Imperial College London

Warped AdS Spaces from brane intersections in type II String Theory

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Outline

Part 1

Basic Facts of D-branes in String Theory and Black Brane Solutions in Supergravity.

D-Branes, String Theory and Duality

Black Brane Solutions and their relation to D-branes.

Part 2

Warped Supergravity Solutions.

This talk is based on

Orlando, D. and Uruchurtu, L. I.
Warped anti-de Sitter spaces from brane intersections in type II string theory.
arXiv:1003.0712 (To be published in JHEP)

Part I

String Theory & D-Branes

Basic input parameter in string theory:

$$T = \frac{1}{2\pi\alpha'} \equiv \frac{1}{2\pi l_s^2}$$

String excitations will have a good description as point-particle like states on scales longer than l_s

After quantisation, finite masses in spectrum are set by the inverse of l_s and the tower of massive excitations becomes inaccessible at low energies.

This same tower plays a role in the UV, helping circumvent questions of non-renormalisability in quantum gravity.

Conceptually, it is then easy to understand string theory using perturbation theory.

Interestingly enough, String theory determines dynamically the value of its own coupling strength, via the dilaton.

What about strong coupling?

- String Field Theory (?) - Possibility of Field theory of strings?

- String theory might not be a theory of strings

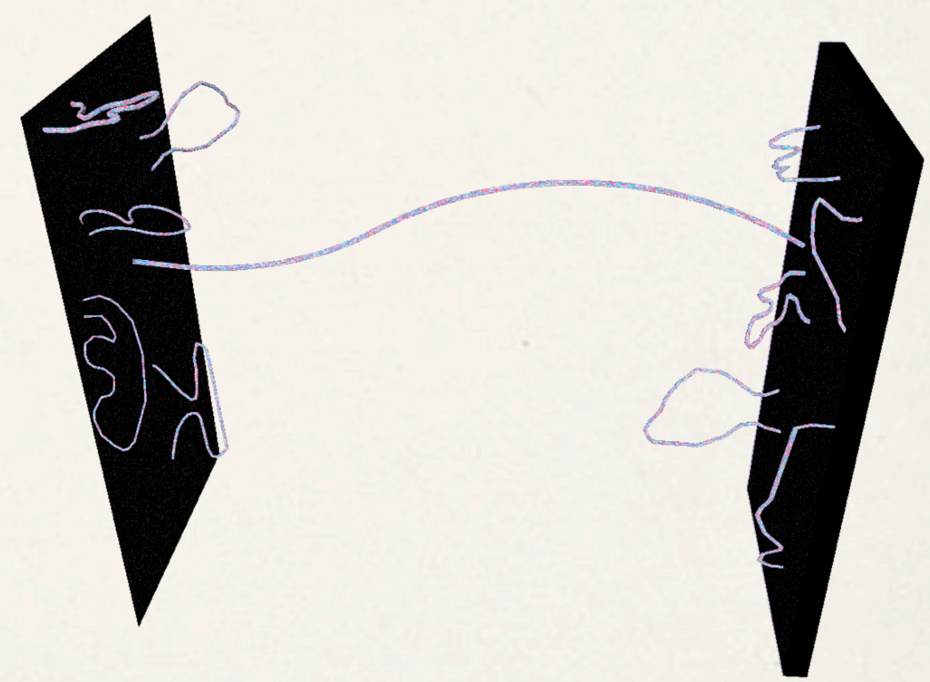
  Appearance of extended objects (**D-Branes**) which became lighter at strong coupling, with their behaviour dominating IR physics.

  Emergence of **dualities** and connection to a M-theory, a theory which doesn't involve string dynamics.

D-Branes

D-Branes are extended objects in which open strings end. More precisely, demanding Dirichlet boundary conditions at the endpoints, specify the coordinates of an hyperplane which is identified with a D-brane.

Dp-Branes have a (p+1)-dimensional worldvolume and in this terminology the 10-dimensional contains a D9-brane filling the space.



D-Branes are dynamical objects and as such respond to the background fields of the theory.

T-Duality

Consider strings propagating on a circle. States are labelled by momentum and winding number.

$$R \to 0$$

Momentum states become heavy and winding states become lighter.

$$R \to \infty$$

Winding states become infinitely massive, while states with w=0 go over to a continuum.

Possible to define a dual theory on a dual circle of radius proportional to *1/R*, in which momentum and winding quantum numbers are interchanged.

