Fukaya category of symplectic manifolds and microlocal sheaf theory

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(Exact) symplectic manifold

Symplectic manifold (M, ω)**:** M is a smooth manifold, ω is a closed, nondegenerate 2-form on M.

- **closed:** $d\omega = 0$.
- **nondegenerate:** The following is an isomorphism:

$$TM \xrightarrow{\sim} T^*M$$
$$X \mapsto \iota_X \omega := \omega(X, \bullet)$$

$\Rightarrow$ M is even dimensional.

Exact symplectic manifold (M, ω, λ)**:** λ is a 1-form on M such that

$$\omega = d\lambda.$$

$\Rightarrow$ M is noncompact.

Examples

- Cotangent bundles are exact symplectic manifolds.

$$M := T^*N = \{(x_1, \dots, x_n, p_1, \dots, p_n)\}$$

$$\omega := \sum_{i=1}^{n} dx_i \wedge dp_i \qquad \lambda := -\sum_{i=1}^{n} p_i dx_i$$

- Orientable surfaces with the area form.
- $\mathbb{CP}^n$ with Fubini-Study symplectic form.
- Complex submanifolds of $\mathbb{C}^n$.
- Smooth complex projective varieties.
- ...

(Exact) Lagrangian submanifold

Lagrangian L in (M, ω): L is n-dimensional submanifold of M^{2n} such that

$$\omega|_L = 0.$$

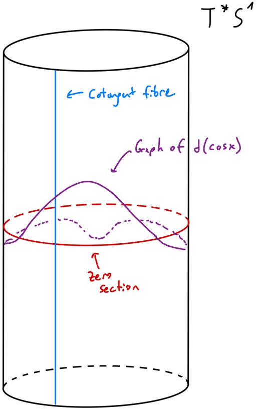
Exact Lagrangian L in (M, ω, λ):

$$\lambda|_L = df$$

for some smooth function $f\colon L \to \mathbb{R}$.

An example: cylinder

$T^*S^1 = \{(x,p)\}$ with $\omega = dx \wedge dp$ (area form), $\lambda = -pdx$.



Every curve is a Lagrangian. Some exact Lagrangians are:

- Zero section $\{p = 0\}$.
- Cotangent fibres $\{x = x_0\}$ for any $x_0 \in S^1$.
- The graph of the differential of $f \colon S^1 \to \mathbb{R}$. Ex: $f(x) = \cos x$.

(Exact) symplectomorphism

Symplectomorphism F **between** (M_1, ω_1) **and** (M_2, ω_2)**:** A diffeomorphism

$$F\colon M_1 \to M_2$$

such that

$$F^*\omega_2 = \omega_1$$

Exact symplectomorphism F **between** $(M_1, \omega_1, \lambda_1)$ **and** $(M_2, \omega_2, \lambda_2)$**:**

$$F^*\lambda_2 = \lambda_1 + df$$

for some smooth function $f\colon M_1 \to \mathbb{R}$.

$\Rightarrow$ (Exact) symplectomorphisms preserves (exact) Lagrangians.

Hamiltonian

Hamiltonian on (M, ω): A smooth function $H\colon M \to \mathbb{R}$.

Hamiltonian vector field X_H on M: Defined by

$$
\begin{aligned}
TM &\xrightarrow{\sim} T^*M \\
X_H &\mapsto dH = \iota_{X_H}\omega := \omega(X_H, \bullet)
\end{aligned}
$$

Hamiltonian flow $\phi_H^t\colon \mathbb{R} \times M \to M$: The flow of X_H.

Hamiltonian symplectomorphism $\phi_H^1\colon M \to M$: Why is it an (exact) symplectomorphism?Cartan's magic formula:

$$
\mathcal{L}_{X_H}\omega = (\iota_{X_H} \circ d + d \circ \iota_{X_H})(\omega) = \iota_{X_H} \circ d\omega + d \circ \iota_{X_H}\omega = d^2 H = 0
$$

$$
\mathcal{L}_{X_H}\lambda = (\iota_{X_H} \circ d + d \circ \iota_{X_H})(\lambda) = \iota_{X_H} \circ d\lambda + d \circ \iota_{X_H}\lambda = d(H + \iota_{X_H}\lambda)
$$

Physical interpretation

Classical mechanics: For simplicity, assume N is 1-dimensional.

