Postdoc Colloquium: M2 & M5 Algebras, Miura Operators, and Coproducts. (joint with D. Gaiotto, M. Rapčák)

- Quantization:

Poisson algebra A, $\{-,-\}$,

Quantization of $A \rightsquigarrow A_\hbar$: flat deformation over $\mathbb{C}[\hbar]$

such that $[x, y] = \hbar\{\bar{x}, \bar{y}\} + O(\hbar^2)$

$\bar{x} = x \mod \hbar$

Example: Moyal product on $A \otimes \mathbb{C}[\hbar]$,

$$x \star y = m\left(e^{\frac{\hbar}{2}\Pi}(x \otimes y)\right) = x \cdot y + \frac{\hbar}{2}\Pi^{ij}\partial_i f\, \partial_j g + O(\hbar^2)$$

usual multiplication

$m(a \otimes b) = a \cdot b$

Poisson tensor

$\Pi^{ij}\partial_i \wedge \partial_j$

V: vector space, then $\left(\mathbb{C}[T^*V][\hbar], \star\right)$ is the Weyl algebra $\mathbb{C}\langle P_i, x^j \rangle / [P_i, x^j] = \hbar\, \delta_i^j$

- **Quantum Hamiltonian reduction:**

$G \curvearrowright A_\hbar$ which induces $\rho : \mathfrak{g} \to \mathrm{End}(A_\hbar)$

Assume: $\exists\, \mu : \mathfrak{g} \to A_\hbar$ quantum moment map

such that $\frac{1}{\hbar}[\mu(x), y] = \rho(x).y$

then we define $A_\hbar /\!\!/_\chi\, \mathfrak{g} = \left(A_\hbar / I_\chi\right)^G$

$I_\chi = A_\hbar \cdot \mathrm{Span}\left(\mu(x) - \chi(x)\right)$ χ: character of $\mathfrak{g}$.

- Quantum Nakajima quiver variety

Q : quiver, dimension vectors $\underline{v}$: vertices (gauge)

$\underline{w}$: framing (flavour)

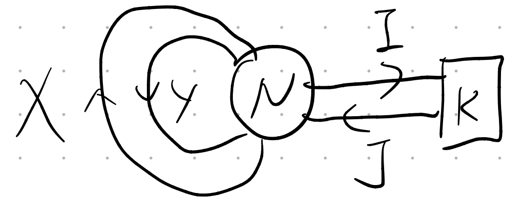
$$\mathrm{Rep}_Q(\underline{v}, \underline{w}) = \mathrm{Hom}_{i \in Q_0}(W_i, V_i) \oplus \mathrm{Hom}_{a \in Q_1}(V_{t(a)}, V_{h(a)})$$

$$\circlearrowleft$$

$$G(\underline{v}) = \prod_{i \in Q_0} GL(V_i)$$

$$\mathbb{C}_\hbar[\mathcal{M}_\chi(Q, \underline{v}, \underline{w})] := \mathrm{Weyl}(\mathrm{Rep}_Q(\underline{v}, \underline{w})) /\!\!/_{\chi}\, \mathfrak{g}(\underline{v})$$

• <u>Jordan quiver case</u>:



X^i_j, Y^i_j, I^a_i, J^j_b $\qquad 1 \le i, j \le N$

$\qquad\qquad\qquad\qquad\qquad\qquad 1 \le a, b \le K$

$$[X^i_j \; Y^k_l] = \varepsilon_1 \delta^k_j \delta^i_l \qquad [J^j_b, I^a_i] = \varepsilon_1 \delta^a_b \delta^j_i$$

$\mathbb{C}_{\varepsilon_1}[M_{\varepsilon_2}(N,K)] = GL_N$-invariants modulo left ideal generated by

$$:[X, Y]^i_j: + I^a_j J^i_a - \varepsilon_2 \delta^i_j$$

normal order

$$X^i_k Y^k_j - X^k_j Y^i_k$$

- M2 brane algebra

Twisted M-theory on

#	Brane	$\mathbb{C}_{\varepsilon_1}$	$TN_k^{\varepsilon_2\varepsilon_3}$	$\mathbb{R}_t$	$\mathbb{C}_z$	$\mathbb{C}_w$
N	M2	X		X		

