Introduction to the moduli space of super Riemann surfaces

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main reference

Witten 2015
Notes on holomorphic string and superstring theory measures in low genus

A Riemann surface is a 1 complex dimensional manifold.

For compact Riemann surfaces, these are equivalently projective algebraic curves.

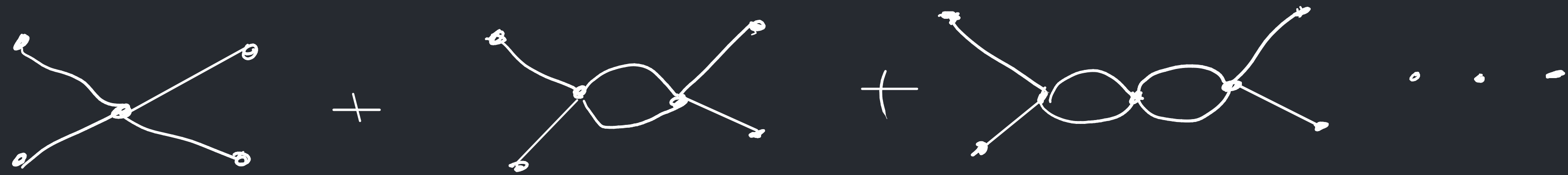
The moduli space of Riemann surfaces $\mathcal{M}_g$ is heavily studied.

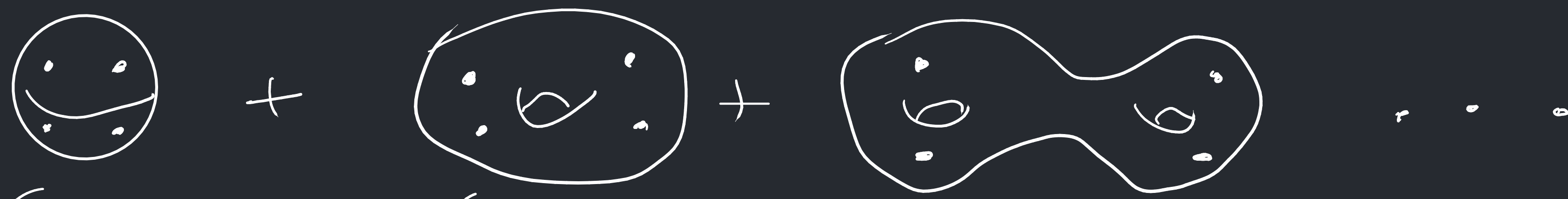
It is natural to consider punctured curves and their moduli space $\mathcal{M}_{g,n}$.

$\mathcal{M}_{g,n}$ is a Deligne-Mumford stack of dim $3g-3+n$.

Bosonic string theory is concerned with computing <u>scattering amplitudes</u>.

Roughly, the bosonic string S^1 traces out worldsheets, and all such worldsheets must be summed over.





$$c^{-2}\int_{\mathcal{M}_{0,4}} d\pi + c^{0}\int_{\mathcal{M}_{1,4}} d\pi + c^{2}\int_{\mathcal{M}_{2,4}} d\pi + \cdots$$

The measure used is the Polyakov measure.

$$Z = \sum_{g=0}^{\infty} c^{2g-2} \int_{\text{metrics}(C)} D\gamma \int_{\text{maps}(C\to M)} Dx \; e^{-J(x,\gamma)}$$

$$J(x,\gamma) = \int_C d^2z \sqrt{|\gamma|}\, \gamma^{ab} \partial_a x^\mu \partial_b x^\mu$$

$$Z = \sum_{g=0}^{\infty} c^{2g-2} \int_{\mathcal{M}_g} \underbrace{d\nu \; \frac{\det{}' \Delta_2}{(\det{}'\Delta_0)^{13}}}_{d\pi_g}$$

Theorem (Belavin-Knizhnik)

$$\boxed{d\pi_g = h(\mu_g \wedge \bar{\mu}_g)}$$

$h : \lambda_1^{-13} \times \overline{\lambda_1}^{-13} \to \mathcal{O}$

μ_g is a holomorphic object

Mumford form

μ_g is a horizontal section of $\lambda_2 \otimes \lambda_1^{-13}$ on $\mathcal{M}_g$

A <u>$m|n$ dimensional complex supermanifold</u> is a m dimensional complex manifold with an expanded structure sheaf locally isomorphic to

$$\mathcal{O}_{\mathbb{C}^m} \otimes \bigwedge(\xi_1, \xi_2, \dots \xi_n) \qquad (\mathbb{Z}/2\mathbb{Z} \text{ graded})$$

A <u>super Riemann surface</u> (or SUSY curve) is a $1|1$ dimensional supermanifold Σ with a maximally nonintegrable distrabution $\mathcal{D} \subset T_\Sigma$ such that $[\mathcal{D}, \mathcal{D}] \cong T_\Sigma/\mathcal{D}$.

$$0 \longrightarrow \mathcal{D} \longrightarrow T_\Sigma \longrightarrow T_\Sigma/\mathcal{D} \cong \mathcal{D}^2 \longrightarrow 0$$

Locally, we can always find <u>superconformal coordinates</u> $(z|\xi)$ such that $\mathcal{D} = \left\langle \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial z} \right\rangle$.

