On a blowup formula for sheaf-theoretic virtual enumerative invariants on projective surfaces

Yuuji Tanaka, Kyoto University

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Plan of talk:

1. Betti numbers of the moduli spaces of semistable vector bundles on a curve/coherent sheaves on a surface
2. Donaldson invariatns on closed four-manifolds
3. Virtual enumerative invariants on projective surfaces
4. A blowup formula for sheaf-theoretic virtual enumerative invariants on projective surfaces

Betti numbers of the moduli spaces of vector bundles

Consider $\mathcal{M}_C^{ss}(n,r)$ the moduli space of semistable vector bundles of rank r with degree n over a complex curve C. Assume for simplicity n and r are coprime to let $\mathcal{M}_C^{ss}(n,r)$ smooth.

- **Harder–Narasimhan** ('75) obtained an inductive formula of and hence computed the Poincaré polynomial of $\mathcal{M}_C^{ss}(n,r)$ via the Weil conjecture (Deligne,'74) by counting in a finite field.
- **Atiyah–Bott** ('82) reproved and also improved the result by Harder–Narashimhan via the infinite dimensional Morse theory.
- **Kirwan** ('83) obtained the Poincaré polynomial of GIT quotient as a finite-dimensional version of the result by Atiyah–Bott.

cf. Later, **Earl–Kirwan** ('00) obtained an inductive formula of the Hodge–Poincaré polynomial of $\mathcal{M}_C^{ss}(n,r)$ without using the Weil conjecture.

The surface case: not only a higher-dimensional analogue

Göttsche ('90) computed the Betti numbers of Hilbert schemes of points $X^{[n]}$ on a projective surface X. Subsequently, **Yoshioka** ('94) did it for $\mathbb{P}^2$ (cf. work by Klyachko ('91)), ruled surfaces ('95), and elliptic surfaces ('96) (both for the semistable=stable and $\mathrm{Ext}^2_0(E,E)=0$ case).

A new feature emerges when one considers it on a complex projective surface, namely, one can consider the **generating series** of e.g. the Euler characteristics by varying c_2 (can not do this on a curve). Surprisingly, one often finds modular forms. E.g., Göttsche obtained:

$$\sum_{k=0}^{\infty} e(X^{[k]})q^k = (q^{-\frac{1}{24}}\eta(q))^{-e(X)},$$

where $\eta(q) := q^{\frac{1}{24}} \prod_{k=1}^{\infty}(1-q^n)$ is the Dedekind eta function and $e(X)$ is the Euler characteristic of X.

The anti-self-dual instanton moduli spaces

For now, let X be a closed, oriented, simply-connected, smooth four-manifold, and let $P \to X$ be a principal $SO(3)$-bundle with the first Pontryagin class p_1 and the second Stiefel–Whitney class w_2.

Fix a Riemannian metric g on X, and consider the Hodge star operator $*_g$ on $\Lambda^2_X := (\Lambda^2 T^* X)$. This satisfies $*_g^2 = 1$, so Λ^2_X decomposes as $\Lambda^2_X = \Lambda^+_X \oplus \Lambda^-_X$.

Definition: A connection A on P is said to be an *anti-self-dual instanton*, if the curvature F_A of A satisfies $F_A^+ := \pi_+(F_A) = 0$, where $\pi_+ : \Gamma(\mathfrak{g}_P \otimes \Lambda^2_X) \to \Gamma(\mathfrak{g}_P \otimes \Lambda^+_X)$ is the projection and $\mathfrak{g}_P$ is the adjoint bundle of P.

We denote by $\mathcal{A}_P$ the set of all connections on P and by $\mathcal{G}_P := \mathrm{Aut}(P)$ the set of all gauge transformations on P, or the *gauge group*.

The gauge group $\mathcal{G}_P$ acts on $\mathcal{A}_P$, and we consider the *anti-self-dual instanton moduli space*:

$$M_{X,g}(w_2, p_1) := \{A \in \mathcal{A}_P \,:\, F_A^+ = 0\}/\mathcal{G}_P.$$

This is an **oriented smooth manifold of expected dimensions** for a generic choice of Riemannian metrics of X, if $b_X^+ > 0$, where b_X^+ is the number of positive eigenvalues of the intersection form on $H^2(X, \mathbb{Z})$.

