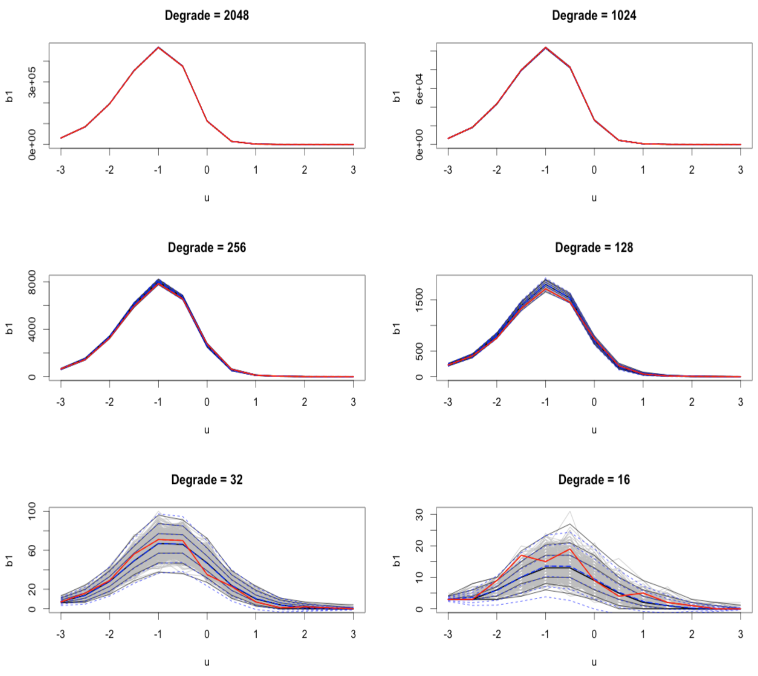
Relative homology – `NPIPE` B-mode							
		χ^2 (empirical)			χ^2 (theoretical)		
Res	FWHM	b_0	b_1	$\mathsf{EC}_{\mathrm{rel}}$	b_0	b_1	$\mathsf{EC}_{\mathrm{rel}}$
threshold = 0.90							
2048	5	0.140	0.708	0.473	0.154	0.708	0.470
1024	10	0.210	0.377	0.315	0.226	0.381	0.324
512	20	**0.000**	0.100	**0.008**	**0.000**	0.097	**0.006**
256	40	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
128	80	**0.005**	0.062	**0.002**	**0.003**	0.053	**0.000**
64	160	0.817	0.172	0.777	0.839	0.183	0.775
32	320	0.587	0.475	0.753	0.594	0.487	0.760
16	640	0.423	0.565	0.540	0.453	0.683	0.569