One of the most reliable ways to construct SRSs is by gluing charts such that

$$\left(\frac{\partial}{\partial \theta_1} + \theta_1 \frac{\partial}{\partial z_1}\right) = \left(\frac{\partial}{\partial \theta_1} + \theta_1 \frac{\partial}{\partial z_1}\right)(\theta_2) \cdot \left(\frac{\partial}{\partial \theta_2} + \theta_2 \frac{\partial}{\partial z_2}\right)$$

(the distribution $\mathcal{D}$ glues)

<u>For example:</u>

$\boxed{\mathbb{P}^{1|1}_{\mathbb{C}}}$ glue $\underset{(z_1 | \theta_1)}{\mathbb{C}^{1|1}}$ and $\underset{(z_2 | \theta_2)}{\mathbb{C}^{1|1}}$

$$z_2 = \frac{1}{z_1} \qquad \theta_2 = \frac{\theta_1}{z_1}$$

The moduli space of SRSs $\mathfrak{M}_g$ may also be considered with punctures, but now they come in two types:

Neveu-Schwarz punctures and Ramond punctures

$$\dim(\mathfrak{M}_g) = (3g-3+n_{NS}+n_R \,|\, 2g-2+n_{NS}+\tfrac{1}{2}n_R)$$

The moduli space of SRSs plays the same role in superstring theory as the classical moduli space in bosonic string theory.

And further the measure analogously may be expressed in terms of the super Mumford form,

μ_g is the canonical horizontal section of $\lambda_{\frac{3}{2}} \otimes \lambda_{\frac{1}{2}}^{-5}$ on $\mathfrak{M}_g$

$$\lambda_{\frac{3}{2}} = \mathrm{ber}(H^0(\Sigma, \omega^3)) \otimes \mathrm{ber}^{-1}(H^1(\Sigma, \omega^3))$$

$$\lambda_{\frac{1}{2}} = \mathrm{ber}(H^0(\Sigma, \omega)) \otimes \mathrm{ber}^{-1}(H^1(\Sigma, \omega))$$

$\omega = \mathrm{ber}(\Omega_\Sigma) =$ canonical bundle

$\mathrm{ber} = \mathrm{sdet} =$ superdeterminant

One hope is that the super Mumford form can be used to calculate the vacuum amplitude (since similar holomorphic methods work for bosonic string theory).

So for genus 1 and genus 2 superstring vacuum amplitudes can be understood using holomorphic methods.

The tour-de-force calculation of D'Hoker and Phong for genus 2 relied on a holomorphic projection

$$\mathfrak{M}_2 \longrightarrow |\mathfrak{M}_2| \cong M_{2,\mathrm{spin}}$$

super period matrix $\longmapsto$ period matrix

Unfortunately, the analogous super period matrix map is only meromorphic for $g=3$.

As well, the important result of Donagi and Witten states that for $g \geq 5$ no such holomorphic projection exists $\mathfrak{M}_g \longrightarrow |\mathfrak{M}_g|$.

($\mathfrak{M}_g$ is not split, nor projected.)

The general idea behind this method is to first integrate over the odd moduli of $\mathfrak{M}_g$, then integrate over the even moduli.

For $\pi : \mathfrak{M}_g \longrightarrow \mathcal{M}_{g,spin}$

Pullback functions $m_1, \dots, m_{3g-3}$ from $\mathcal{M}_{g,spin}$.

Choose odd functions $\eta_1, \dots \eta_{2g-2}$ on $\mathfrak{M}_g$ to form a local coordinate system $(m_1, \dots m_{3g-3} | \eta_1, \dots, \eta_{2g-2})$.

Then $\mu_g = [m_1, \dots m_{3g-3} | \eta_1, \dots \eta_{2g-2}] \cdot \sigma$

σ is a holomorphic section of λ_1^{-5}

Expansion in local coordinates:

$$\mu_g = \sigma^{(0)} + \sum_{i,j} \eta_i \eta_j \sigma_{ij}^{(1)} + \dots + \underline{\eta_1 \eta_2 \cdots \eta_{2g-2} \sigma^{(g-1)}}$$

only this term survives in integrating over $\mathfrak{M}_g$

However, if π and σ <u>cannot be globally</u> defined on $\mathcal{M}_{g,spin}$, then $\sigma^{(g-1)}$ is only <u>locally</u> defined.

Further considerations:

The use of a super period matrix leads naturally to the super Schottky problem.

classically

$\mathrm{Jac}: \mathcal{M}_g \longrightarrow \mathcal{A}$

$3g-3$ $\quad$ $\frac{g(g+1)}{2}$

super

$s\mathrm{Jac}: \mathfrak{M}_{g,+} \dashrightarrow \mathcal{A}_g$

$3g-3|2g-2$ $\quad$ $\frac{g(g+1)}{2}$

Both the classical and super Schottky problems have implicit solutions in terms of KP and super KP hierarchies. (Shiota 1986, Mulase 1991)

This way to characterize the (super) Schottky locus hints at the (super) Sato Grassmannian as a replacement for $\mathcal{A}_g$. (Maxwell 2022)

Recent results about the super Schottky problem are found in Codogni, Viviani 2019
Felder, Kazhdan, Polishchuk 2021

$$I^d_{Sch} \subset I_{sSch} \qquad d = g \text{ or } g-1$$

As well, work on the measure on $\mathfrak{M}_g$ continues:

Dirscoll 2019
Felder, Kazhdan, Polishchuk arXiv 2006.13271
Witten arXiv 1501.02499
Donagi, Ott arXiv 2208.02478

- formula for μ_g around Ramond punctures
- $\overline{\mathfrak{M}}_g$ is a Deligne-Mumford stack
- regularity of μ_g along theta divisor for $g \leq 11$
- definition and properties of the super period matrix in the presence of Ramond punctures

Thank you!