Defects, branes and 3D lattice model

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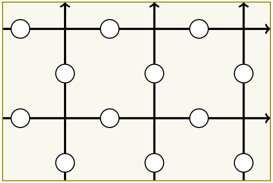
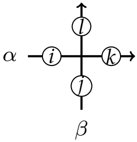
6-vertex model in classical statistical mechanics, placed on $\mathbb{T}^2$:



Each line $\mathcal{L}_\alpha$ of the lattice has a spectral parameter $z_\alpha \in \mathbb{C}$

Spins $\bigcirc$ on edges interact at vertices, with Boltzmann weights

$$R(z_\alpha - z_\beta)_{ij}^{kl} =$$



$$\in \mathbb{C}\,,$$

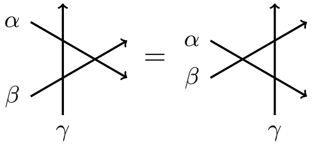
Partition function

$$Z = \sum_{\text{configs}} \prod_{\text{vertices}} R$$

Simplest case: $\bigcirc \in \{+, -\}$ and only 6 configs allowed

6-vertex model R-matrix satisfies Yang–Baxter equation

$$R_{\alpha\beta}(z_\alpha - z_\beta)R_{\alpha\gamma}(z_\beta - z_\gamma)R_{\beta\gamma}(z_\beta - z_\gamma)$$
$$= R_{\beta\gamma}(z_\beta - z_\gamma)R_{\alpha\gamma}(z_\beta - z_\gamma)R_{\alpha\beta}(z_\alpha - z_\beta)$$



YBE implies 6-vertex model is integrable.

6-vertex model ↔ Heisenberg XXZ quantum spin chain

Commuting conserved charges on the spin chain Hilbert space

6-vertex model appears in many supersymmetric QFTs:

- 2D $\mathcal{N} = (2,2)$ gauge theories [Nekrasov–Shatashvili]
- 4D $\mathcal{N} = 2$ gauge theories [Nekrasov–Shatashvili, Chen–Dorey–Hollowood–Lee]
- 3D $\mathcal{N} = 4$ gauge theories

  [Bullimore–Dimofte–Gaiotto, Braverman–Finkelberg–Nakajima, Dedushenko–Gaiotto]
- 4D $\mathcal{N} = 1$ gauge theories (with surface defects)

  [Gaiotto–Rastelli–Razamat, Gadde–Gukov, Gaiotto–Razamat, Maruyoshi–Y, Y]
- 4D $\mathcal{N} = 1$ gauge theories from brane tilings

  [Spiridonov, Bazhanov–Sergeev, Yamazaki, Y, ...]
- 4D Chern–Simons theory (= deformed 6D MSYM)

  [Costello, Costello–Witten–Yamazaki]

All of these have brane construction in string theory.

They are related by string dualities. [Costello–Y]

Today: Lift this story to one dimension higher

- Brane setups in string theory $\to$ brane setup in M-theory
- 6-vertex model $\to$ integrable 3D lattice model

The two sides are related by dualities

The 3D model reduces to 6v model via reduction on circle

Underlying principle:

topological quantum field theory

$+$

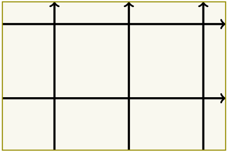
defects

$+$

"extra dimensions"

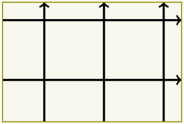
Place a 2D TQFT with line defects on $\mathbb{T}^2$.

Wrap line defects $\mathcal{L}_\alpha$, $\alpha = 1, \ldots, M+N$, around 1-cycles of $\mathbb{T}^2$:



Task: calculate the correlator of the $M \times N$ lattice of line defects

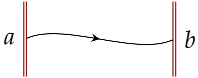
$$\left\langle \prod_\alpha \mathcal{L}_\alpha \right\rangle = Z \left(\quad \right)$$



Strategy: chop up the torus into squares.

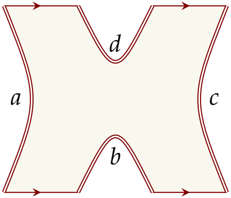
We use the formalism of open-closed TQFT.