What about open strings?

Open strings don't have winding no. so when the circle becomes small, the non-zero momentum states become heavy and since there is no new continuum of states, we are left with a theory with one dimension fewer.

Closed strings are defined on D dimensions but open strings only in D-1. The subtlety has to do with the endpoints, as they are now restricted to a D-1 dimensional hyperplane (Dirichlet-Neumann BC exchange)

The dynamics of a D-brane can be described in terms of the fields which live on its worldvolume.

$$S_p = -T_p \int d^{p+1}\xi e^{-\phi} \{\det G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}\}^{1/2}$$

This action respects T-duality, hence B and F appearing on the det.

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From SUSY considerations, it is possible to deduce that D-branes are **BPS objects** and as such must carry conserved charges. RR charges have the correct Lorentz properties, since the volume of a p-brane naturally couples to a (p+1)-form potential.

The Field strengths to which Dp-branes and D(6-p)-branes couple are dual to one another:

$$F_{p+2} = *F_{8-p}$$

In this way, one says that D-branes carry **RR charge.**

Type IIB string theory: Contains even RR forms, hence D1, D3, D5,...

Type IIA string theory: Contains off RR forms, hence D0, D2, D4,...

Under T-duality on a compact circle, type IIB becomes IIA and viceversa.

Supergravity & Black Branes

We can truncate consistently the spectrum to focus on the massless sector of the string theories (low energy limit, $\alpha' \to 0$). We can then write a low energy effective field theory action for these fields.

$$S_{II} = \frac{1}{2\kappa_0^2} \int d^{10}x (-G)^{1/2} \left\{ e^{-2\phi} \left[R + 4(\nabla\phi)^2 - \frac{1}{12} H_{(3)}^2 \right] + RR \right\}$$

$$2\kappa^2 \equiv 2\kappa_0^2 g_s^2 = (2\pi)^7 \alpha'^4 g_s^2$$

There is an interesting family of solutions which source gravity, dilaton and RR potentials.

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String Frame

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There is an interesting family of solutions which source gravity, dilaton and RR potentials.

$$ds^2 = H_p^{-1/2}(r)\left(-K(r)dt^2 + dx_i^2\right) + H_p^{1/2}(r)\left(\frac{dr^2}{K(r)} + r^2 d\Omega_{8-p}^2\right)$$

$$H_p(r) = 1 + \alpha_p\left(\frac{r_p}{r}\right)^{7-p} \qquad K(r) = 1 - \left(\frac{r_H}{r}\right)^{7-p}$$

$$e^{2\phi} = H_p(r)^{\frac{3-p}{2}} \qquad C_{p+1} = g_s^{-1}(H_p(r)^{-1} - 1)$$

These represent p-dim objects which are localised in the (9-p) directions transverse to them. They are aligned along the $(x^1, x^2, \cdots, x^p)$ and move in time so they have a (p+1) dim worldvolume.

The extremal limit of the solution is given by taking $\alpha_p = 1$ and the resulting solution is BPS with respect to the 10d SUSY algebra.

Solutions are RR charged.

Since D-branes are the basic sources of RR fields, it is natural to suppose that there is a connection between these solutions and D-branes.

D-Branes & Black Branes

Adding a D-brane adds a boundary to the set-up, and open string perturbation theory is good as long as $g_s N < 1$. In this regime the supergravity solution fails to be valid since:

$$R^2 \sim \left(\frac{r_p}{r}\right)^{7-p} \sim g_s N \left(\frac{\sqrt{\alpha'}}{r}\right)^{7-p}$$

On the other hand, the supergravity solution is weakly curved when $g_s N > 1$ in which the D-brane picture breaks down.

Complimentary description. In sum, for a large enough no. of coincident D-branes, spacetime gets deformed according to the geometry on the previous slide.

We can study the action of T-duality on Dp-branes using their representation as supergravity solutions (up to some subtleties!).

Part II

- Preliminaries & Motivation
  - Warped / Squashed Geometries
  - Sightings in String Theory.
- The construction
- Supersymmetry
- Conclusions and Outlook

Preliminaries & Motivation

3D Anti de Sitter space

Although does not have propagating gravitons, AdS3 Einstein gravity has nontrivial black hole solutions (BTZ black holes) thus providing a simple but rich arena for addressing questions about quantum gravity in general.

[Banados, Teitelboim, Zanelli], [Banados, Henneaux,...]...