A-twist: $\underbrace{\mathbb{C}_{\varepsilon_1} \;\; TN_k^{\varepsilon_2\varepsilon_3} \;\; \mathbb{R}_t}$ B-twist: $\underbrace{\mathbb{C}_z \;\; \mathbb{C}_w}$

Reduce $S^1_{\varepsilon_2}$ $\downarrow$

IIA on

#	Brane	$\mathbb{C}_{\varepsilon_1}$	$\mathbb{R}^3$	$\mathbb{R}_t$	$\mathbb{C}_z$	$\mathbb{C}_w$
N	D2	X		X		
K	D6	X		X	X	X

D2 : 3d $\mathcal{N}=4$ ADHM quiver gauge theory

D6 : 7d $U(k)$ SYM

$$\Big\downarrow \ \Omega_{\varepsilon_1}\text{-deformation}$$

1D

ADHM quantum mechanics

$$\frac{1}{\varepsilon_1}\int_{\mathbb{R}_t} \mathrm{Tr}\left(X d_\alpha Y + J d_\alpha I - \varepsilon_2 \alpha\right)$$

5D

topological-holomorphic Chern-Simons

$$\frac{1}{\varepsilon_1}\int_{\mathbb{R}_t\times\mathbb{C}^2_{z,w}} dz\, dw\, \mathrm{Tr}\left(A \underset{\varepsilon_2}{\star} dA + \frac{2}{3} A \underset{\varepsilon_2}{\star} A \underset{\varepsilon_2}{\star} A\right)$$

$\underset{\varepsilon_2}{\star}$: Moyal product on $\mathbb{C}^2_{z,w}$, $\Pi = \varepsilon_2\, \partial_z \wedge \partial_w$

Algebra of Local Observables

$$\mathbb{C}_{\varepsilon_1}[\mathcal{M}_{\varepsilon_2}(N,K)]$$

$$C^\bullet_{\varepsilon_1}\left(\mathfrak{gl}_K \otimes \mathbb{C}_{\varepsilon_2}[z,w]\right)$$

Theorem [Costello] $C^\bullet_{\varepsilon_1}(\mathfrak{gl}_K \otimes \mathbb{C}_{\varepsilon_2}[z,w])$ is Koszul dual to $\underbrace{\text{large } N \text{ limit of } \mathbb{C}_{\varepsilon_1}[\mathcal{M}_{\varepsilon_2}(N,K)]}_{\text{denote by } \mathcal{A}^{(K)}}$

Remark: It is known that

$$C^\bullet_{\varepsilon_1}(\mathfrak{gl}_K \otimes \mathbb{C}_{\varepsilon_2}[z,w])^! \simeq U_{\varepsilon_1}(\mathfrak{gl}_K \otimes \mathbb{C}_{\varepsilon_2}[z,w])$$

Conjecture [Costello] $U_{\varepsilon_1}(\mathfrak{gl}_K \otimes \mathbb{C}_{\varepsilon_2}[z,w])$ is isomorphic to the $\underbrace{\text{deformed double current algebra}}_{\text{denote by } D^{(K)}}$ of $\mathfrak{gl}_K$.

for generic ε_1, ε_2.