Open string stretched between branes a and b



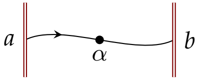
Hilbert space V_{ab}

Scattering of two open strings



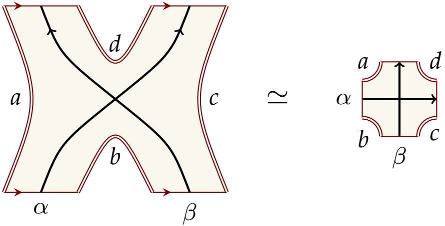
$: V_{ab} \otimes V_{bc} \to V_{ad} \otimes V_{dc}$

Open string with particle α attached, between branes a and b:



Hilbert space $V_{ab,\alpha}$

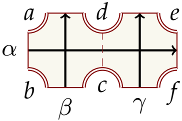
Scattering



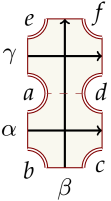
defines R-matrix

$$\check{R}_{\alpha\beta}\begin{pmatrix} a & d \\ b & c \end{pmatrix} : V_{ab,\alpha} \otimes V_{bc,\beta} \to V_{ad,\beta} \otimes V_{dc,\alpha}$$

Gluing pieces = composition of R-matrices:

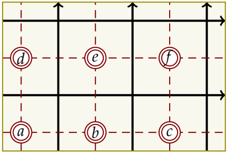


$$= \check{R}_{\alpha\gamma}\begin{pmatrix} d & e \\ c & f \end{pmatrix} \circ_{V_{dc,\alpha}} \check{R}_{\alpha\beta}\begin{pmatrix} a & d \\ b & c \end{pmatrix}$$



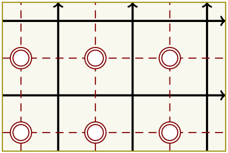
$$= \check{R}_{\alpha\gamma}\begin{pmatrix} d & e \\ c & f \end{pmatrix} \circ_{V_{ad,\beta}} \check{R}_{\alpha\beta}\begin{pmatrix} a & d \\ b & c \end{pmatrix}$$

Try to reconstruct the torus:

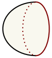


$$= \mathrm{Tr}\, \overset{\mathrm{Tr}}{\begin{pmatrix} \check{R} & \check{R} & \cdots & \check{R} \\ \vdots & \vdots & \ddots & \vdots \\ \check{R} & \check{R} & \cdots & \check{R} \\ \check{R} & \check{R} & \cdots & \check{R} \end{pmatrix}}$$

We got a slice of Swiss cheese!

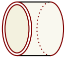


To fill the holes, we make use of the vacuum state



$: 1 \mapsto |1\rangle$

and boundary states

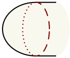


a $: 1 \mapsto |a\rangle$

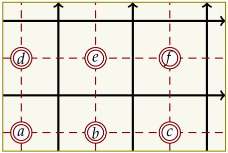
Assume: we have enough kinds of branes so that

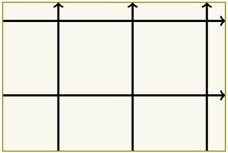
$$|1\rangle = \sum_a c_a |a\rangle$$

Then

$$\sum_a c_a \Big(a$$  $$\Big) = \sum_a c_a \Big(|a\rangle$$  $$\Big) = |1\rangle$$  $$=$$ 

Summation over branes fill the holes:

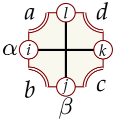
$$\sum_{a,b,c,d,e,f}$$ 

$$=$$ 

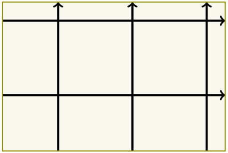
(We have set $c_a = 1$ by redefining $\check{R}$.)