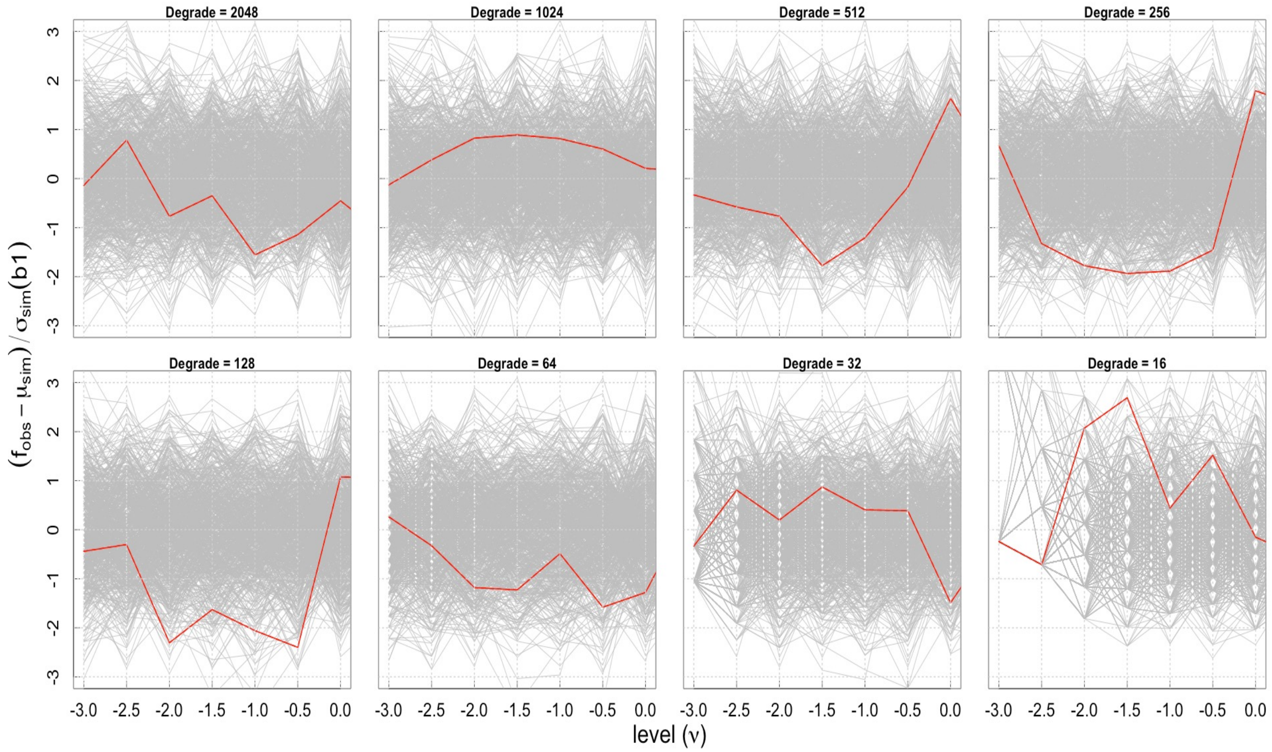




Q maps

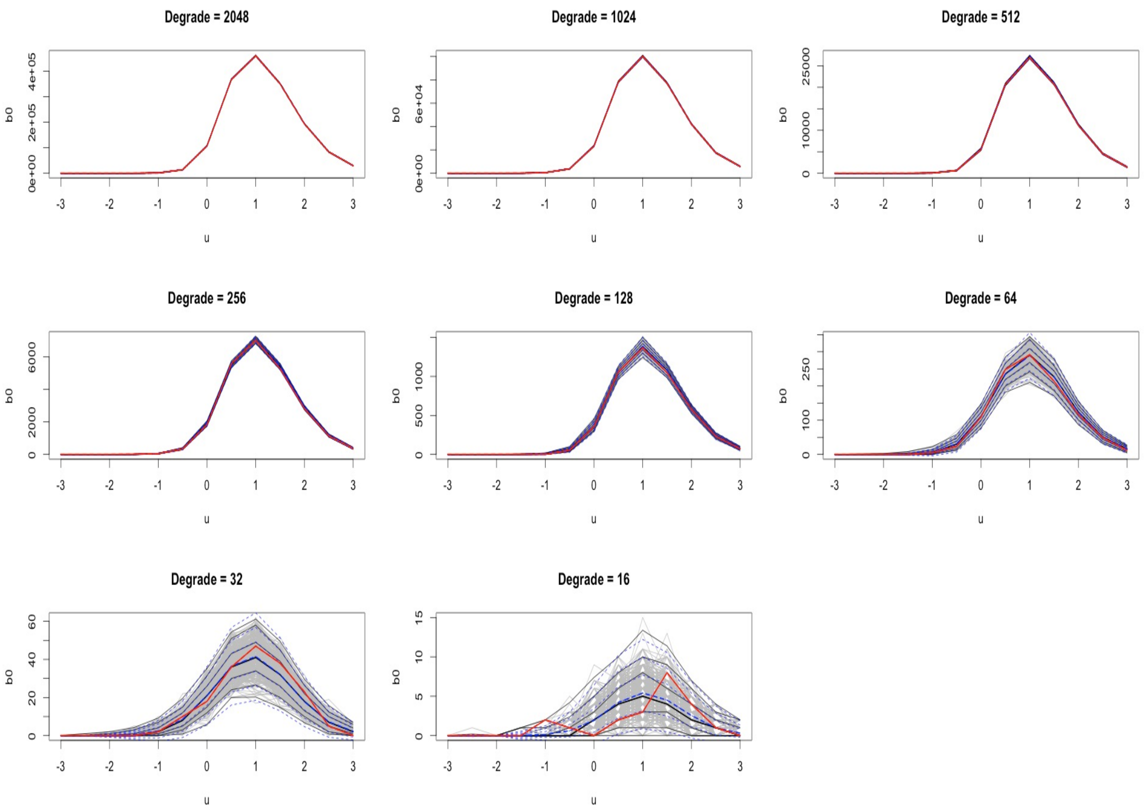
Isolated objects (betti 0)