<u>Remark</u>: It is known, by the work of Etingof, and Kalinov et. al., that $D^{(K)}$ is large N limit of spherical gl_k-extended rational Cherednik algebras $B_{\varepsilon_1,\varepsilon_2}(N,k)$

$$B_{\varepsilon_1\varepsilon_2}(N,K) = e\, H_{\varepsilon_1,\varepsilon_2}(N,k)\, e \qquad e = \frac{1}{N!}\sum_{g\in S_N} g$$

$$H_{\varepsilon_1,\varepsilon_2}(N,K) = \mathbb{C}[S_N] \ltimes \left(\mathbb{C}\langle x_1 \cdots x_N, y_1 \cdots y_N\rangle \otimes gl_k^{\otimes N}\right)$$

relations $$[y_i, x_j] = \delta_{ij}\Big(\varepsilon_2 - \varepsilon_1 \sum_{l\neq i} S_{il}\Omega_{il}\Big) + (1-\delta_{ij})\,\varepsilon_1 S_{ij}\Omega_{ij}$$

$$\Omega_{ij} = E^a_{b,i} \otimes E^b_{a,j} \, ,\qquad S_{ij}: (i,j)\text{-permutation} \in S_N$$

• What is $\mathcal{A}^{(k)}$?

① Define $\mathbb{C}[\varepsilon_1, \varepsilon_2]$-subalgebra $\mathcal{A}^{(k)}_N \subset \mathbb{C}_{\varepsilon_1}[\mathcal{M}_{\varepsilon_2}(N,K)][\varepsilon_1^{-1}]$

generated by

$$e^a_{b;n,m} = \frac{1}{\varepsilon_1} I^a \, \mathrm{Sym}(x^n y^m)\, J_b$$

$$t_{n,m} = \frac{1}{\varepsilon_1} \mathrm{Tr}\left(\mathrm{Sym}(x^n y^m)\right)$$

② Some of Relations

Easy

(1) $e^a_{a;n,m} = \varepsilon_2 \cdot t_{n,m}$

(2) $t_{0,0}$ is central

(3) $[e^a_{b;0,0}, e^c_{d;n,m}] = \delta^c_b e^a_{d;n,m} - \delta^a_d e^c_{b;n,m}$

(4) $[t_{1,0}, e^a_{b;n,m}] = m\, e^a_{b;n,m-1}$, $[t_{0,1}, e^a_{b;n,m}] = n \cdot e^a_{b;n-1,m}$

(5) $[t_{2,0}, e^a_{b;n,m}] = 2m\, e^a_{b;n+1,m-1}$, $[t_{0,2}, e^a_{b;n,m}] = -2n\, e^a_{b;n-1,m+1}$

$[t_{1,1}, e^a_{b;n,m}] = (m-n) e^a_{b;n,m}$

(6) $[t_{3,0}, e^a_{b;1,0}] = 0$

(7) $$[e^a_{b;1,0}, e^c_{d;0,n}] = \delta^c_b e^a_{d;1,n} - \delta^a_d e^c_{b;1,n} - \frac{\epsilon_3 n}{2}\left(\delta^c_b e^a_{d;0,n-1} + \delta^a_d e^c_{b;0,n-1}\right) - n\epsilon_1 \delta^c_d e^a_{b;0,n-1}$$
$$- \epsilon_1 \sum_{m=0}^{n-1} \frac{m+1}{n+1} \delta^a_d e^c_{f;0,m} e^f_{b;0,n-1-m} - \epsilon_1 \sum_{m=0}^{n-1} \frac{n-m}{n+1} \delta^c_b e^a_{f;0,m} e^f_{d;0,n-1-m}$$
$$+ \epsilon_1 \sum_{m=0}^{n-1} e^a_{d;0,m} e^c_{b;0,n-1-m}$$

Hard

found using "Spin-Calogero" representation

$K\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$

(8) $$[t_{3,0}, t_{0,n}] = 3n t_{2,n-1} + \frac{n(n-1)(n-2)}{4}(\epsilon_1^2 - \epsilon_2\epsilon_3) t_{0,n-3}$$
$$- \frac{3\epsilon_1}{2} \sum_{m=0}^{n-3} (m+1)(n-2-m)(e^a_{c;0,m} e^c_{a;0,n-3-m} + \epsilon_1\epsilon_2 t_{0,m} t_{0,n-3-m}).$$

③ Define $\mathcal{A}^{(K)} = \mathbb{C}[\varepsilon_1, \varepsilon_2]$-algebra generated by $e^a_{b;n,m}$, $t_{n,m}$ with relations (1)-(8).