Represent matrix elements of $\check{R}$ by

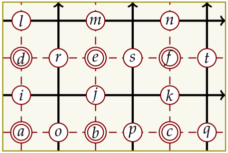
$$\check{R}_{\alpha\beta}\begin{pmatrix} a & d \\ b & c \end{pmatrix}^{lk}_{ij} =$$



We have found



$$= \sum_{a,b,c,d,e,f} \; \sum_{\substack{i,j,k,l,m,n \\ o,p,q,r,s,t}}$$



RHS is the partition function of a 2D lattice model!

$$\left\langle \prod \mathcal{L}_i \right\rangle = Z_{\text{2D lattice model}}$$

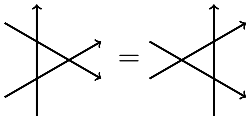
Spins ○ and ◎ interact at vertices with Boltzmann weights $\check{R}$.

Now suppose there are hidden extra dimensions C:

- Spacetime is really $\mathbb{T}^2 \times C$
- The theory is topological on $\mathbb{T}^2$ but not on C
- Line defects $\mathcal{L}_\alpha$ are located at points $u_\alpha \in C$

Implications:

- $\mathcal{L}_\alpha = \mathcal{L}_\alpha(u_\alpha)$, $\check{R}_{\alpha\beta} = \check{R}_{\alpha\beta}(u_\alpha, u_\beta)$
- No singularities when lines are deformed: YBE holds



Conclusion: TQFT + defects + extra dimensions imply the integrability of the lattice model

Example: 4D Chern–Simons theory

- Action:

$$S = \int_{\mathbb{T}^2 \times C} \mathrm{d}z \wedge \mathrm{CS}(A)\,, \quad A = A_x\,\mathrm{d}x + A_y\,\mathrm{d}y + A_{\bar{z}}\,\mathrm{d}\bar{z}$$

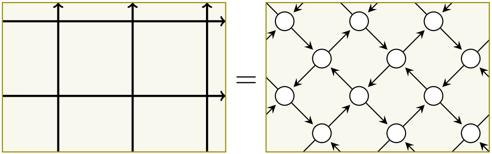
  where $C = \mathbb{C}$, cylinder or torus
- Topological on $\mathbb{T}^2$, holomorphic on C
- Wilson lines $\mathcal{L}_\alpha$ wrapping 1-cycles $\times \{z_\alpha\} \subset \mathbb{T}^2 \times C$

$\check{R}_{\alpha\beta}(z_\alpha, z_\beta) = \check{R}_{\alpha\beta}(z_\alpha - z_\beta)$ is a rational/trigonometric/elliptic solution of YBE.

For $G = \mathrm{SU}(2)$, $C = \mathbb{C}^\times$ and $\mathcal{L}_\alpha$ Wilson lines in the fundamental rep, we get the 6-vertex model.

Example: Brane tiling theories

- N D5s on $\mathbb{T}^2 \times E \times \mathbb{R}^2 \subset \mathbb{T}^2 \times E \times \mathbb{R}^6$, where E is a torus
- Line defects: NS5s intersecting D5s along 1-cycles $\times E \times \mathbb{R}^2$
- TQFT structure: partition function is a supersymmetric index, invariant under deformations
- Extra dimensions: S-duality + T-duality on E turn NS5s into D3s supported at points on $E^\vee$
- Lattice model: 4D $\mathcal{N} = 1$ quiver



Seiberg duality implies YBE.

The same argument holds in higher dimensions.

Setup:

- $(m+n)$-dim QFT on $\mathbb{T}^m \times M_n$
- Topological on $\mathbb{T}^m$ but not on M_n
- Defects $\mathcal{D}_\alpha$ making a lattice in $\mathbb{T}^m$, separated in M_n

This structure implies

- Correlator is the partition function of a lattice model:

$$\left\langle \prod_\alpha \mathcal{D}_\alpha \right\rangle = Z_{\text{lattice model}}$$

- Integrability: m-dim analog of Yang–Baxter equation

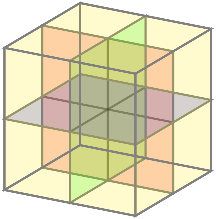
We will construct an integrable 3D lattice model using branes.