Uhlenbeck compactification:

$$\overline{M}_{X,g}(w_2, p_1) := \coprod_{\ell=0}^{-p_1/4} M_{X,g}(w_2, p_1 + 4\ell) \times S^\ell X,$$

where $S^\ell X$ is the ℓ-th symmetric product of X. This is equipped with a natural topology, and it is a compact Hausdorff space.

Donaldson's polynomial invariants

Assume $b_X^+ > 0$. Denote by 2d the expected dimension of $M_{X,g}(w_2, p_1)$ and by A_d the symmetric algebra of degree d generated by $H_2(X)$ and $H_0(X)$ with $\alpha \in H_2(X)$ degree 1 and $p \in H_0(X)$ degree 2.

One can define $\bar{\mu} : H_p(X, \mathbb{Z}) \to H^{4-p}(\overline{M}_{X,g}(w_2, p_1), \mathbb{Z})$ for $p = 0, 2$ via a universal bundle on the moduli space. We then define Donaldson's polynomial invariant $q_{X,d} : A_d \to \mathbb{Z}$ of degree d by

$$q_{X,d}(\sigma_1, \dots, \sigma_n, p^m) := \int_{[\overline{M}_{X,g}(w_2,p_1)]} \bar{\mu}(\sigma_1) \cup \cdots \cup \bar{\mu}(\sigma_n) \cup \bar{\mu}(p)^m,$$

where $\sigma_1, \dots, \sigma_n \in H_2(X)$, $p \in H_0(X)$ and $d = n + 2m$. This is independent of the choice of Riemannian metrics of X, if $b_X^+ > 1$.

If $w_2 \neq 0$, then there is no trivial bundle in lower strata of the Uhlenbeck compactification, and the fundamental class $[\overline{M}_{X,g}(w_2, p_1)]$ is well-defined. However, if $w_2 = 0$, then there is a trivial bundle in a lower stratum of the compactification. In this case, we introduce the notion of *stable range* (e.g. one requires $-p_1$ is sufficiently large) to have a well-defined fundamental class $[\overline{M}_{X,g}(w_2, p_1)]$.

Blowup formula by Friedman–Morgan: Consider the blowup $\widehat{X} \to X$ at a point in X and denote by e its exceptional divisor. Then Friedman and Morgan prove:

$$q_{\widehat{X},d+1}(\sigma_1, \dots, \sigma_d, e, e, e, e) = -2q_{X,d}(\sigma_1, \dots, \sigma_d).$$

This can be used to define the Donaldson invariants for *unstable range*.

Moreover, blowup formulae played fantastic roles in figuring out algebraic structures in the theory of the Donaldson invariants.

Kronheimer–Mrowka ('95) proved the structure theorem for the Donaldson invariants. Denote by $q_X : \oplus_d A_d \to \mathbb{Q}$ the polynomial invariants. They considered the following Donaldson series:

$$\sum \frac{q_X(\Sigma^d)}{d!} + \frac{1}{2} \sum \frac{q_X(p\Sigma^d)}{d!}.$$

under the assumption of X *simple type*, and proved that this formal series can be written by a finite collection of *basic classes* by developing a theory of singular connections and a blowup formula.

Fintsushel–Stern ('96) gave an alternative proof of the above structure theorem by improving the blowup formula. They further obtained a universal form of blowup formula, surprisingly, it is expressed in terms of modular forms.

Göttsche ('96) determined the wall-crossing term under the assumption that *Kotschick–Morgan conjecture* is true by an effective use of the blowup formula, and it turned out to be written in terms of modular forms.

Supersymmetric Yang–Mills theories

Witten ('88) reformulated the Donaldson invariants in terms of a topologically-twisted theory of $\mathcal{N} = 2$ super Yang–Mills theory.

Seiberg–Witten ('94) subsequently discovered the Donaldson series can be written in terms of the intersection form and a finite collection of Seiberg–Witten invariants (cf. the work of Kronheimer and Mrowka) which can be defined through the moduli spaces of pairs consisting of a $U(1)$-connection and a positive spinor on X, via a generalisation of electro-magnetic duality which is believed to exist in the theory.

Moore–Witten ('97) generalised the work by Seiberg–Witten to the case including wall-crossing phenomena and clarified more the origin of the modularity in the theory by means of the *u-plane*.

Vafa–Witten ('94) considered a more symmetric model: a topological twist of $\mathcal{N} = 4$ Super Yang–Mills theory.

