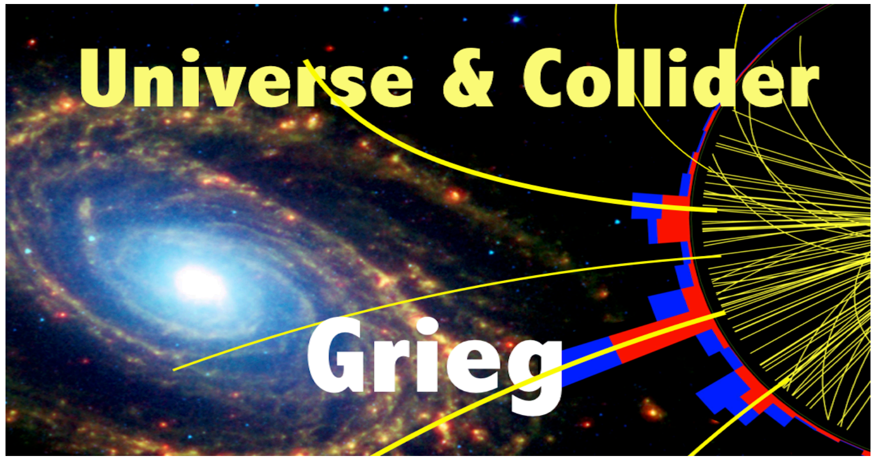
Quantum theory of dark matter scattering

Ayuki Kamada (University of Warsaw)



UNIVERSITY OF WARSAW

Based on
AK, Hee Jung Kim and Takumi Kuwahara, JHEP, 2020
AK, Takumi Kuwahara and Ami Patel, in preparation

Feb. 17, 2023 @ APEC seminar

Contents

Dark matter phenomenology

- long-range force
- Sommerfeld enhancement and self-scattering

Scattering state of quantum mechanics

- different limits of single state determine the above two
- tight correlation is expected and indeed found

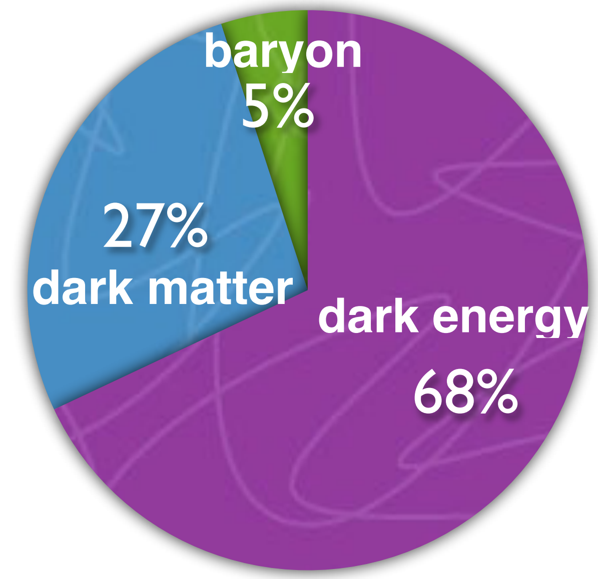
Formulation of the correlation

- Watson’s theorem and Omnès solution
- effective range theory around resonances
- Levinson’s theorem

Dark matter

Dark matter

- evident from cosmological observations
    - cosmic microwave background (CMB)…
- one of the biggest mysteries
    - astronomy, cosmology, particle physics…