- $M := T^*N = \{(x,p)\}$ is the phase space: $x \in N$ is position, p is momentum.
- $H(x,p) = \frac{p^2}{2m} + \phi(x)$ is the total energy of a particle at x with momentum p, where $\phi(x)$ is the potential energy at x.
- Path of a particle between time $t = 0$ and $t = 1$:

$$\gamma \colon [0,1] \to M$$
$$t \mapsto \gamma(t) = (x(t), p(t))$$

Newton's 2nd law $F = ma$	$\iff$	Hamiltonian equations $\frac{\partial H}{\partial x} = -\frac{dp}{dt}$ and $\frac{\partial H}{\partial p} = \frac{dx}{dt}$	$\iff$	$\frac{d\gamma}{dt} = X_H$

Conclusion: Hamiltonian vector field tells where the particle will go!

Stationary-action principle

Define the symplectic action functional

$$S\colon \Omega \to \mathbb{R}$$
$$\gamma \mapsto \int_0^1 -\gamma^*\lambda - H(\gamma(t))dt$$

on the space of paths of the form $\gamma\colon [0,1] \to M$ with **some boundary conditions**.

Theorem: Assume that the boundary conditions are given by $\gamma(0) \in L_0$ and $\gamma(1) \in L_1$ such that L_0 and L_1 are exact Lagrangians, then γ solves the Hamiltonian equations if and only if γ is a critical point of S .

An example

Question: Given a Hamiltonian H, find all paths of motion $\gamma(t) = (x(t), p(t))$ in the phase space such that $x(0) = x_0$ and $x(1) = x_1$.

An approach: Here, boundary conditions are $\gamma(0) \in \{x = x_0\}$ and $\gamma(1) \in \{x = x_1\}$ which are cotangent fibres, hence exact Lagrangians.

Solution: Minimise the symplectic action functional to find solutions.

Minimum number of solutions

Question: Given Lagrangian boundary conditions, is there a lower bound of solutions?

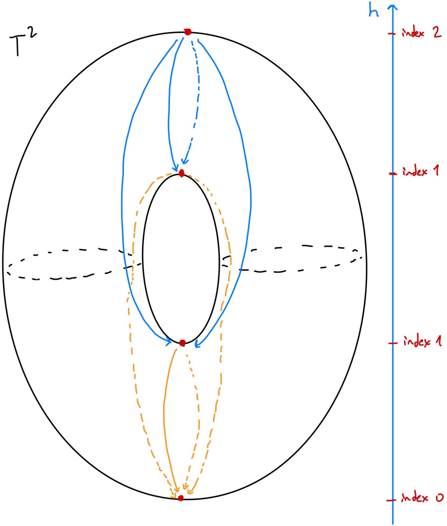
Equivalent question: Is there a lower bound for the number of critical points of the symplectic action functional S?

Morse theory: The total dimension of the homology gives a lower bound for the number of critical points of smooth functions on compact manifolds.

Problem: Ω is infinite-dimensional.

Morse theory

Example: Torus T^2 with height function $h\colon T^2 \to \mathbb{R}$.



Morse chain complex:

$$\mathbb{R}[2] \xrightarrow{1-1+1-1} (\mathbb{R} \oplus \mathbb{R})[1] \xrightarrow{1-1+1-1} \mathbb{R}[0]$$

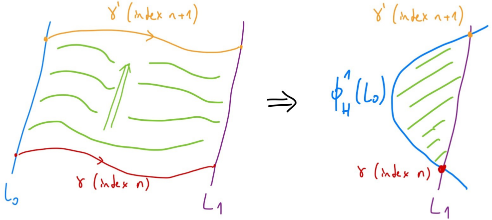
Its homology is the singular homology $H_*(M; \mathbb{R})$, which doesn't depend on h. Hence,

$$\#\text{crit points of } h \geq \dim H_*(M; \mathbb{R}) = 4$$

for any smooth $h\colon T^2 \to \mathbb{R}$.