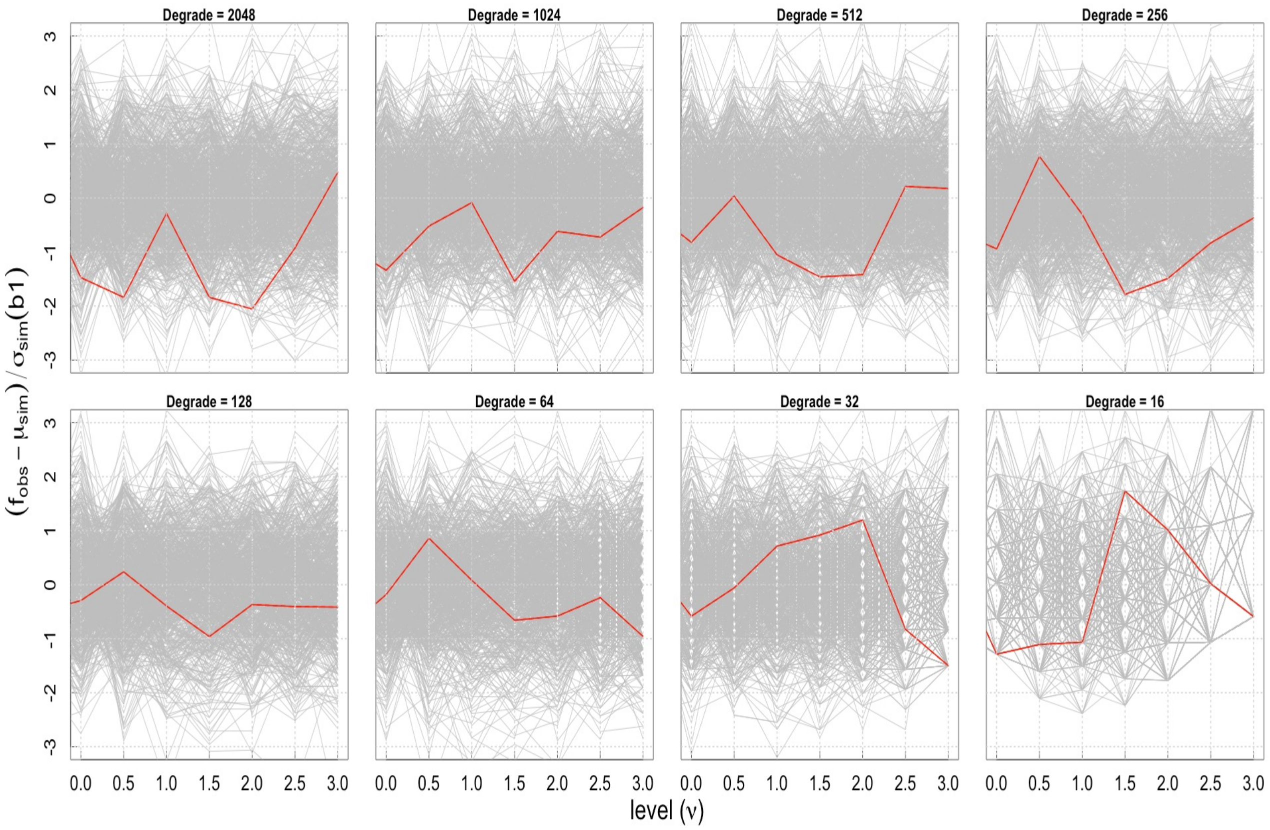




Q maps