- All of the solutions to the e.o.m for pure AdS3 Einstein gravity are locally AdS and are obtained as quotients of global AdS3 by a subgroup of its SO(2,2) isometry.

- AdS3 can arise in the near horizon geometry of various D–brane configurations and in some cases they can be shown to be exact string backgrounds.

- AdS spaces are also important in the context of the AdS/CFT correspondence. AdS3 quantum gravity is conjectured to be dual to a 2D CFT.

Recently the topic of 3d AdS solutions has been revived with interest in the context of 3D Topological Massive Gravity with negative cosmological constant. The theory admits a family of asymptotically AdS solutions parametrized by the value of the Chern–Simons coupling.

[Li, Song, Strominger], [Anninos, et.al.], [Compere, Detournay] ...

$$I_{TMG} = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left[(R + \frac{2}{l^2}) + \frac{1}{\mu} I_{CS} \right]$$

For every value of the CS coupling, TMG has a classical AdS3 solution with radius l. Theories for which the couplings satisfy $\mu l \neq 1$ yield unstable AdS3 vacua but have 2 "warped" AdS solutions for every value of μ

[Li, Song, Strominger], [Cho, Pope, Sezgin.], ...

A Note on Terminology: Squashed versus Warped

$$ds^2 = R^2 \left[d\omega^2 - \cosh\omega^2 d\tau^2 + \frac{1}{\cosh{\Theta_w}^2} (d\beta + \sinh\omega d\tau)^2 \right]$$

In this context a **warped AdS3 geometry** is obtained by changing the radius of the S1 fiber over AdS2. In a suitable coordinate system, we are considering Minkowskian 3–manifolds endowed with a one–parameter family of metrics where Θ_w is the deformation parameter that interpolates between AdS3 for $\Theta_w = 0$ and AdS2 ×S1 for $\Theta_w \to \infty$

"Squashed", in particular in the context of "squashed spheres", is universally used to mean a parameter has been changed from the value that would give the usual "round" geometry, with the parameter remaining constant. One could use the terminology "squashed AdS" in analogous fashion, as it can be obtained from squashed S^3 by analytic continuation.

Strominger et al. have called such deformations **warped AdS3**, whereas the word "warped" usually means in the literature a deformation that varies, rather than being constant. They also use **stretched** and **squashed** for different values of the squashing parameter. We will stick with **warped** bearing in mind that we mean a constant deformation to the U(1) fiber.

- The SL(2,R) × SL(2,R) isometry group of AdS3 is broken down to SL(2,R) × U(1). There are solutions with both timelike and spacelike U(1) isometries. At the critical value $\mu l = 3$ the two solutions have a null U(1) isometry.

- It is also possible to consider identifications of the above geometries under discrete subgroups of their isometries, á la BTZ.

More Warped AdS Sightings

- Topological Massive ED coupled to Gravity.
- Near horizon of extremal Kerr black holes with specific conditions.
- String Theory

String Theory Sightings

- Null WAdS3 has appeared as the non-trivial part of the geometry discussed in the search for a dual theory to non-relativistic CFT's.

[Hull, et.al.], ...

- Squashed WAdS3 (in the form of Gödel BH’s) can be lifted to a full string theory solution in ten dimensions.

[Compere, Detournay, Romo]

[Rooman, Spindel], [Isräel, Kounnas, Petropoulos]

- Black strings with near horizons given by WAdS3 ×S3 were shown to be **Hopf-T duals** of the dyonic black string in six-dimensions, supported by both **NS-NS** and **R-R** charges, which has an AdS3 × S3 near horizon.

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In TMG: For $\mu > 1$, the timelike fiber is stretched, and the space gives Gödel's geometry. Gödel's original solution corresponds to setting $\mu = 2$.

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Can the AdS/CFT correspondence be used to study String theory on these three-dimensional backgrounds?

It was proposed that quantum gravity in spacelike WAdS3 is holographically dual to a 2D CFT (Left and right moving central charges were explicitly written down by looking at thermodynamic properties of the corresponding BHs).

[Li, Song, Strominger], [Anninos, et.al]

The values of these central charges have been verified by studying the asymptotic symmetries á la **Brown-Henneaux** for specific deformation parameter regimes.