cosmic energy budget

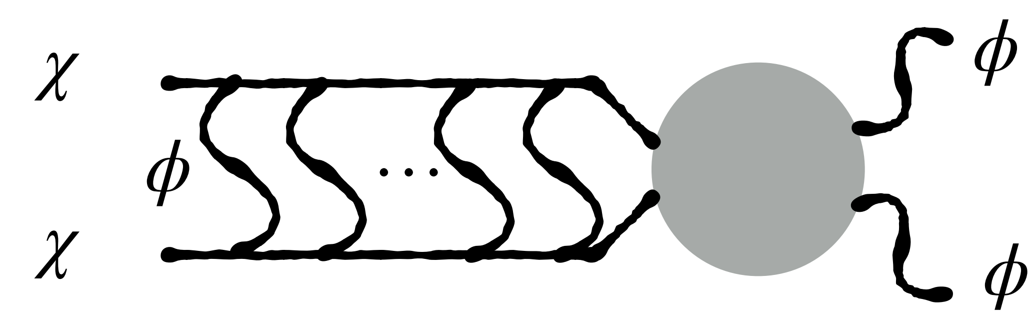
Long-range force

- mediator lighter than the dark matter
- electroweak-scale or lighter dark matter
    - new dark force (e.g., dark photon)
- TeV-scale dark matter (e.g., weak multiplet)
    - weak force

Sommerfeld enhancement

Distortion of wave function

- multiple exchanges of a mediator
- non-perturbative but described by the Schrödinger equation (later)



Enhanced annihilation

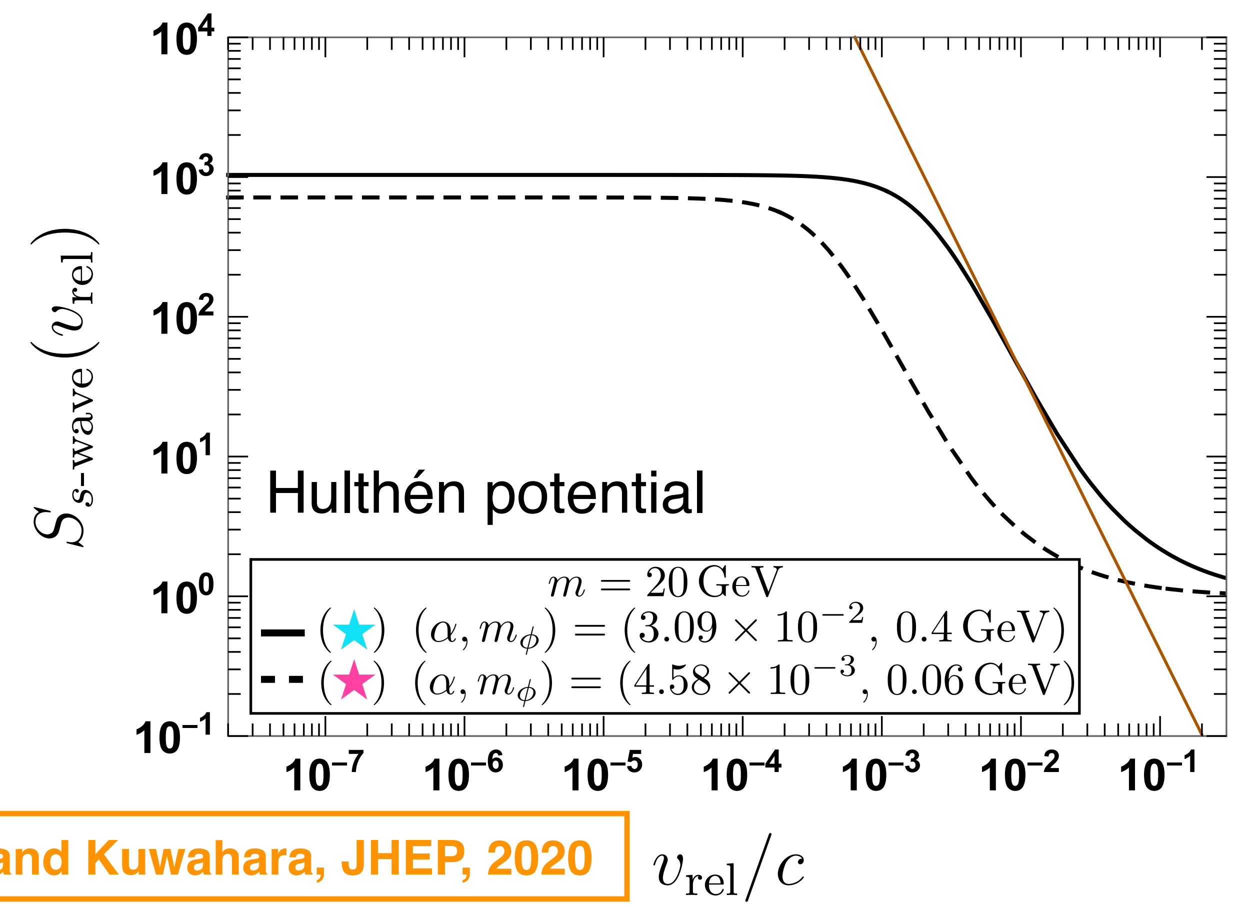
- annihilation cross section is enhanced at low velocity