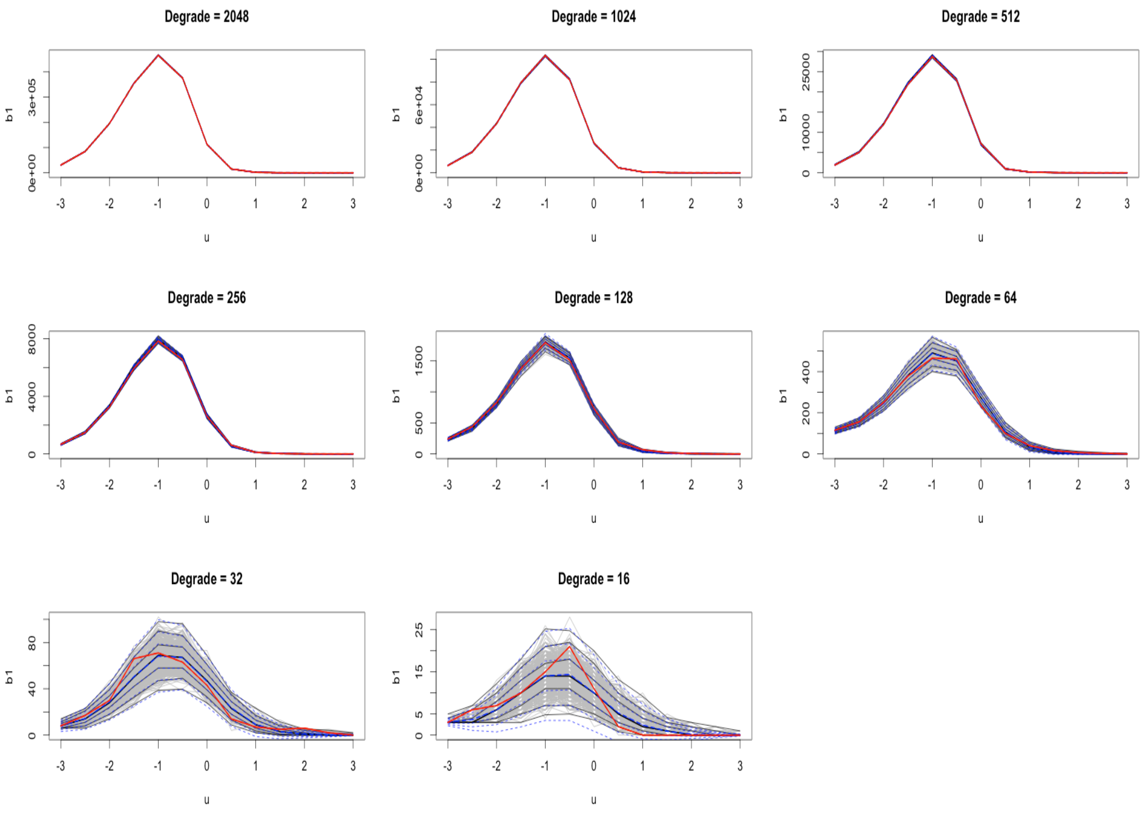
Loops(betti 1)