Floer theory

Floer: Morse theory for $S \colon \Omega \to \mathbb{R}$ on infinite dimensional space Ω.



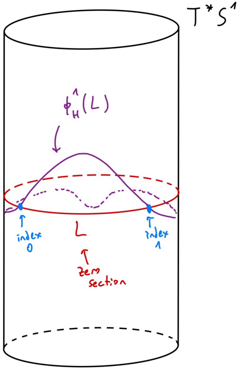
Critical points: Paths from L_0 to L_1 satisfying Hamiltonian equations. They are in fact the intersection points $\phi^1_H(L_0) \cap L_1$.

Flow lines: Pseudoholomorphic disks (for some almost complex structure J on M).

Floer cochain complex: $CF^*(L_0, L_1)$ for compact L_0, L_1. Floer cohomology $HF^*(L_0, L_1)$ doesn't depend on H or J. Invariant of M, L_0, L_1.

Some properties and implications

- #crit points of $S \geq \dim HF^*(L_0, L_1)$.
- $CF^*(L, L) = ?$
  Example: $L \subset T^*S^1$ is a zero section, $H = \cos x \Rightarrow X_H = (\sin x)\frac{\partial}{\partial p}$.



Floer cochain complex $CF^*(L, L)$:

$$\mathbb{R}[0] \xrightarrow{1-1} \mathbb{R}[1]$$

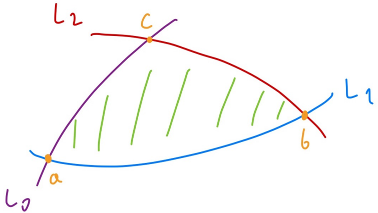
Then $HF^*(L, L) = H^*(L; \mathbb{R})$.
This is true for any compact exact L.

- **Implication:** Nondisplacibility of compact exact Lagrangians.

Fukaya categories

Fuk(M) for an exact symplectic manifold M:

- **Objects:** Compact exact Lagrangians.
- **Morphisms between L_0 and L_1:** $CF^*(L_0, L_1)$ (with differential).
- **Composition:** $CF^*(L_0, L_1) \otimes CF^*(L_1, L_2) \to CF^*(L_0, L_2)$.



contributes $a \otimes b \mapsto c$.

Not associative: $(ab)c - a(bc) = d(\mu^3(a, b, c))$

$\Rightarrow$ Fuk(M) is an A-infinity category with operations μ^1 (differential), μ^2 (composition), $\mu^3, \mu^4, \ldots$ It is an invariant of M.

Some properties and conjectures

- If we allow noncompact Lagrangians: wrapped Fukaya category $\mathrm{Fuk}(M) \subset \mathrm{WFuk}(M)$. Hamiltonians should be "quadratic at infinity". $\mathrm{WFuk}(T^*N)$ is generated by a cotangent fibre (Abouzaid).

- **Homological mirror symmetry conjecture:**

$$\mathrm{Fuk}(M) \simeq \mathrm{Coh}(X)$$

  for some complex variety X, where $\mathrm{Coh}(X)$ is coherent sheaves on X (Kontsevich).

- **Nearby Lagrangian conjecture:** If N is a closed manifold, any closed exact Lagrangian in T^*N is Hamiltonian isotopic to the zero section. **Current progress:** "simple-homotopic" (Abouzaid-Kragh using Fukaya categories).

Microlocal sheaf theory

An example: $\mathrm{Fuk}(T^*S^1) \simeq \mathrm{Loc}(S^1) \simeq \mathrm{Repr}(\bullet \circlearrowleft \mathrm{inv}) \simeq \mathrm{Coh}(\mathbb{C}^*)$.