Vafa–Witten equations: Let X be a closed, oriented, smooth four-manifold, and let $P \to X$ be a principal G-bundle over X, where G is a connected Lie group. Fix a Riemannian metric on X. For $(A, B, C) \in \mathcal{A}_P \times \Gamma(\mathfrak{g}_P \otimes \Lambda^+_X) \times \Gamma(\mathfrak{g}_P \otimes \Lambda^0_X)$, we consider the following equations:

$$d_A^* B + d_A C = 0,\ F_A^+ + [B, C] + [B.B] = 0,$$

where $[B.B] \in \Gamma(\mathfrak{g}_P \otimes \Lambda^+_X)$.

These equations with a gauge fixing equation form an elliptic system with index always zero.

(T ('13, '15) Taubes ('13, '17), and others study the compactification problem for the moduli space of solutions to these equations, but it looks very difficult at the moment.)

Vafa and Witten's "invariant" has the following form:

$$VW := \chi(\overline{M}_{X,g}(w_2, p_1)) + \sum_{\xi \in M_{VW} \setminus M_{X,g}(w_2,p_1)} \varepsilon(\xi),$$

where χ is the "Euler characteristic", M_{VW} is the moduli space of solutions to the Vafa–Witten equations, and $\varepsilon(\cdot)$ is a "signed counting" of points in $M_{VW} \setminus M_{X,g}(w_2, p_1)$. They conjectued the generating series of this could be written in terms of modular forms. They checked it when $(B, C) \equiv 0$, in this case, one does not have the second term in the above and the generating series is that of the Euler characteristics of the instanton moduli spaces.

A far-reaching explanation of the modularity: Hiraku Nakajima constructs a representation of e.g the Heisenberg algebra on the homology of moduli space of the Hilbert scheme of points on a surface ('97), then the modularity follows since the generating series is the character of the representation.

Mochizuki's virtual Donaldson invariants

The Hitchin–Kobayashi correspondence tells the instanton moduli space is the same as the moduli space of semistable sheaThe ves on a complex projective surface. (cf. Algebraic Donaldson invariants.)

Then there is one other way of defining Donaldson type invariant from the moduli space of semistable sheaves by using *virtual techniques* developed by **Li–Tian** and **Behrend–Fantechi**. in Gromov–Witten theory. (cf. **Li–Tian** and **Fukaya–Ono** in Symplectic Geometry).

Takuro Mochizuki ('09) constructed a perfect obstruction theory on the moduli space. Then, **Donaldson–Mochizuki invariants** are defined as the integrations over the virtual fundamental class.

Mochizuki performed this in full generality (even with parabolic structures). More precisely, Mochizuki

- formulates the invariants where the moduli space may have strictly semistable sheaves;
- proves a weak wall-crossing formula for his invariants; and
- expresses them in terms of Seiberg–Witten invariants.

These lead to the determination of the wall-crossing terms and a resolution of Witten's conjecture $D = SW$ on a projective surface both by **Göttsche–Nakajima–Yoshioka** ('08, '11) both with analysis (blowup formulae) on the Nekrasov partition function.

One may think about other insertions. **Fantechi–Göttsche** and **Ciocan-Fontanine–Kapranov**, for instance, define the *virtual Euler characteristic* of $\mathcal{M}$ by

$$e^{\mathrm{vir}}(\mathcal{M}) := \int_{[\mathcal{M}]^{\mathrm{vir}}} c_{vd}(T^{\mathrm{vir}}_{\mathcal{M}}),$$

where $T^{\mathrm{vir}}_{\mathcal{M}}$ is the virtual tagent sheaf coming form the perfect obstruction theory on $\mathcal{M}$.

More generally, they define the *virtual* χ_y*-genus* of $\mathcal{M}$ by

$$\chi_{-y}^{\mathrm{vir}}(\mathcal{M}) := \int_{[\mathcal{M}]^{\mathrm{vir}}} (1 - y\mathrm{ch}(\mathrm{T}^{\mathrm{vir}})^{-1}) \cdot \mathrm{td}(\mathrm{T}^{\mathrm{vir}})) \in \mathbb{Q}[y].$$

These virtual Euler characteristic and virtual χ_y-genus of the moduli space of semistable sheaves can be thought of as the *instanton part* of the Vafa-Witten invariant or a refinement of it.

Göttsche–Kool forms conjectures that the generating series of them and also other virtual enumerative invariants such as *virtual Segre* and *Verlinde numbers* of the moduli spaces could be written in terms of modular forms and Seiberg–Witten invariants.