[Compere, Detournay]

$$CFT_1 \longleftrightarrow AdS_3 \times S^3_1 \longleftrightarrow WAdS_3 \times S^3_2 \overset{?}{\longleftrightarrow} CFT_2$$

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Hopf T-duality

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[Anninos]

String Theory Solutions & Marginal Deformations

- Squashed spacetime geometries are not new and have been studied in the context of deformed CFT's which were partially motivated by the search for black holes that generalised BTZ–type backgrounds.

[Israël, Kounnas, Orlando, Petropoulos], [Israël], [Detournay, Orlando, Petropoulos, Spindel], ...

- String theory realisations of metrics including 3-spheres and warped AdS3 spaces have been obtained as exact marginal deformations of SU(2) and SL(2, R) Wess–Zumino–Witten models, thus providing by construction a worldsheet theory.

- Such configurations rely on a non–vanishing NS–NS field to support the geometry.

- What about geometries sourced by RR backgrounds (S-dual configurations)?
- AdS/CFT is friendlier to string theories with RR backgrounds.
- Until now, there was no D–brane interpretation of the Hopf T–dual black string with near horizon geometry WAdS3 ×S3.

A step towards that direction was given in [Levi, Raeymaekers, et.al.], where Gödel space emerged from an M–theory compactification of the form Gödel × S2 × CY3, which was interpreted as coming from the backreaction of M2–branes wrapping the S2.

Our aim was then to improve our understanding of these vacua by their explicit realisation in string theory.

The Construction

Ingredients

- Standard brane intersection rules for building supergravity solutions
- Application of a form of Hopf T–duality at the level of the ten–dimensional theory.

Our starting point will be the IIB setup obtained by T–dualizing the D = 4 extremal dyonic black string. The configuration can be obtained by the superposition of a D1/D5 system with a magnetic monopole and a plane wave.

$$
\begin{aligned}
ds^2 &= H_1^{1/2}H_5^{1/2}\Big(H_1^{-1}H_5^{-1}(dudv+K du^2)+H_5^{-1}(dy_1^2+dy_2^2+dy_3^2+dy_4^2)+ \\
&+ V^{-1}(d\psi+A_i dx^i)^2+V(dx_1^2+\cdots+dx_3^2)\Big)
\end{aligned}
$$

$$
e^{2\phi}=H_1^{-1}H_5 \qquad\qquad F_{[3]}=H_1^{-1}dt\wedge du\wedge dv-B_i dx^i\wedge d\psi
$$

[Tseytlin], [Boostra, Peeters, Skenderis],...

where the harmonic functions depend on the transverse coordinates and

$$dB = *dH_5 \qquad dA = - * dV$$

Passing to spherical coordinates for the transverse coordinates and taking the near horizon limit one gets AdS3 x S3 x T4 just like in the standard D1/D5 system. Some of the variables are periodic by construction and one can impose a discrete identification in the anti–de Sitter part, leading to a BTZ black hole.

$$\begin{aligned} ds^2 &= Q_m Q_1^{1/2} Q_5^{1/2} \left(-d\tau^2 + d\omega^2 + Q_w d\sigma^2 + 2Q_w^{1/2} \sinh\omega d\sigma d\tau\right) \\ &+ Q_m Q_1^{1/2} Q_5^{1/2} (d\theta^2 + d\phi^2 + d\psi^2 + 2\cos\theta d\psi + d\phi) + Q_1^{1/2} Q_5^{-1/2} (dy_1^2 + \cdots + dy_4^2) \end{aligned}$$

$$e^{2\phi} = Q_1^{-1} Q_5 \qquad F_{[3]} = Q_m Q_1^{1/2} Q_5^{1/2} (\cosh\omega d\tau \wedge d\omega \wedge d\sigma + \sin\theta d\phi \wedge d\psi \wedge d\theta)$$

with

$$\begin{aligned} \psi &= \psi + 4\pi \\ \sigma &= \sigma + 4\pi \\ y_i &= y_i + 2\pi \end{aligned}$$

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Plane Wave

$$\begin{aligned} ds^2 &= Q_m Q_1^{1/2} Q_5^{1/2} \left(-d\tau^2 + d\omega^2 + Q_w d\sigma^2 + 2Q_w^{1/2} \sinh\omega d\sigma d\tau\right) \\ &+ Q_m Q_1^{1/2} Q_5^{1/2} (d\theta^2 + d\phi^2 + d\psi^2 + 2\cos\theta d\psi + d\phi) + Q_1^{1/2} Q_5^{-1/2} (dy_1^2 + \cdots + dy_4^2) \end{aligned}$$