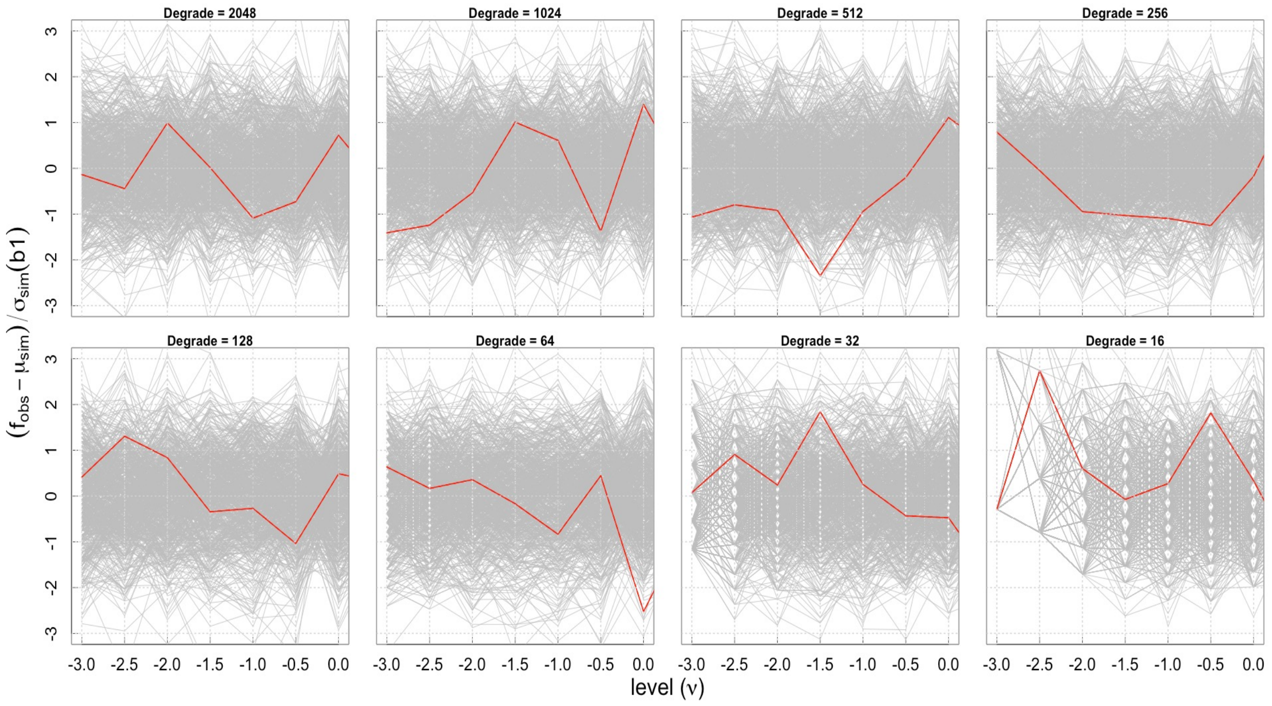




U maps

Isolated objects (betti 0)





U maps
Loops(betti 1)