$$(\sigma_{\text{ann}} v_{\text{rel}}) = S(\sigma_{\text{ann}}^{(0)} v_{\text{rel}})$$

- without potential

- Sommerfeld enhancement factor

- larger cross section in the late Universe than the thermal one



AK, Kim and Kuwahara, JHEP, 2020

Indirect detection

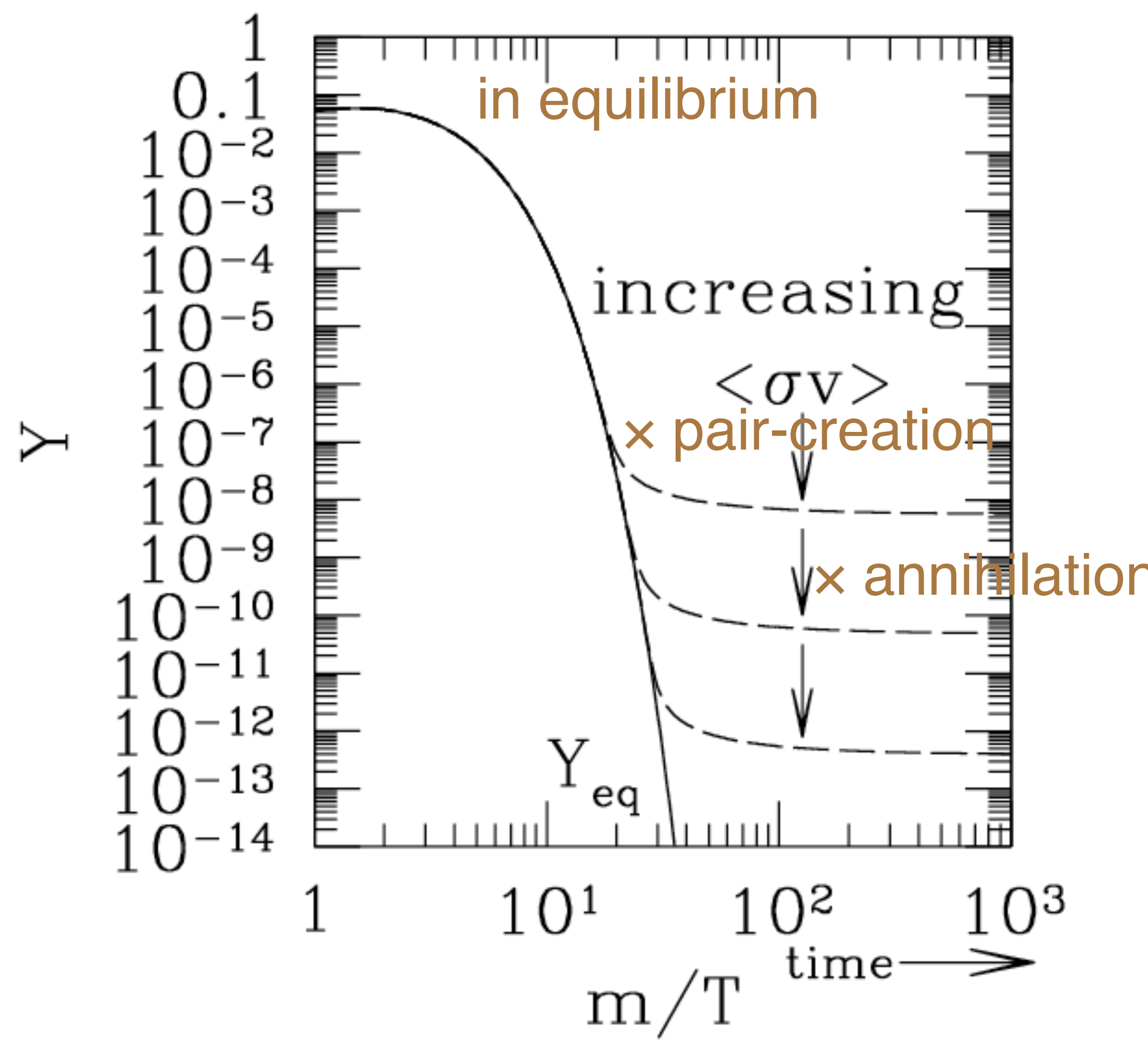