<u>Theorem</u>: $\mathcal{A}^{(K)}$ is isomorphic to the large N limit algebra define in Costello's work.

- $\mathcal{A}^{(K)} \simeq D^{(k)}$

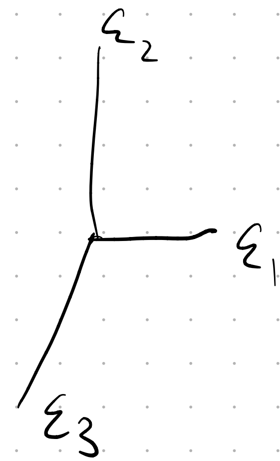
① <u>Lemma</u>: Assume ε_2 invertible, then $\exists$ surjective homomorphism $\mathcal{A}^{(K)} \twoheadrightarrow B_{\varepsilon_1,\varepsilon_2}(N,K)$, generated by:

$$t_{2,0} \longmapsto \frac{1}{\varepsilon_2} \sum_{i=1}^{N} y_i^2 \qquad e^a_{b;0,n} \longmapsto \sum_{i=1}^{N} x_i^n \otimes E^a_{b,i}$$

Proof uses Dunkl representation

② Use the above lemma, and compare with generators in Kalinov's work on large N limit of $B_{\varepsilon_1,\varepsilon_2}(N,K)$, we can show an explicit isomorphism $\mathcal{A}^{(K)} \simeq D^{(k)}$ (Assume $\varepsilon_2 \cdot \varepsilon_3$ is invertible)

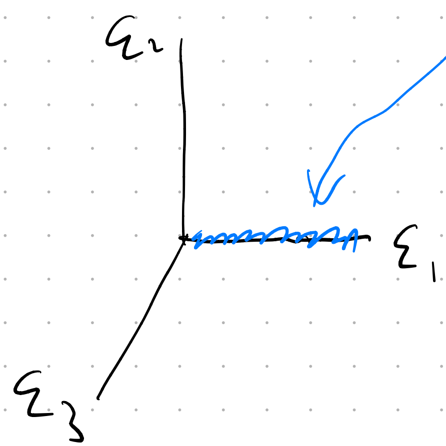
- **Physics Meaning of $A^{(k)} \simeq D^{(k)}$**

toric CY_3 : $\mathbb{C}_{\varepsilon_1} \times TN_K^{\varepsilon_2 \varepsilon_3}$ $=$ 

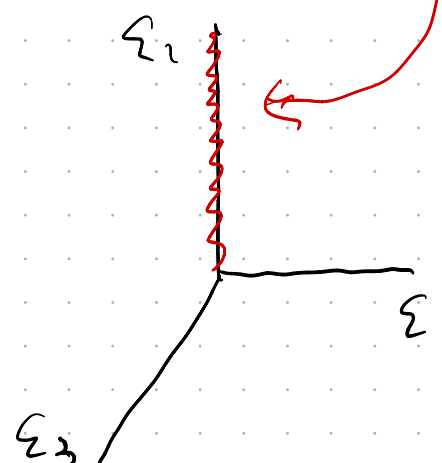
$K\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$

$$TN_K^{\varepsilon_2 \varepsilon_3} = \frac{\mathbb{C}_{\varepsilon_2} \times \mathbb{C}_{\varepsilon_3}}{\mathbb{Z}_K}$$

① For $A^{(k)}$, M2 wraps



② For $D^{(k)}$, M2 wraps



Single M2 wrap $\mathbb{C}_{\varepsilon_2}$ $\xrightarrow[\text{to 5D}]{\text{reduce}}$ Fundamental Wilson line in 5D CS