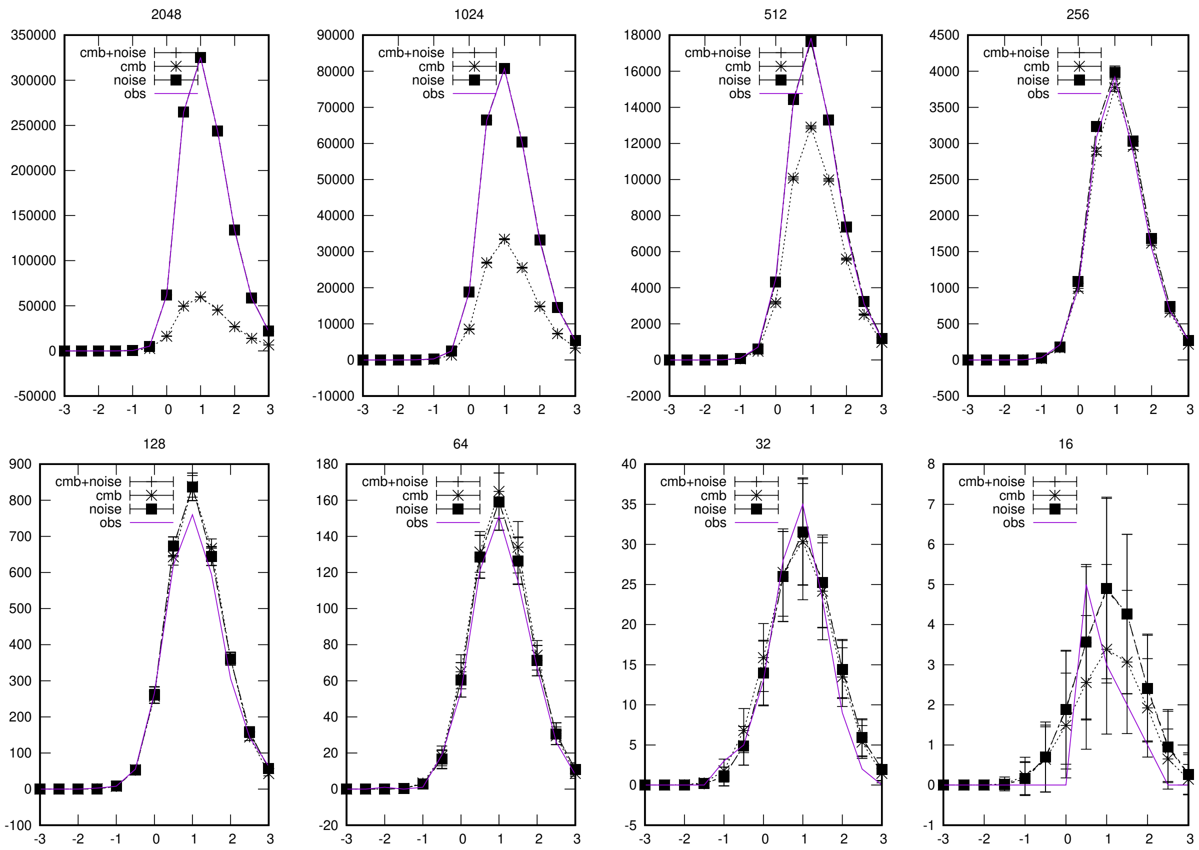
Relative homology										
		χ^2 (empirical)			χ^2 (theoretical)			Tukey Depth		
Res	FWHM	b_0	b_1	EC_{rel}	b_0	b_1	EC_{rel}	b_0	b_1	EC_{rel}
threshold = 0.90										
2048	5	0.578	0.332	0.583	0.584	0.334	0.574	0.540	0.360	0.485
1024	10	0.517	0.957	0.838	0.503	0.950	0.830	0.513	0.948	0.870
512	20	0.480	0.363	0.812	0.468	0.376	0.799	0.487	0.338	0.795
256	40	**0.038**	0.102	0.090	**0.048**	0.097	0.088	**0.000**	**0.000**	**0.000**
128	80	0.178	0.168	0.430	0.201	0.169	0.427	**0.000**	**0.000**	0.462
64	160	0.103	0.503	0.707	0.102	0.494	0.698	**0.000**	0.505	0.750
32	320	0.802	0.587	0.825	0.781	0.617	0.828	0.780	0.558	0.798
16	640	0.178	0.103	0.228	0.177	0.061	0.217	**0.000**	**0.000**	**0.000**
summary	NA	0.582	**0.017**	**0.057**	0.569	**0.011**	**0.060**	**0.000**	**0.000**	**0.000**

Table 1: Table displaying the two-tailed p-values for relative homology obtained from parametric (Mahalanobis distance) and non-parametric (Tukey depth) tests, for different resolutions and smoothing scales for the `NPIPE` dataset Q component. The last entry is the p-value for the summary statistic computed across all resolutions. Marked in boldface are p-values 0.05 or smaller.