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with

$$\begin{aligned} \psi &= \psi + 4\pi \\ \sigma &= \sigma + 4\pi \\ y_i &= y_i + 2\pi \end{aligned}$$

We introduce the following variables

$$\psi = \alpha + 2y_i \qquad \sigma = \beta + 2y_2$$

and rewrite the metric as

$$\begin{aligned} ds^2 &= R^2\left[d\omega^2 - \cosh^2\omega d\tau^2 + \frac{1}{\cosh^2\Theta_w}\left(d\beta + \sinh\omega d\tau\right)^2\right] \\ &+ R^2\left[d\theta^2 + \sin^2\theta d\phi^2 + \frac{1}{\cosh^2\Theta_m}\left(d\alpha + \cos\theta d\phi\right)^2\right] + \frac{R^2}{\sinh^2\Theta_m}\left(dy_3^2 + dy_4^2\right) \\ &+ \left(dz_w + R\tanh\Theta_w\left(d\beta + \sinh\omega d\tau\right)\right)^2 + \left(dz_m + R\tanh\Theta_m\left(d\alpha + \cos\theta d\phi\right)\right)^2 \end{aligned}$$

where

$$R^2 = Q_m\sqrt{Q_1 Q_5} \qquad \sinh^2\Theta_m = 4Q_m Q_5 \qquad \sinh^2\Theta_w = 4Q_w Q_m Q_5$$

and

$$z_m = \frac{R}{\tanh\Theta_m}y_1 \qquad z_w = \frac{R}{\tanh\Theta_w}\tilde{y}_2$$

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where

Def. parameter not continuous ↙

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Notice:

- Both the AdS3 and S3 geometries can be understood as Hopf fibrations (respectively of AdS2 and S2), and performing straightforward T-duality in the direction of the fiber can undo the structure.

- Notice that the y coordinates stop describing an external torus when we introduce the coordinates α and β, which are linear combinations of the y coordinates and angular coordinates in AdS and the sphere respectively.

We now perform T-duality along the z directions. Since we only have RR fields turned on, we cannot use the Büscher rules directly but have to reduce to 9D, rewrite the fields in terms of their IIA counterparts and oxidise to 10D, along the T-dual coordinate.

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We are interested in performing T-duality in a direction that is a **mixture** of one of the fiber directions on the compact space, and one of the Torus directions.

Squashed Sphere

We perform T-duality along the z_m direction:

$$\begin{aligned} ds^2_{10} &= AdS_3[R] + T^3 + R^2\left[d\theta^2 + \sin^2\theta d\phi^2 + \frac{1}{\cosh^2\Theta_m}\left(d\alpha + \cos\theta d\phi\right)^2\right] + d\xi_m^2 \\ F_4 &= F_3^{(4)} \wedge d\xi_m = \left[\omega_{AdS} + \frac{R^2}{\cosh^2\Theta_m}\sin\theta d\theta \wedge d\phi \wedge d\alpha\right] \wedge d\xi_m \\ F_2 &= F_2^{(2)} = R\tanh\Theta_m \sin\theta d\theta \wedge d\phi \\ H_3 &= F_2^B \wedge d\xi_m = R\tanh\Theta_m \sin\theta d\theta \wedge d\phi \wedge d\xi_m \end{aligned}$$

It is worthwhile to emphasise that by construction α is 4π-periodic and the geometry is the one of a respectable squashed three-sphere.

Squashed Sphere

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Def. parameter

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Def. parameter

T-dual coordinate

It is worthwhile to emphasise that by construction α is 4π-periodic and the geometry is the one of a respectable squashed three-sphere.

Remarks

- The construction here presented differs from that of Duff, et.al. given that in that scenario, the starting configuration had both NS and RR fluxes turned on. Then T-duality was considered along one of the sphere isometries, yielding a Lens space $S^3/\mathbb{Z}_p$ or a squashed version when only NS fluxes were present, where p and the squashing depend on the values of the charges.

- If only NS fluxes were turned off in that case, the resulting T-dual geometry is that of S2 x S1.

Consider the S3 × S1 part. The geometry can be understood as the fibration

$$\begin{array}{ccc} S^1 \times S^1 & \longrightarrow & S^3 \times S^1 \\ & & \downarrow \\ & & S^2 \end{array}$$

where one of the directions in the torus fibration is the Hopf fiber in S3.