Algebra of local operators $= \mathfrak{gl}_K \otimes \mathbb{C}_{\varepsilon_2}[z, w][\varepsilon_1]$

$= \mathcal{H}_{\varepsilon_1, \varepsilon_2}(1, K)$

N stacks of M2 $\rightsquigarrow \mathcal{H}_{\varepsilon_1, \varepsilon_2}(N, K)$

$A^{(K)} \simeq D^{(K)}$: different brane setting realizing

Same duality : $\mathrm{Obs}(5d\ CS)^! \simeq \mathrm{Obs}\left(\begin{array}{c}\text{Universal line}\\ \text{defect}\end{array}\right)$

• $\varepsilon_2 \leftrightarrow \varepsilon_3$ Duality

M2 can also wrap C_{ε_3} $\rightsquigarrow$ antifund Wilson in 5D CS

Symmetry $C_{\varepsilon_2} \leftrightarrow C_{\varepsilon_3}$

Algebraically, this is

$$A^{(K)} \xrightarrow{\sim} A^{(K)}$$

$$\varepsilon_2 \longmapsto \varepsilon_3$$

$$X \mapsto X \qquad\qquad t_{n,m} \longmapsto t_{n,m}$$

$$Y \mapsto Y$$

$$I^a_i \mapsto J^i_a \qquad\qquad e^a_{b;n,m} \longmapsto -e^b_{a;n,m} - \varepsilon_1 \delta^b_a t_{n,m}$$

$$J^j_b \mapsto -I^b_j$$

Remark: If $K=1$, duality is enhanced to triality

$$\begin{tikzcd}[column sep=small] & \varepsilon_1 \arrow[dl, dash] \arrow[dr, dash] & \\ \varepsilon_2 \arrow[rr, dash] & & \varepsilon_3 \end{tikzcd}$$

- B-Algebra & Yangian:

① If A is $\mathbb{Z}$-graded algebra, define B-algebra

$$B(A) = A_0 \Big/ \Big(\sum_{i>0} A_i \cdot A_{-i} \Big)$$

grading on $A^{(K)}$: $\deg(e^a_{b;n,m}) = \deg(t_{n,m}) = n-m$

② Conjecture [Costello] $B(A^{(K)}) \simeq Y_{\varepsilon_1}(gl_K)$ for generic ε_2.

③ Idea of proof: Observe that $T^a_b[n] \mapsto \frac{1}{\varepsilon_1} I^a (Y \cdot X)^n \cdot J_b$

gives an embedding of $Y_{\varepsilon_1}(gl_K) \hookrightarrow A^{(K)}$, degree zero.

Check: $Y_{\varepsilon_1}(gl_K) \to B(A^{(K)})$ is linear isomorphism,

if $\varepsilon_2 \neq 0$.

• M5 Branes:

M5 can wrap (Toric 4-cycles $\subseteq CY_3$) $\times \mathbb{C}_w$

$\Big\}$ reduce to 5D

Surface defect in 5D CS

Expectation: Obs(5D Gl_K CS on $\mathbb{C}_z^\times \times \mathbb{C}_w \times \mathbb{R}_t$)!

$\cong$

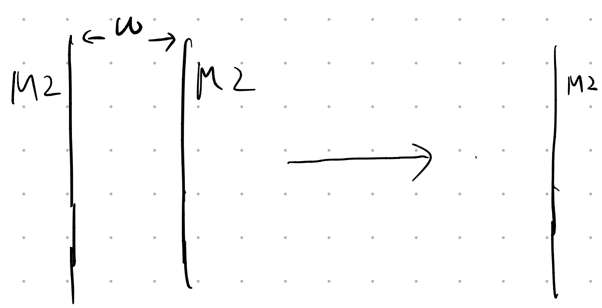
Mode algebra (gl_K-extended $W_{1+\infty}$)