M5-brane system in M-theory:

	$\mathbb{R}_0$	$\mathbb{S}_1$	$\mathbb{S}_2$	$\mathbb{S}_3$	$\mathbb{R}_4$	$\mathbb{R}_5$	$\mathbb{R}_6$	$\mathbb{R}_7$	$\mathbb{R}_8$	$\mathbb{R}_9$	$\mathbb{R}_\natural$
$M5_{1+}$	$-$	$\cdot$	$-$	$-$	$-$	$\cdot$	$\cdot$	$-$	$-$	$\circ$	$\circ$
$M5_{2+}$	$-$	$-$	$\cdot$	$-$	$\cdot$	$-$	$\cdot$	$-$	$-$	$\circ$	$\circ$
$M5_{3+}$	$-$	$-$	$-$	$\cdot$	$\cdot$	$\cdot$	$-$	$-$	$-$	$\circ$	$\circ$

- $-$: M5 extends in that direction
- $\cdot$: M5 is located at any point
- $\circ$: M5 is located at the origin

M5s make an $L \times M \times N$ cubic lattice in $\mathbb{T}^3_{123}$



	$\mathbb{R}_0$	$\mathbb{S}_1$	$\mathbb{S}_2$	$\mathbb{S}_3$	$\mathbb{R}_4$	$\mathbb{R}_5$	$\mathbb{R}_6$	$\mathbb{R}_7$	$\mathbb{R}_8$	$\mathbb{R}_9$	$\mathbb{R}_\natural$
$\mathrm{M5}_{1+}$	—	·	—	—	—	·	·	—	—	○	○
$\mathrm{M5}_{2+}$	—	—	·	—	·	—	·	—	—	○	○
$\mathrm{M5}_{3+}$	—	—	—	·	·	·	—	—	—	○	○

The system has 4 supercharges. Two of them, $Q_\pm$, satisfy[1]

$$Q_+^2 + Q_-^2 = \frac{1}{2}(H - Z^{(2)}_{14} - Z^{(2)}_{25} - Z^{(2)}_{36})$$
$$[Q_\pm, J_{78} - J_{9\natural}] = 0$$

- H: translation generator in $\mathbb{R}_0$
- $Z^{(2)}$: 2-form charge
- J_{78} & $J_{9\natural}$: rotation generators on $\mathbb{R}^2_{78}$ & $\mathbb{R}^2_{9\natural}$

[1] RHS of the first equation also contains components of a 5-form charge but they are irrelevant in the present discussion.

	$\mathbb{S}_0$	$\mathbb{S}_1$	$\mathbb{S}_2$	$\mathbb{S}_3$	$\mathbb{R}_4$	$\mathbb{R}_5$	$\mathbb{R}_6$	$\mathbb{R}_7$	$\mathbb{R}_8$	$\mathbb{R}_9$	$\mathbb{R}_\natural$
$M5_{1+}$	—	$\cdot$	—	—	—	$\cdot$	$\cdot$	—	—	$\circ$	$\circ$
$M5_{2+}$	—	—	$\cdot$	—	$\cdot$	—	$\cdot$	—	—	$\circ$	$\circ$
$M5_{3+}$	—	—	—	$\cdot$	$\cdot$	$\cdot$	—	—	—	$\circ$	$\circ$

Compactify $\mathbb{R}_0$ with twist:

$$\mathbb{R}_0 \times \mathbb{R}^2_{78} \times \mathbb{R}^2_{9\natural} \to [0,\beta] \times \mathbb{R}^2_{78} \times \mathbb{R}^2_{9\natural}/(0,z,w) \sim (\beta, e^{i\theta}z, e^{-i\theta}w)$$

The partition function is the supersymmetric index

$$Z_{\mathrm{M5}} = \mathrm{Tr}_{\mathcal{H}}\big((-1)^F e^{\mathrm{i}\theta(J_{78}-J_{9\natural})} e^{-\beta H}\big)$$

It's a q-series:

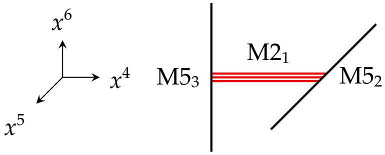
$$Z_{\mathrm{M5}} = \sum_{j=-\infty}^{\infty} q^j \, \mathrm{Tr}_{\mathcal{H}^j}\big((-1)^F e^{-\beta H}\big) ,$$

$$q = e^{\mathrm{i}\theta} , \quad \mathcal{H}^j = \ker(J_{78} - J_{9\natural} - j)$$