They also raised conjectures on blowup formulae for them. The one for the virtual Euler characteristics of the moduli spaces resembles **Li–Qin**'s one for the virtual Hodge polynomials, whose origin is the work by Vafa–Witten, where the interesting appearance of modular forms is explained by means of Conformal Field theory, or Vertex Algebras. (cf. **Nakajima**'s pioneering works for these subjects)

Vafa–Witten invariants on projective surfaces

The Hitchin–Kobayashi correspondence (by Álvarez-Cónsul and García-Prada ('03), T ('13)) tells the moduli space of solutions to the Vafa–Witten equations is the same as the moduli space $\mathcal{N}$ of pairs (E, ϕ) consisting of a coherent sheaf E and a morphism $\phi : E \to E \otimes K_X$ with a certain stability condition on a complex projctive surface X, where K_X is the canonical bundle of X.

Richard Thomas and the speaker ('17) constructed a symmetric perfect obstruction theory on the above moduli space $\mathcal{N}$.

There is a $\mathbb{C}^*$-action on $\mathcal{N}$ induced by the multiplication $\phi \mapsto \lambda\phi$ by $\lambda \in \mathbb{C}^*$. The fixed loci $\mathcal{N}^{\mathbb{C}^*}$ of this action is:

$$\mathcal{N}^{\mathbb{C}^*} \cong \mathcal{M}_X^{ss}(p, r) \sqcup \mathcal{M}^{Higgs},$$

where $\mathcal{M}_X^{ss}(p, r)$ is the moduli space of semistable sheaves of rank r with fixed Hilbert polynomial p on X.

The moduli space $\mathcal{N}$ is not proper, but these fixed loci are proper, so we define:

$$VW := \int_{[\mathcal{N}^{\mathbb{C}^*}]^{vir}} \frac{1}{e^{\mathbb{C}^*}(N^{vir}_{\mathcal{N}^{\mathbb{C}^*}})}.$$

By a result of **Jiang–Thomas** ('14), one sees:

$$\begin{aligned} VW &:= \int_{[\mathcal{N}^{\mathbb{C}^*}]^{vir}} \frac{1}{e^{\mathbb{C}^*}(N^{vir}_{\mathcal{N}^{\mathbb{C}^*}})} \\ &= \int_{[\mathcal{M}^{ss}_X(p,r)]^{vir}} \frac{1}{e^{\mathbb{C}^*}(N^{vir}_{\mathcal{M}^{ss}_X(p,r)})} + \int_{[\mathcal{M}^{Higgs}]^{vir}} \frac{1}{e^{\mathbb{C}^*}(N^{vir}_{\mathcal{M}^{Higgs}})} \\ &= \underbrace{\int_{[\mathcal{M}^{ss}_X(p,r)]^{vir}} c_{vd}(T^{vir}_{\mathcal{M}^{ss}_X(p,r)})}_{e^{vir}(\mathcal{M}^{ss}_X(p,r))} + \underbrace{\int_{[\mathcal{M}^{Higgs}]^{vir}} \frac{1}{e^{\mathbb{C}^*}(N^{vir}_{\mathcal{M}^{Higgs}})}}_{\text{the contribution from the Higgs fields}} \quad . \end{aligned}$$

This does resemble the form that Vafa and Witten envisaged, which we discussed earlier in this talk. In fact, this matches with the conjecture by Vafa and Witten.

For the K3 surface case, Thomas and the speaker ('17) obtained the following, assuming a conjecture by Yukinobu Toda ('11) in Donaldson–Thomas theory (it was later proved by Maulik and Thomas ('18)), which also matches with the conjecture by Vafa and Witten:

$$\sum_{c_2} VW_{r,c_2} q^{c_2} = \sum_{d|r} \frac{d}{r^2} \sum_{j=0}^{d-1} \eta \left(e^{\frac{2\pi i j}{d}} q^{\frac{r}{d^2}} \right)^{-24}.$$

To be precise, the above is for the $SU(r)$ Vafa–Witten invariants, namely, we fix the first Chern class of sheaves and consider ϕ with $\mathrm{tr}\phi = 0$, let us denote them by $VW_{r,c_2}^{SU(r)}$. Also, the *Vafa–Witten partition function* is defined to be:

$$Z_{VW}^{SU(r)}(q) := r^{-1} q^{\lambda} \sum_{c_2} q^{\frac{1}{2r}\mathrm{vd}} (-1)^{\mathrm{vd}} VW_{r,c_2}^{SU(r)},$$

where $\lambda := -\frac{1}{2}\chi(\mathcal{O}_X) + \frac{r}{24}K_X^2$, and vd is the virtual dimension of the moduli space $\mathcal{M}_X^{ss}(p, r)$.