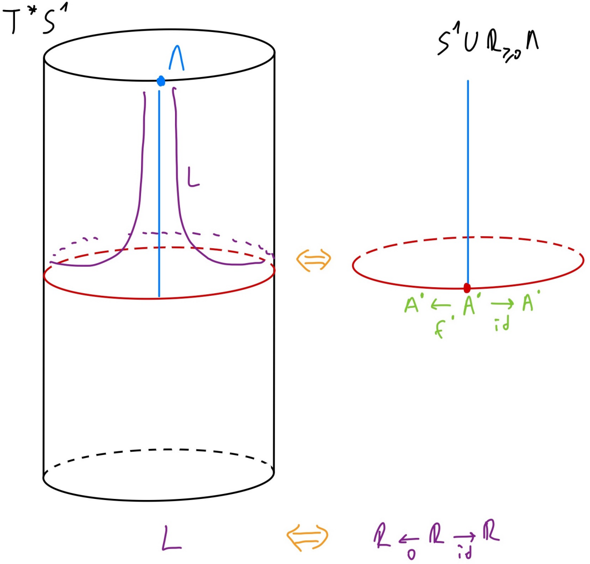
Kashiwara-Schapira, Nadler-Zaslow, Ganatra-Pardon-Shende...

$$\mathrm{Fuk}(T^*N, \Lambda) \simeq \mu\mathrm{Sh}(N \cup \mathbb{R}_{\geq 0}\Lambda)$$

where $\mathrm{Fuk}(T^*N, \Lambda)$ contains noncompact Lagrangians asymptotic to Λ, and $\mu\mathrm{Sh}(K)$ is the dg category of microlocal sheaves on K.

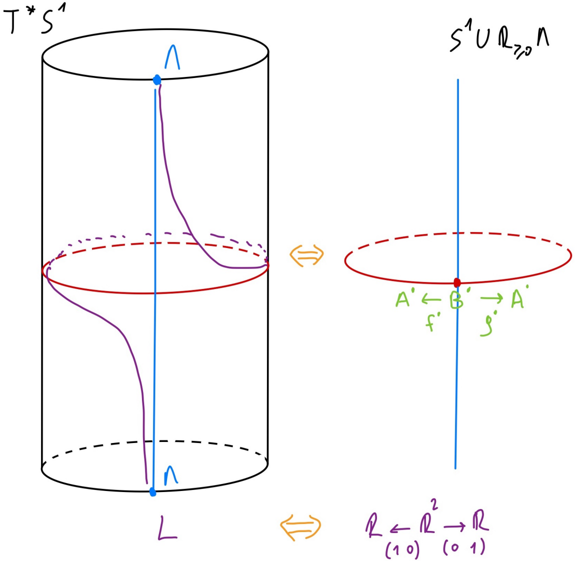
An example

$\mathrm{Fuk}(T^*S^1, \Lambda = \{1 \text{ pt}\}) \simeq \mu\mathrm{Sh}(S^1, \mathbb{R}_{\geq 0}\Lambda) \simeq \mathrm{Repr}(\bullet \circlearrowleft) \simeq \mathrm{Coh}(\mathbb{C})$.



Another example

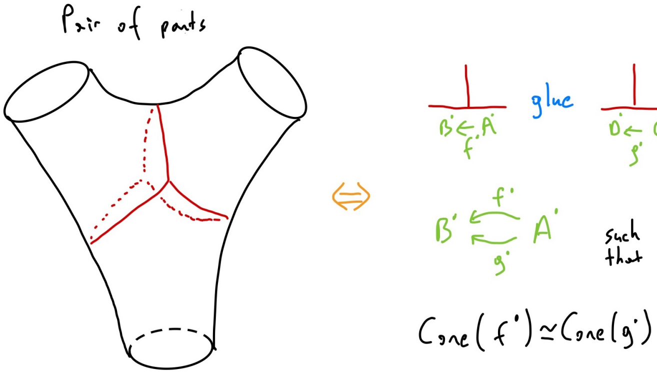
$$\mathrm{Fuk}(T^*S^1, \{2 \text{ pts}\}) \simeq \mu\mathrm{Sh}(S^1, \mathbb{R}_{\geq 0}\Lambda) \simeq \mathrm{Repr}(\bullet \rightrightarrows \bullet) \simeq \mathrm{Coh}(\mathbb{P}^1).$$



Liouville and Weinstein manifolds

For a more general class of exact symplectic manifolds (Liouville & Weinstein manifolds):Locally see them as cotangent bundles with stops and then glue.

An example:



Thank you!