Canonical cross section

- thermal freeze-out (annihilation in the early Universe) $v_{\rm rel} \simeq 1/2$

$$\Omega h^2 = 0.1 \times \frac{3\times10^{-26}\,{\rm cm^3/s}}{\langle\sigma_{\rm ann} v\rangle}$$

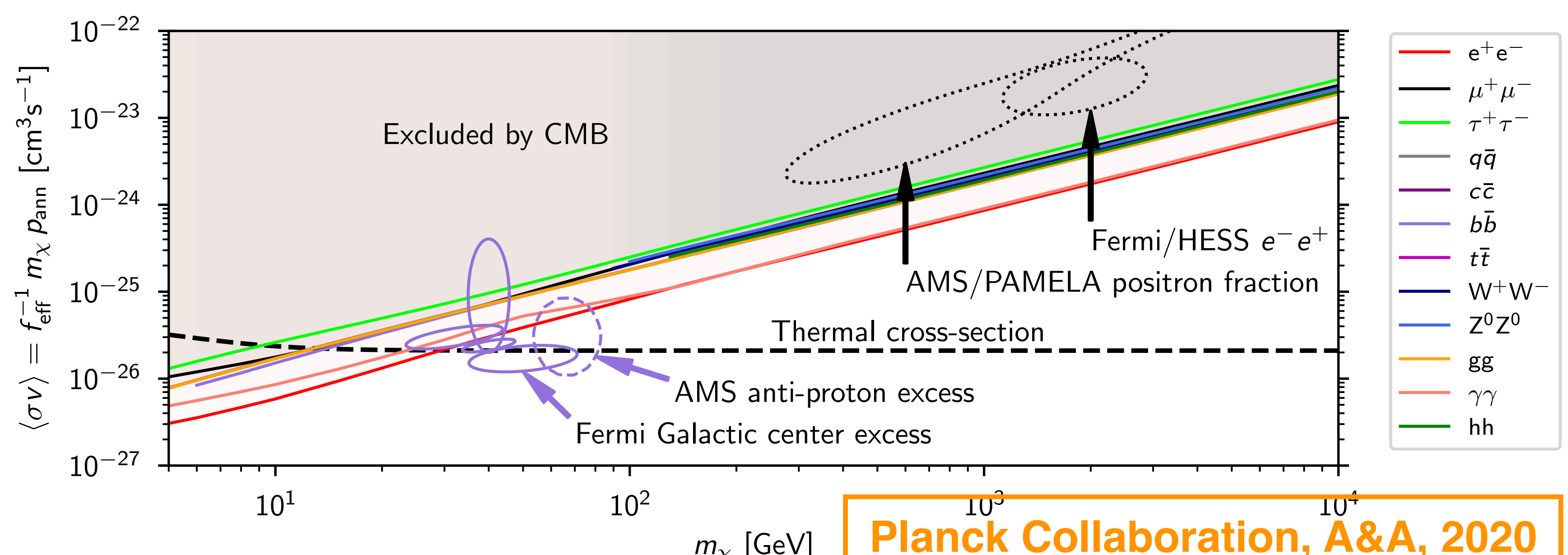
- requires a weak-scale annihilation cross section