Later, **Jiang and Kool** ('20) proved the *S-duality conjecture* for $K3$ surfaces by constructing the Langlands dual side of the story, when the rank r is prime:

$$Z_{VW}^{SU(r)}(-1/\tau) = (-1)^{(r-1)\chi(\mathcal{O}_X)} \left(\frac{r\tau}{i}\right)^{-\frac{e(X)}{2}} Z_{VW}^{{}^L SU(r)}(\tau),$$

where $q = e^{2\pi i\tau}$, and the ${}^L SU(r)$ Vafa–Witten partition function $Z_{VW}^{{}^L SU(r)}$ on the right-hand side is defined through the moduli spaces of twisted sheaves by Yoshioka.

Furthermore, Göttsche–Kool–Laarakker proved possible closed formulae for the rank 4 and 5 (vertical) Vafa–Witten partition function. Interestingly, e.g. the rank 5 case involves **Rogers–Ramanujan's continued fraction**. They then raise an interesting new conjecture on the virtual Euler characteristics of the moduli spaces of semistable sheaves on a surface for the rank 4 and 5 cases by using the S-duality argument.

A blowup formula for virtual enumerative invariants

We establish a Nakajima–Yoshioka style blowup formula for virtual enumerative invariants on a projective surface, which include the virtual Euler characteristics, virtual χ_y-genera, virtual Segre and Verlinde numbers of the moduli spaces of semistable sheaves, and the Donaldson–Mochizuki invariants, by constructing perfect obstruction theories on the moduli spaces of m-stable sheaves which interpolate the moduli space of semistable sheaves on a complex projective surface and that on its blowup at a point, via enhanced master spaces.

Nakajima–Yoshioka's *m*-stable sheaves. Let $p : \hat{X} \to X$ be the blowup at a point of a projective surface X. Fix a cohomology class $c := r + c_1 + \mathrm{ch}_2$ of X and consider $\widehat{c} = p^*c + ke$ of $\widehat{X}$, where $e := \mathrm{ch}(\mathcal{O}_C(-1))$ with C the exceptional divisor.

We would like to compare the (virtual) fundamental classes coming from the moduli spaces $\mathcal{M}_X(c)$ and $\mathcal{M}_{\widehat{X}}(\widehat{c})$ of semistable sheaves on X and $\hat{X}$ respectively.

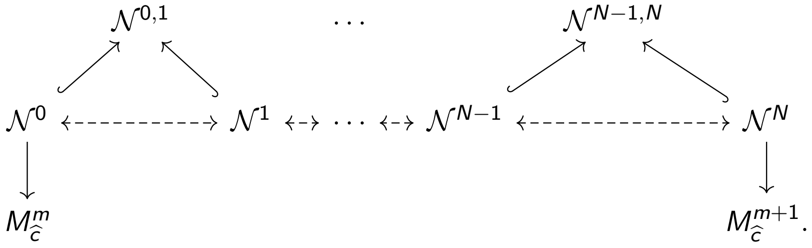
To do so, we consider the moduli spaces $M^m_{\widehat{c}}$ of m-stable sheaves on $\widehat{X}$ introduced by Nakajima–Yoshioka on the blowup, which interpolate the moduli spaces of semistable sheaves on the blowup and on the original surface in the following manner:

$$\begin{array}{ccccccccc} M^0_{\widehat{c}} & \leftarrow\dashrightarrow & M^1_{\widehat{c}} & \leftarrow\dashrightarrow & \cdots & \leftarrow\dashrightarrow & M^{m-1}_{\widehat{c}} & \leftarrow\dashrightarrow & M^m_{\widehat{c}} \\ \downarrow{\scriptstyle p_*} & & & & & & & & \downarrow \\ \mathcal{M}_X(c) & & & & & & & & \mathcal{M}_{\widehat{X}}(\widehat{c}), \end{array}$$

where the right vertical arrow becomes an isomorphism if m is sufficiently large, while so is the left vertical arrow if $\widehat{c} = p^*c$.