Relative homology										
		χ^2 (empirical)			χ^2 (theoretical)			Tukey Depth		
Res	FWHM	b_0	b_1	EC_{rel}	b_0	b_1	EC_{rel}	b_0	b_1	EC_{rel}
threshold = 0.90										
2048	5	**0.045**	0.768	0.295	**0.049**	0.767	0.309	0.000	0.785	**0.000**
1024	10	0.593	0.275	0.192	0.620	0.264	0.171	0.625	0.252	**0.000**
512	20	0.593	0.150	0.283	0.602	0.145	0.285	0.577	**0.000**	**0.000**
256	40	0.298	0.747	0.932	0.314	0.747	0.925	0.335	0.730	0.887
128	80	0.933	0.700	0.760	0.936	0.690	0.755	0.953	0.567	0.832
64	160	0.823	**0.040**	0.298	0.831	**0.043**	0.302	0.787	**0.000**	**0.000**
32	320	0.682	0.435	0.447	0.679	0.439	0.444	0.502	0.362	**0.000**
16	640	0.312	0.177	0.667	0.320	0.139	0.700	0.255	0.355	0.827
summary	NA	**0.023**	**0.008**	**0.010**	**0.022**	**0.003**	**0.011**	**0.000**	**0.000**	**0.000**

Table 2: Table displaying the two-tailed p-values for relative homology obtained from parametric (Mahalanobis distance) and non-parametric (Tukey depth) tests, for different resolutions and smoothing scales for the `NPIPE` dataset U component. The last entry is the p-value for the summary statistic computed across all resolutions. Marked in boldface are p-values 0.05 or smaller.

Signal-to-Noise



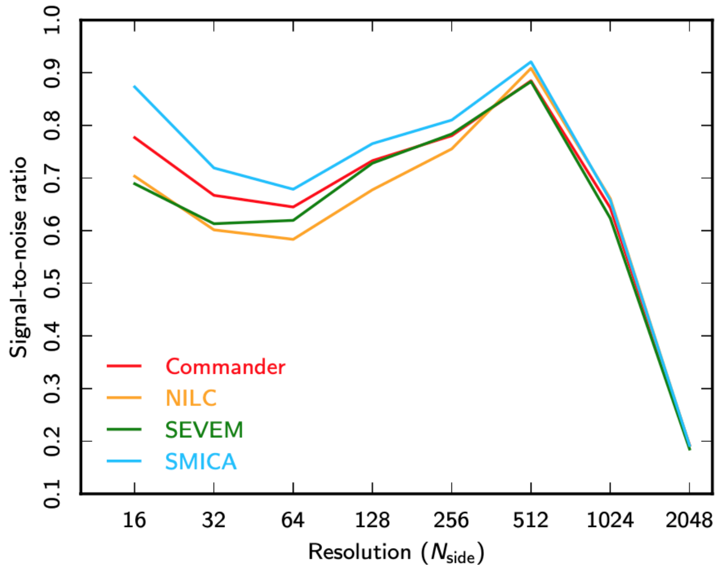


Fig. 5. Signal-to-noise ratio for the variance estimator in polarization for `Commander` (red), `NILC` (orange), `SEVEM` (green), and `SMICA` (blue), obtained by comparing the theoretical variance from the *Planck* FFP10 fiducial model with an MC noise estimate (right-hand term of Eq. (7)). Note that the same colour scheme for distinguishing the four component-separation maps is used throughout this paper.

Conclusions

- Topology and geometry powerful means of characterizing data.

- CMB data exhibits mild to significant anomaly in both temperature and polarization data.
    - Temperature: full sky (large-scale); hemisphere (Degree-scale)
    - Polarization: full-sky (degree-scale)

- Results are based on legitimate mathematical foundations, and not a case of over-exploitation of data through tailor-made statistics.

- Hints of violation of cosmological principal, or other late time mechanisms at play (including doppler boosting/dipolar modulation)?