$$\langle\sigma_{\rm ann} v\rangle \simeq 1\,{\rm pb}\times c$$



CMB constraints

- energy deposit around the last scattering



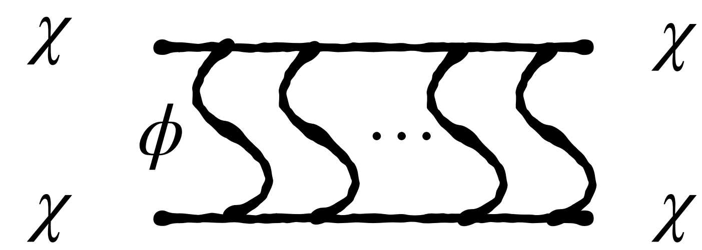
Planck Collaboration, A&A, 2020

Self-scattering

The same light mediator

- non-perturbative (multiple exchanges) when the distortion of wave function is significant

- again described by the Schrödinger equation (later)

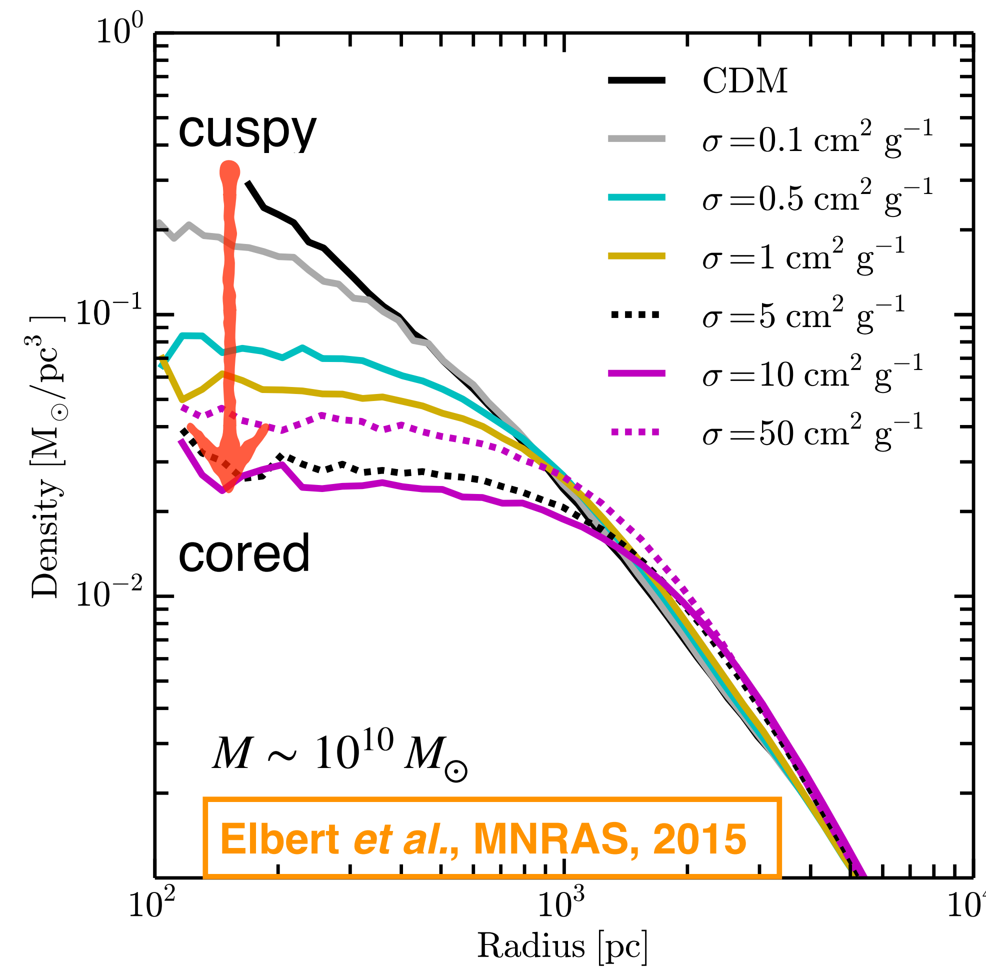