Wall-crossing. In order to compare things on $M_{\widehat{c}}^{m}$ with those on $M_{\widehat{c}}^{m+1}$, one may embed them in an intermediate space, the moduli space of $(m, m+1)$-semistable sheaves.

But, the moduli space of them has highly positive-dimensional automorphism groups, so we use Mochizuki's technique, the *enhanced moduli stack* $\mathcal{N}^{\ell}$, which is a Deligne–Mumford stack, together with the intermediate Artin stack $\mathcal{N}^{\ell,\ell+1}$, which has only one-dimensional stabiliser group, satisfying the following diagram:



We then construct the Kiem–Li style master space $\mathcal{Z}$ equipped with a $\mathbb{C}^*$-action, and its set-theoretical $\mathbb{C}^*$-fixed loci are

$$|\mathcal{Z}|^{\mathbb{C}^*} = |\mathcal{N}^{\ell}| + |\mathcal{N}^{\ell+1}| + \text{the rest.}$$

The stack-theoretical version of this with virtual integrations over the fibres of $\mathcal{N}^0$ and $\mathcal{N}^N$ leads to a wall-crossing formula we desire.

Blowup formula for Donaldson–Mochizuki invariants. One application of our wall-crossing formula is the Friedman–Morgan style blowup formula for the virtual invariants:

$$D_{\widehat{X},p^*c_1}(\sigma_1 \dots \sigma_n[C]^4) = -2D_{X,c_1}(\sigma_1, \dots, \sigma_n).$$

This leads to a direct proof of the equivalence between Mochizuki's virtual Donaldson invariants and the classical ones in our setting.

Nakajima–Yoshioka style structure theorem. Let $\Phi(\mathcal{E})$ be a Chow cohomology class depending on the universal sheaf on $X \times \mathcal{M}$ and cohomology classes of X such as the insertions for virtual Euler characteristics, virtual χ_y-genera, virtual Segre and Verlinde numbers of the moduli spaces of semistable sheaves on X, or the Donaldson–Mochizuki invariants. We then obtain:

Theorem (Kuhn–T) Fix a Chern class $\widehat{c} = p^*c - ke$ for some $k \geq 0$ on $\widehat{X}$, where $c = r + c_1 + c_2$ is that on X. Then there exist universal power series $\Omega_n \in \mathbb{Q}[[\nu_2, \ldots, \nu_r]]$ depending only on r, Φ and k, which satisfy:

$$\int_{[\mathcal{M}_{\widehat{X}}(\widehat{c})]^{\mathrm{vir}}} \Phi(\mathcal{E}) = \sum_{n=0}^{\infty} \int_{[\mathcal{M}_X(c+n[pt])]^{\mathrm{vir}}} \Phi(\mathcal{E})\Omega_n(\mathcal{E}).$$

From the construction, these Ω_n can be determined by calculating them on the moduli spaces of framed sheaves on $\mathbb{P}^2$.
This resolve conjectures by **Göttsche–Kool** on the structures of various virtual eumerative invariants of projective surfaces.

For example, we obtain the following blowup formula for the generating series of the virtual χ_y-genera of the moduli spaces:

Theorem (Kuhn–Leigh–T)

$$\sum_{\mathrm{ch}_2} \chi^{\mathrm{vir}}_{-y}(\mathcal{M}_{\hat{X}}(\hat{c}))q^{\mathrm{vd}\,\mathcal{M}_{\hat{X}}(\hat{c})} = Y_k(q,y) \sum_{\mathrm{ch}_2} \chi^{\mathrm{vir}}_{-y}(\mathcal{M}_X(c))q^{\mathrm{vd}\,\mathcal{M}_X(c)},$$

with $$Y_k(q,y) := \frac{(q^{2r}y^r)^{r/24}}{\eta(q^{2r}y^r)^r} \left(\sum_{v \in \mathbb{Z}^{r-1} + \frac{k}{r} I} (q^{2r}y^r)^{v^tAv}\, y^{v^tAI} \right),$$

where $A = (a_{ij})$ is the $(r-1) \times (r-1)$-matrix with entries $a_{ij} = 1$ for $i \leq j$ and $a_{ij} = 0$ otherwise, and I is the column vector of length $r-1$ with all entries equal to one, η is the Dedekind eta function.

This coincides with Göttsche's conjecture on the generating series of topological χ_y-genera of the moduli spaces and the one by Göttsche and Kool for the virtual one.