A S1 sub-bundle A of the torus, obtained as a rational linear combination of the two directions, describes a fibration

$$\begin{array}{ccc} A & \longrightarrow & S^3 \times S^1 \\ & & \downarrow \\ & & \mathrm{Sq}S^3 \end{array}$$

where the squashing parameter depends on the coefficients in the linear combination. Just like in the pure S3 case described in Duff, et.al., since we only have RR fields, performing a Hopf-T-duality in the A direction will "unwind" the fiber and lead to a geometry which is the direct product SqS3 ×S1.

The same statements can be applied to the AdS3 ×S1 part.

In principle these constructions can be extended to other group manifold geometries (e.g. the obvious choice leading to a squashed AdS3) but in any case one should start from a configuration with RR fields, *since the absence of NS–NS antisymmetric fields is the key ingredient for the trivialisation of the fiber bundle.*

$$
\begin{aligned}
ds^2 &= R^2\left[d\omega^2 - \cosh^2\omega d\tau^2 + \frac{1}{\cosh^2\Theta_w}\left(d\beta + \sinh\omega d\tau\right)^2\right] + d\zeta_w^2 + S^3[R] + T^3 \\
F_4 &= \left[R^2\,\omega_{\mathrm{S}} + \frac{R^2}{\cosh^2\Theta_w}\cosh\omega d\omega\wedge d\tau\wedge d\beta\right]\wedge d\zeta_w \\
F_2 &= R\tanh\Theta_w\cosh\omega d\omega\wedge d\tau \\
H_3 &= R\tanh\Theta_w\cosh\omega d\omega\wedge d\tau\wedge d\zeta_w
\end{aligned}
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Warped AdS

$$
\begin{aligned}
ds^2 &= R^2\left[d\omega^2 - \cosh^2\omega d\tau^2 + \frac{1}{\cosh^2\Theta_w}(d\beta + \sinh\omega d\tau)^2\right] + d\zeta_w^2 + S^3[R] + T^3 \\
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\end{aligned}
$$

Isometries

In order to understand the isometries of the squashed and warped spaces it is convenient to describe their geometry in algebraic terms. Both the 3–sphere and AdS3 are group manifolds, respectively for SU(2) and SL(2, R).

$$ds^2 = R^2 \mathrm{Tr}[g^{-1} dg g^{-1} dg] = R^2 \sum_a J^a \otimes J^a$$

with $J^a = \mathrm{Tr}[t^a g^{-1} dg]$

$$ds^2[\Theta] = R^2 \left(J^1 \otimes J^1 + J^2 \otimes J^2 + \frac{1}{\cosh \Theta^2} J^3 \otimes J^3 \right)$$

The effect of the T–duality amounts to adding an extra term to the metric proportional to J3^2 (In the sl2 case, the J3 is the hyperbolic generator).

The initial group manifold has G × G isometry, generated by Ja and bar Ja but only part of this symmetry remains after the T-duality. To be precise, while the J bar generators are preserved (they commute with the current J3), both J1 and J2 are not Killing vectors anymore, as one can verify with a direct calculation of the Lie derivative of the metric:

$$\mathcal{L}_{J^1}[ds^2[\Theta]] = 2R^2 \tanh[\Theta] J^2 \otimes J^3 \qquad \mathcal{L}_{J^2}[ds^2[\Theta]] = 2R^2 \tanh[\Theta] J^3 \otimes J^1$$

Even though the squashed sphere is not a group manifold, we can still use techniques borrowed from group theory. In particular, the isometry group for this part of the metric is SU(2) × U(1) and that the spectrum of the scalar Laplacian is

$$\Delta_{\Theta_m} Y_{ij} = \frac{1}{R^2}\left[l(l+1) + \sinh{\Theta_m}^2 j^2\right] Y_{ij} \qquad l = 0, 1/2, \cdots ; \qquad j = -l \cdots l$$

In a similar way, one finds that the Laplacian spectrum on WAdS3. It is given by the sum of the Laplacian of AdS3 and an extra component.

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The resulting isometry group is G × U(1).

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Supersymmetry

T–duality transformations can break supersymmetries preserved by D–brane solutions.

This has already been observed in the context of Hopf T-dualities on 6D supergravity backgrounds of the form AdS3 ×S3. The phenomenon is akin to the breaking of supersymmetry by compactification on a circle, so the resulting supersymmetries will be those that survive the circle compactification.