By supersymmetry, only BPS states contribute to Z_{M5}:

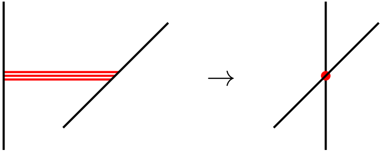
$$|\Psi\rangle \text{ is BPS} \iff Q_{\pm}|\Psi\rangle = 0 \iff H|\Psi\rangle = (Z_{14}^{(2)} + Z_{25}^{(2)} + Z_{36}^{(2)})|\Psi\rangle$$

BPS states are M2s stretched between M5s:



	$\mathbb{S}_0$	$\mathbb{S}_1$	$\mathbb{S}_2$	$\mathbb{S}_3$	$\mathbb{R}_4$	$\mathbb{R}_5$	$\mathbb{R}_6$	$\mathbb{R}_7$	$\mathbb{R}_8$	$\mathbb{R}_9$	$\mathbb{R}_\natural$
M5$_{1+}$	—	•	—	—	—	•	•	—	—	○	○
M5$_{2+}$	—	—	•	—	•	—	•	—	—	○	○
M5$_{3+}$	—	—	—	•	•	•	—	—	—	○	○
M2$_1$	—	—	•	•	•—•	•	•	○	○	○	○
M2$_2$	—	•	—	•	•	•—•	•	○	○	○	○
M2$_3$	—	•	•	—	•	•	•—•	○	○	○	○

Adjust M5 positions so that they go through the origin of $\mathbb{R}^3_{456}$:



Then BPS M2s have zero area and $H = 0$:

$$Z_{\text{M5}} = \sum_{j=-\infty}^{\infty} q^j \operatorname{Tr}_{\mathcal{H}^j}\left((-1)^F e^{-\beta H}\right) \to \mathring{Z}_{\text{M5}} = \sum_{j=-\infty}^{\infty} q^j \operatorname{Tr}_{\mathcal{H}^j_{\text{BPS}}}(-1)^F$$

The coefficient of q^j is the Witten index of $\mathcal{H}^j$.

Generalization:

- Replace $\mathbb{T}^3_{123} \to M_3$ and $\mathbb{T}^3_{123} \times \mathbb{R}^3_{456} \to T^*M_3$
- Wrap M5s on conormals $N^*\Sigma_2 \subset T^*M_3$

BPS M2s still have zero area; $\mathring{Z}_{\mathrm{M5}}$ is still given by Witten indices.

Witten indices are invariant under deformations of M_3 and M5 worldvolumes

We get a structure of a TQFT on M_3 + surface defects.[2]

[2] To really get a QFT, replace $\mathbb{R}^4_{789\natural}$ with a Taub–NUT and reduce on the circle fiber. Then we get a D6-brane which produces 7D SYM.

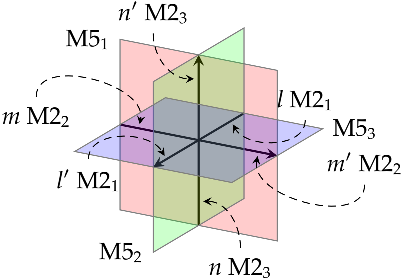
For $M_3 = \mathbb{T}^3$ and surface defects making a cubic lattice,

$$\mathring{Z}_{\text{M5}} = Z_{\text{3D lattice model}}$$

Spin variables: #M2s stretched between M5s (span Fock spaces)

Local Boltzmann weights: 3D R-matrix $\mathcal{R}$

$$\mathcal{R}^{l'm'n'}_{lmn} =$$



Partition function:

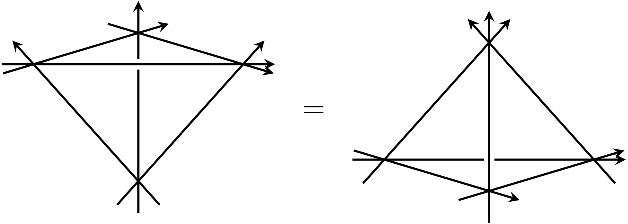
$$\mathring{Z}_{\text{M5}} = \sum_{\text{spin configs}} \prod_{v \in \text{vertices}} \mathcal{R}^{l'_v m'_v n'_v}_{l_v m_v n_v}$$

	$\mathbb{S}_0$	$\mathbb{S}_1$	$\mathbb{S}_2$	$\mathbb{S}_3$	$\mathbb{R}_4$	$\mathbb{R}_5$	$\mathbb{R}_6$	$\mathbb{R}_7$	$\mathbb{R}_8$	$\mathbb{R}_9$	$\mathbb{R}_\natural$
$\mathrm{M5}_{1+}$	—	•	—	—	—	•	•	—	—	◦	◦
$\mathrm{M5}_{2+}$	—	—	•	—	•	—	•	—	—	◦	◦
$\mathrm{M5}_{3+}$	—	—	—	•	•	•	—	—	—	◦	◦

M5s can be shifted in $\mathbb{R}^3_{456}$ and $\mathring{Z}_{\mathrm{M5}}$ still makes sense. Witten indices are invariant, hence so is $\mathring{Z}_{\mathrm{M5}}$.

$\mathbb{R}^3_{456}$ are extra dimensions!

3D analog of YBE is Zamolodchikov's tetrahedron equation:



Difference with 2D case: 3D R-matrix has no spectral parameter

This is because $\mathring{Z}$ is independent of the M5 positions in $\mathbb{R}^3_{456}$.

The original partition function

$$Z_{\mathrm{M5}} = \sum_{j=-\infty}^{\infty} q^j \, \mathrm{Tr}_{\mathcal{H}^j_{\mathrm{BPS}}} \left((-1)^F e^{-\beta (Z^{(2)}_{14} + Z^{(2)}_{25} + Z^{(2)}_{36})} \right)$$

does depend on these positions since $Z^{(2)}$ does.

It turns out

$$Z_{\mathrm{M5}} = Z_{\text{3D lattice model with twisted B.C.}}$$

M5 positions in $\mathbb{R}^3_{456}$ determine twist of boundary conditions.

We want to identify the 3D R-matrix.

The brane construction implies a few key properties:

- Normalization: $R_{000}^{000} = 1$
- Parity reversal symmetry: $R_{lmn}^{l'm'n'} = R_{nml}^{n'm'l'}$
- Involutivity: $R = R^{-1}$
- Charge conservation: $R_{lmn}^{l'm'n'} = 0$ unless $l + m = l' + m'$ and $m + n = m' + n'$

The most remarkable is the behavior under reduction to 2D

	$\mathbb{S}_0$	$\mathbb{S}_1$	$\mathbb{S}_2$	$\mathbb{S}_3$	$\mathbb{R}_4$	$\mathbb{R}_5$	$\mathbb{R}_6$	$\mathbb{R}_7$	$\mathbb{R}_8$	$\mathbb{R}_9$	$\mathbb{R}_\natural$
L M5_{1+}	—	•	—	—	—	○	○	—	—	○	○
M M5_{2+}	—	—	•	—	○	—	○	—	—	○	○
N M5_{3+}	—	—	—	•	○	○	—	—	—	○	○

Reduction on $\mathbb{S}_0$ + T-duality on $\mathbb{S}_3$:

	$\mathbb{S}_1$	$\mathbb{S}_2$	$\check{\mathbb{S}}_3$	$\mathbb{R}_4$	$\mathbb{R}_5$	$\mathbb{R}_6$	$\mathbb{R}_7$	$\mathbb{R}_8$	$\mathbb{R}_9$	$\mathbb{R}_\natural$
L D3_{1+}	•	—	•	—	•	•	—	—	○	○
M D3_{2+}	—	•	•	•	—	•	—	—	○	○
N D5_{3+}	—	—	—	•	•	—	—	—	○	○

N D5s produce 6D MSYM with $G = \text{U}(N)$, deformed due to twisting of $\mathbb{S}_0 \times \mathbb{R}^2_{78} \times \mathbb{R}^2_{9\natural}$ ("Ω-deformation").