$\cong$

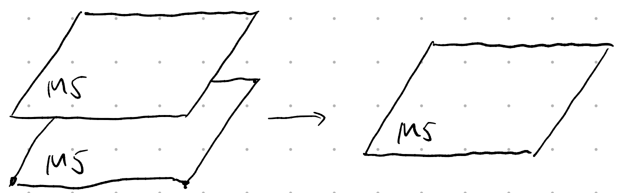
Affine Yangian of gl_K

M5 algebra
$Y^{(K)}$

• Defect Fusions

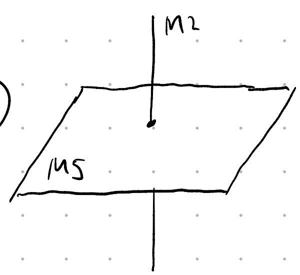
① 

$$\Delta_A(w) : A^{(k)} \to A^{(k)} \otimes A^{(k)}((w^{-1}))$$

② 

$$\Delta_{Y,Y} : Y^{(k)} \longrightarrow Y^{(k)} \otimes Y^{(k)}$$

[See Kodera-Ueda]

③ 

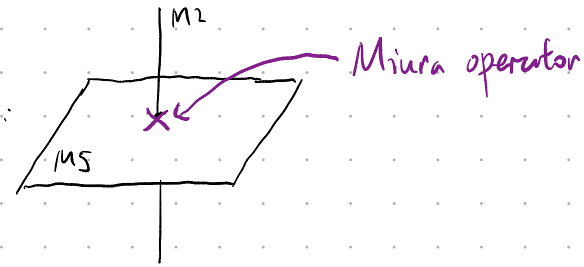
gauge invariance implies

$$\Delta_{A,Y} : A^{(k)} \longrightarrow A^{(k)} \otimes Y^{(k)}$$

We worked out explicit formulae for these coproducts and checked various coassociativity conditions

• M2 - M5 intersection

We propose that:



Consider $\left(\alpha\,\partial_w - J^a_b(w)\cdot E^b_a\right)\cdot|0\rangle \in \underbrace{V^{k_\alpha}(gl_k)\otimes B_{\varepsilon_1\varepsilon_2}(1,k)}_{\mathcal{A}^{(k)}\otimes \mathcal{Y}^{(k)} - \mathcal{A}^{(k)} \text{ bimodule}}$

Left action $\mathcal{A}^{(k)} \twoheadrightarrow B_{\varepsilon_1,\varepsilon_2}(1,k)$

$\mathcal{Y}^{(k)} \longrightarrow \text{Mode alg}\left(V^{k_\alpha}(gl_k)\right)$ [See Kodera-Ueda]

Right action $\mathcal{A}^{(k)} \twoheadrightarrow B_{\varepsilon_1,\varepsilon_2}(1,k)$

Then $\Delta_{\mathcal{A}\mathcal{Y}}(x)\cdot\left(\alpha\,\partial_w - J^a_b(w)\,E^b_a\right)\cdot|0\rangle = \left(\alpha\partial_w - J^a_b(w)E^b_a\right)|0\rangle\cdot x$

$\forall x \in \mathcal{A}^{(k)}$

Remark 1. If $C_A : A^{(k)} \to \mathbb{C}[\varepsilon_1, \varepsilon_2]$ is the augmentation, then

$(C_A \otimes Id) \circ \Delta_{A,Y} : A^{(k)} \to Y^{(k)}$ is the standard

embedding of DDCA inside affine Yangian [See Guay]

Remark 2

$$\begin{tikzcd} A^{(k)} \arrow[r, "\Delta_{A,Y}"] \arrow[d, hook] & A^{(k)} \otimes Y^{(k)} \arrow[d, hook] \\ Y^{(k)} \arrow[r, "\Delta_{Y,Y}"] & Y^{(k)} \otimes Y^{(k)} \end{tikzcd}$$

commutes

Remark 3: We actually have $\Delta_Y(w) : Y^{(k)} \longrightarrow A^{(k)} \otimes Y^{(k)}((w^{-1}))$

such that

$$\begin{tikzcd} A^{(k)} \arrow[r, "\Delta_A(w)"] \arrow[d, hook] & A^{(k)} \otimes A^{(k)}((w^{-1})) \arrow[d, hook] \\ Y^{(k)} \arrow[r, "\Delta_Y(w)"] & A^{(k)} \otimes Y^{(k)}((w^{-1})) \end{tikzcd}$$