We start studying the properties of these new backgrounds by computing the explicit expressions of the Killing spinors.

- Under T–duality, the IIA backgrounds with squashed/warped spacetimes preserve 1/4 of the supersymmetries of the original D1/D5 background.

- For specific values of the deformation parameter some supersymmetries are restored and generically, IIB backgrounds containing both warped AdS and squashed spheres preserve no supersymmetry.

When $\mathrm{sech}\Theta_m = 0$ or $\mathrm{sech}\Theta_w = 0$

There are no spinors depending on the T–dual coordinate and supersymmetry is restored to 1/2-BPS.

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This corresponds to the case in which the S3 becomes S2 × S1 or AdS3 becomes AdS2 x S1. This had already been anticipated in studies of the S-dual configuration, via marginal deformations of WZW.

Concluding remarks and Outlook

Summary

We have explicitly constructed string theory backgrounds (at the level of sugra) which include 3D squashed spheres and/or warped anti-de Sitter spaces. They can be seen as Hopf–T dual brane intersections of D1/D5 systems with monopole and/or plane waves.

- The deformation parameters of the squashed/warped spaces are interpreted in terms of the charges of the original background.

Under T–duality, IIA backgrounds containing SqS3 or WAdS3 preserve eight susys for generic values of the deformation parameter.

- In the special cases $\cosh\Theta_{w,m} \to \infty$, some supersymmetries are restored and the backgrounds preserve sixteen supersymmetries.

These results were obtained in the context of supergravity. It is well-known that spacetime susys that are manifest in some string background might very well be hidden in their T–dual.

[Bakas, Sfetsos], [Sfetsos]

There are examples in the literature in which supersymmetry that seemed to be destroyed by duality could actually be restored by a non–local realization.

[Hassan], [Ricci, Tseytlin, Wolf]

- To clarify these issues, it would be necessary to study the form of Hopf-T duality here discussed at the level of the GS action for a IIB AdS3 ×S3 × T4.

- If a dual CFT can be defined, an analysis from this perspective would be enlightening as all worldsheet and spacetime supersymmetries should remain symmetries of the underlying CFT

Generalisations

- The same construction works for any metric that posses an S1 fibration and formally also for S3 fibers, even though in this case, some modifications would need to be included (in particular in order to generalize the T–duality to non–Abelian fields).

- Another possibility would be to follow the procedure along the time-like S1 fiber in AdS3, which would lead to a hyperbolic plane (H2) geometry in type II* backgrounds with negative kinetic terms.

- We have not explored T–dualizing the dyonic black string background along a direction that mixes the two initial coordinates ψ and σ. The resulting geometry is described by the metric interpolating between WAdS3 ×S3 and AdS3 × SqS3.

Holography (?)

It would be interesting to use these solutions to compute holographic data, thus formulating in a more precise way the duality chain. This would also be interesting from the black hole microstate motivation.

For the case in which the AdS space becomes warped after Hopf T-dualizing, an appropriate embedding in an asymptotically AdS background would be required.

In the realisation of Gödel geometry via M2-brane probes, the authors considered enclosing the region by a domain wall, cancelling the charge. Then, as in three dimensions all vacuum spacetimes are locally AdS3, on the other side of the wall it is guaranteed to find a local AdS3 spacetime.

In other words, they tried to **glue** the Gödel's space through a domain wall to AdS3.

Interesting quantities to determine: Holographic Stress Tensor, Partition Functions, Entropy?

More on String Theory in WAdS

It would be interesting to study the backgrounds here constructed from the full string theory, via its sigma model action. For this, the knowledge of the Hopf T-dual action would be required.

- Usual CFT methods do not work for the RR AdS3 sigma-model.
- AdS3 backgrounds with RR flux are as complicated as their higher-dimensional counterparts AdS5 × S5 or AdS4 × CP3, but are exactly solvable due to their integrability properties.

Integrability in AdS3/CFT2 via GS action on AdS3 × S3, and from the symmetries of the giant magnons on this background. What about the properties of the WAdS3 string theory?

[Rahmfeld, Rajaraman], [Chen, Zhang, Son], [Adam, Dekel, Mazzucato, Oz], [David, Sahoo]

[Babichenko, Stefanski, Zarembo],... [Alday, Arutyunov, Frolov], [Ricci, Tseytlin, Wolf]

Thank you