	$\mathbb{S}_1$	$\mathbb{S}_2$	$\check{\mathbb{S}}_3$	$\mathbb{R}_4$	$\mathbb{R}_5$	$\mathbb{R}_6$	$\mathbb{R}_7$	$\mathbb{R}_8$	$\mathbb{R}_9$	$\mathbb{R}_\natural$
L $\mathrm{D3}_{1+}$	•	—	•	—	•	•	—	—	○	○
M $\mathrm{D3}_{2+}$	—	•	•	•	—	•	—	—	○	○
N $\mathrm{D5}_{3+}$	—	—	—	•	•	—	—	—	○	○

BPS sector of the 6D theory is 4D CS on $\mathbb{T}^2_{12} \times \check{\mathbb{S}}_3 \times \mathbb{R}_6$. [Costello–Y]

Each D3 creates a line defect with Hilbert space

$$\text{Fock space}^{\otimes N} = \bigoplus \text{symmetric tensor reps of } \mathrm{GL}(N)$$

Crossing of two such lines gives a 2D R-matrix in this rep:

$$\mathcal{S}(z)^{\{l'_1,\dots l'_N\}\{m'_1,\dots m'_N\}}_{\{l_1,\dots l_N\}\{m_1,\dots m_N\}} = \sum_{n_1,\dots,n_N} z^{n_1} \mathcal{R}^{l'_N m'_N n_1}_{l_N m_N n_N} \cdots \mathcal{R}^{l'_3 m'_3 n_3}_{l_2 m_2 n_2} \mathcal{R}^{l'_2 m'_2 n_2}_{l_1 m_1 n_1}$$

A solution of TE that has all of these properties is known!

It was found by Kapranov–Voevodsky and Bazhanov–Sergeev:

$$\mathcal{R}_{lmn}^{l'm'n'} = \delta_{l+m}^{l'+m'} \delta_{m+n}^{m'+n'} \times \sum_{\substack{\lambda,\mu\in\mathbb{Z}_{\geq 0} \\ \lambda+\mu=m'}} (-1)^{\lambda} q^{l(n'-m)+(n+1)\lambda+\mu(\mu-n)} \frac{(q^2)_{n'+\mu}}{(q^2)_{n'}} \binom{l}{\mu}_{q^2} \binom{m}{\lambda}_{q^2}$$

where

$$(q)_n = \prod_{k=1}^{n}(1-q^k)\,, \quad \binom{m}{n}_q = \frac{(q)_m}{(q)_{m-n}(q)_n}$$

Relation to Bethe/gauge correspondence

	$\mathbb{S}_1$	$\mathbb{S}_2$	$\check{\mathbb{S}}_3$	$\mathbb{R}_4$	$\mathbb{R}_5$	$\mathbb{R}_6$	$\mathbb{R}_7$	$\mathbb{R}_8$	$\mathbb{R}_9$	$\mathbb{R}_\natural$
L D3_{1+}	•	—	•	—	•	•	—	—	○	○
M D3_{2+}	—	•	•	•	—	•	—	—	○	○
N D5_{3+}	—	—	—	•	•	—	—	—	○	○

$\downarrow$ S-duality and T-duality on $\mathbb{S}_1$

	$\mathbb{S}_1$	$\mathbb{S}_2$	$\check{\mathbb{S}}_3$	$\mathbb{R}_4$	$\mathbb{R}_5$	$\mathbb{R}_6$	$\mathbb{R}_7$	$\mathbb{R}_8$	$\mathbb{R}_9$	$\mathbb{R}_\natural$
L D4_{1+}	—	—	•	—	•	•	—	—	○	○
M D2_{2+}	•	•	•	•	—	•	—	—	○	○
N NS5_{3+}	—	—	—	•	•	—	—	—	○	○

D4–NS5 produces a 4D $\mathcal{N} = 2$ linear quiver theory on $\mathbb{T}^2_{12} \times \mathbb{R}_{78}$, connecting to 4D Bethe/gauge correspondence.