Commutes

Remark 4 Two points of view on $\Delta(w)$

① $\exists$

$$\begin{array}{ccc} B(N_1+N_2, K) & \longrightarrow & B(N_1, K) \otimes B(N_2, K)((w^{-1})) \\ \downarrow & & \downarrow \\ \mathbb{B}(N_1+N_2, K) & \longrightarrow & B(N_1, K) \otimes \mathbb{B}(N_2, K)((w^{-1})) \end{array}$$

trignometric Cherednik

$$\xRightarrow{N\to\infty} \Delta_A(w), \quad \Delta_y(w)$$

② For a vertex algebra $\mathcal{V}$, we can construct

$$\begin{array}{ccc} \widetilde{\mathcal{U}}_+(\mathcal{V}) & \xrightarrow{\Delta_{\mathcal{V}}(w)} & \widetilde{\mathcal{U}}_+(\mathcal{V}) \otimes \widetilde{\mathcal{U}}_+(\mathcal{V})((w^{-1})) \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{U}}(\mathcal{V}) & \xrightarrow{\Delta_{\mathcal{V}}(w)} & \widetilde{\mathcal{U}}_+(\mathcal{V}) \otimes \widetilde{\mathcal{U}}(\mathcal{V})((w^{-1})) \end{array}$$

$\tilde{\mathcal{U}}(\mathcal{V})$: Modification of mode algebra of $\mathcal{V}$

$\cup$

$\tilde{\mathcal{U}}_+(\mathcal{V})$ positive part

Assume $\mathcal{W}^{(k)}_{1+\infty}$ exists, then

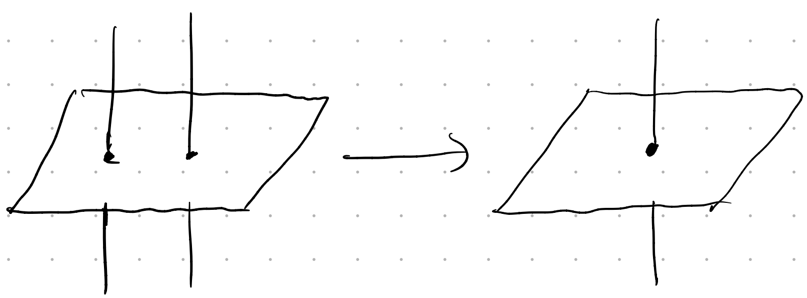
$$\mathcal{A}^{(k)} \sim \tilde{\mathcal{U}}_+ \left(\mathcal{W}^{(k)}_{1+\infty} \right)$$

$$\mathcal{Y}^{(k)} \sim \tilde{\mathcal{U}} \left(\mathcal{W}^{(k)}_{1+\infty} \right)$$

- Compatibility between $\Delta(\omega)$ & Δ_{AY}

Physics	Mathematics

Physics



$$\left(\Delta_{AY}\otimes\Delta_{AY}\right)\circ\Delta_{A}(\omega) = \left(\Delta_{A}(\omega)\otimes\Delta_{Y}(\omega)\right)\circ\Delta_{AY}$$

Mathematics

$\Delta_{\mathcal{V}}(\omega)$ is functorial with respect to $\mathcal{V}_1\to\mathcal{V}_2$

$$\mathcal{V}_1 = \mathcal{W}^{(k)}_{1+\infty}$$

$$\mathcal{V}_2 = \mathcal{W}^{(k)}_{1+\infty}\otimes\mathcal{W}^{(k)}_{1+\infty}$$