M2 point of view leads to 2D Bethe/gauge correspondence.

Relation to brane tilings

	$\mathbb{S}_0$	$\mathbb{S}_1$	$\mathbb{S}_2$	$\mathbb{S}_3$	$\mathbb{R}_4$	$\mathbb{R}_5$	$\mathbb{R}_6$	$\mathbb{R}_7$	$\mathbb{R}_8$	$\mathbb{R}_9$	$\mathbb{R}_\natural$
$M5_{1+}$	—	•	—	—	—	•	•	—	—	○	○
$M5_{2+}$	—	—	•	—	•	—	•	—	—	○	○
$M5_{3+}$	—	—	—	•	•	•	—	—	—	○	○

$\downarrow$ Reduce on $\mathbb{S}_3$, compactify $\mathbb{R}_6 \to \mathbb{S}_6$, T on $\mathbb{S}_6$ and S

	$\mathbb{S}_0$	$\mathbb{S}_1$	$\mathbb{S}_2$	$\mathbb{R}_4$	$\mathbb{R}_5$	$\check{\mathbb{S}}_6$	$\mathbb{R}_7$	$\mathbb{R}_8$	$\mathbb{R}_9$	$\mathbb{R}_\natural$
$NS5_{1+}$	—	•	—	—	•	—	—	—	○	○
$NS5_{2+}$	—	—	•	•	—	—	—	—	○	○
$D5_{3+}$	—	—	—	•	•	—	—	—	○	○

This is a brane tiling for a 4D $\mathcal{N} = 1$ theory on $\mathbb{S}_0 \times \check{\mathbb{S}}_6 \times \mathbb{R}^2_{78}$.

We can add more M5s while preserving $Q_\pm$:

	$\mathbb{S}_0$	$\mathbb{S}_1$	$\mathbb{S}_2$	$\mathbb{S}_3$	$\mathbb{R}_4$	$\mathbb{R}_5$	$\mathbb{R}_6$	$\mathbb{R}_7$	$\mathbb{R}_8$	$\mathbb{R}_9$	$\mathbb{R}_\natural$
L_+ $\mathrm{M5}_{1+}$	—	$\bullet$	—	—	—	$\bullet$	$\bullet$	—	—	$\circ$	$\circ$
M_+ $\mathrm{M5}_{2+}$	—	—	$\bullet$	—	$\bullet$	—	$\bullet$	—	—	$\circ$	$\circ$
N_+ $\mathrm{M5}_{3+}$	—	—	—	$\bullet$	$\bullet$	$\bullet$	—	—	—	$\circ$	$\circ$
L_- $\mathrm{M5}_{1-}$	—	$\bullet$	—	—	—	$\bullet$	$\bullet$	$\circ$	$\circ$	—	—
M_- $\mathrm{M5}_{2-}$	—	—	$\bullet$	—	$\bullet$	—	$\bullet$	$\circ$	$\circ$	—	—
N_- $\mathrm{M5}_{3-}$	—	—	—	$\bullet$	$\bullet$	$\bullet$	—	$\circ$	$\circ$	—	—

Spin variables: #M2s along $\mathrm{M5}_{i\sigma} \cap \mathrm{M5}_{j\sigma'} \in \begin{cases} \mathbb{Z} & (\sigma = \sigma') \\ \{0,1\} & (\sigma \neq \sigma') \end{cases}$

This gives a $\mathbb{Z}_2$-graded generalization

R-matrix with expected properties is known. [Sergeev, Yoneyama]

Related to 4D CS with gauge supergroup and Bethe/gauge correspondence for superspin chains [Ishtiaque–Moosavian–Raghavendran–Y]

Summary

- TQFT + codim-1 defects produce lattice models.
- Extra dimensions implies integrability.
- Integrable 3D lattice model can be constructed from branes.
- Related by dualities to 6-vertex models appearing in QFTs

Open problems:

- Quantitative determination of the 3D R-matrix?
- Other duality frames in which 3D lattice model appear?
- Beyond partition functions?