Self-interacting dark matter

- interactions among dark matter particles

$$\sigma/m \sim 1\,\mathrm{cm}^2/\mathrm{g} \sim 1\,\mathrm{barn/GeV}$$

- dark matter density profile inside a halo turns from cuspy to cored



Velocity dependence

Self-interacting dark matter

- cored profile "appear to" provide better fit to astronomical data

- "data" points from astrophysical observations of various size halos

- MW satellites

$M_{\rm infall} \sim 10^9\, M_\odot$

- dwarf spiral galaxies

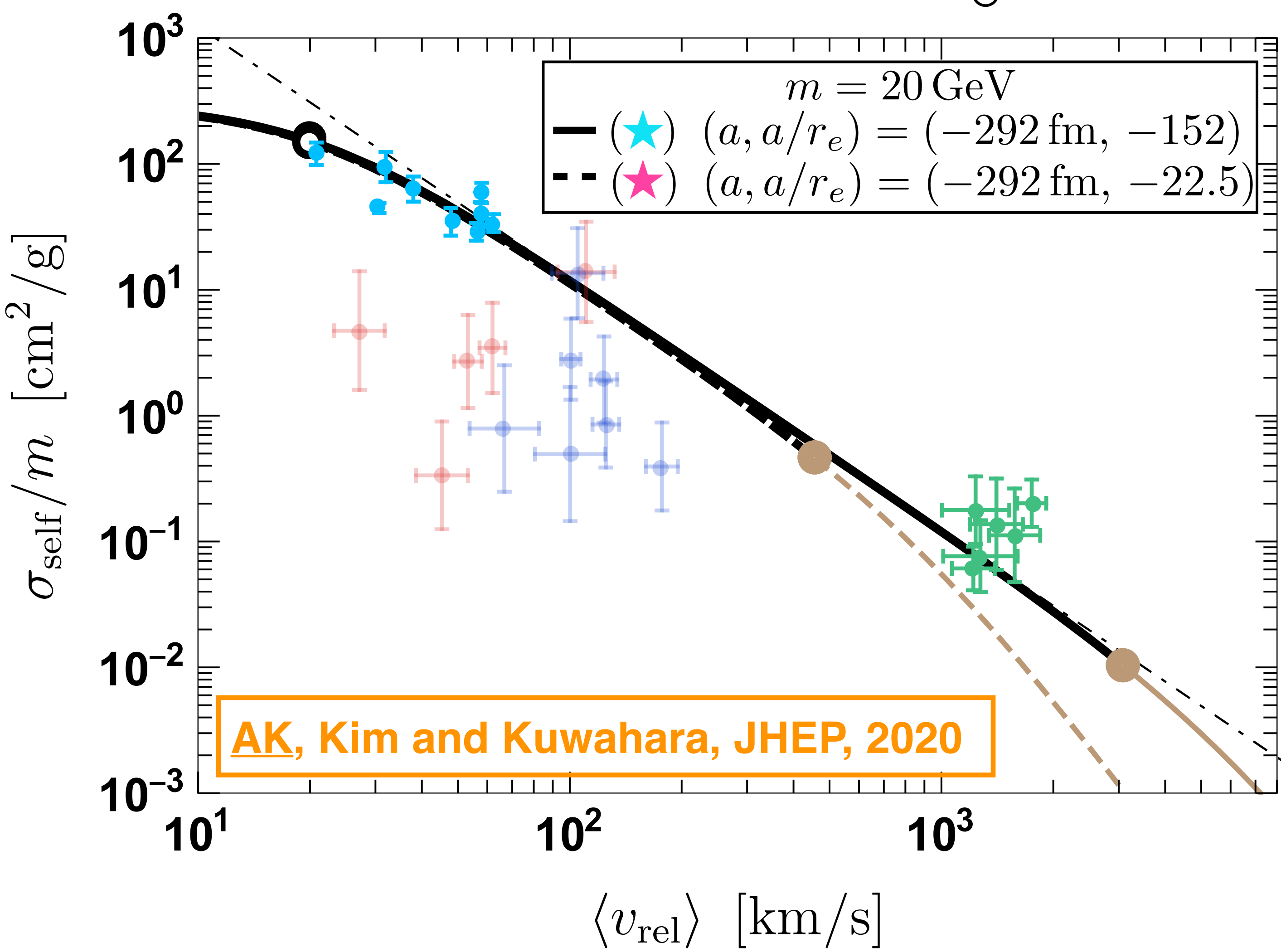
$M \sim 10^{11}\, M_\odot$

- galaxy clusters

$M \sim 10^{14}\, M_\odot$

Light mediator

- introduce a velocity dependence, which is compatible with "data"

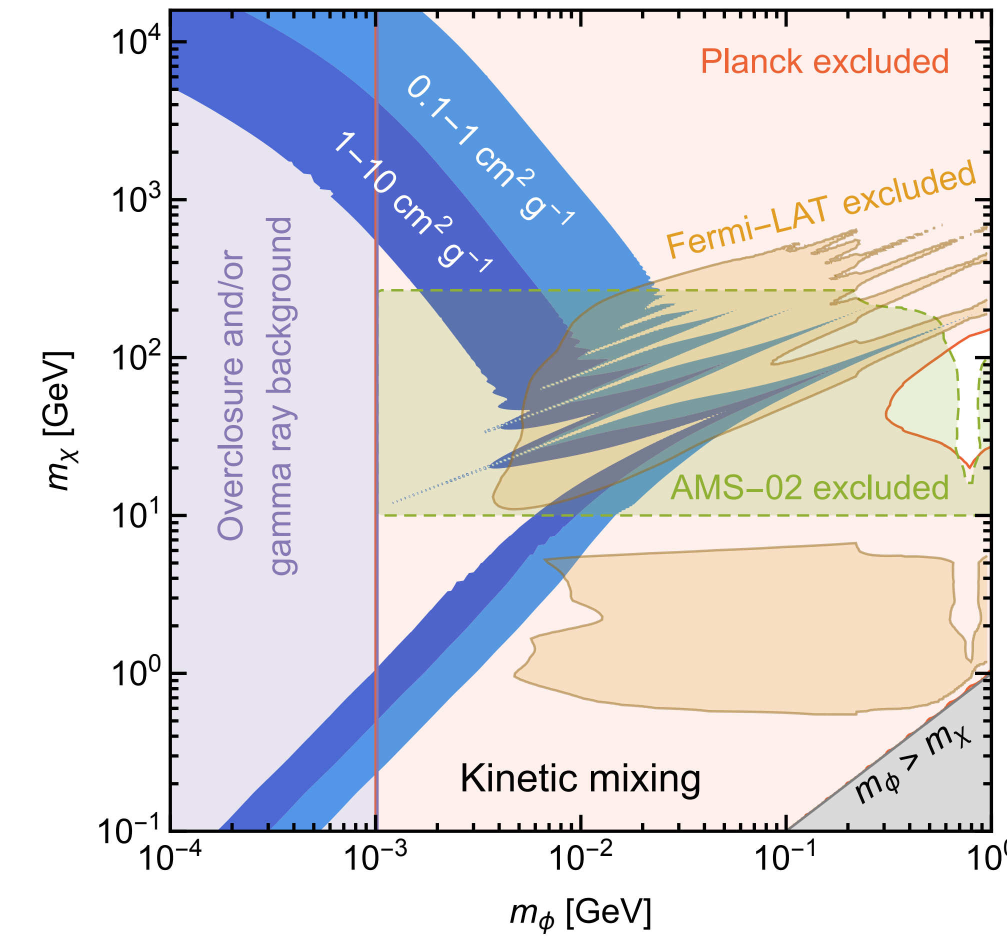


Correlation

Bringmann, Kahlhoefer, Schmidt-Hoberg and Walia, JHEP, 2020

Sommerfeld enhancement and self-scattering

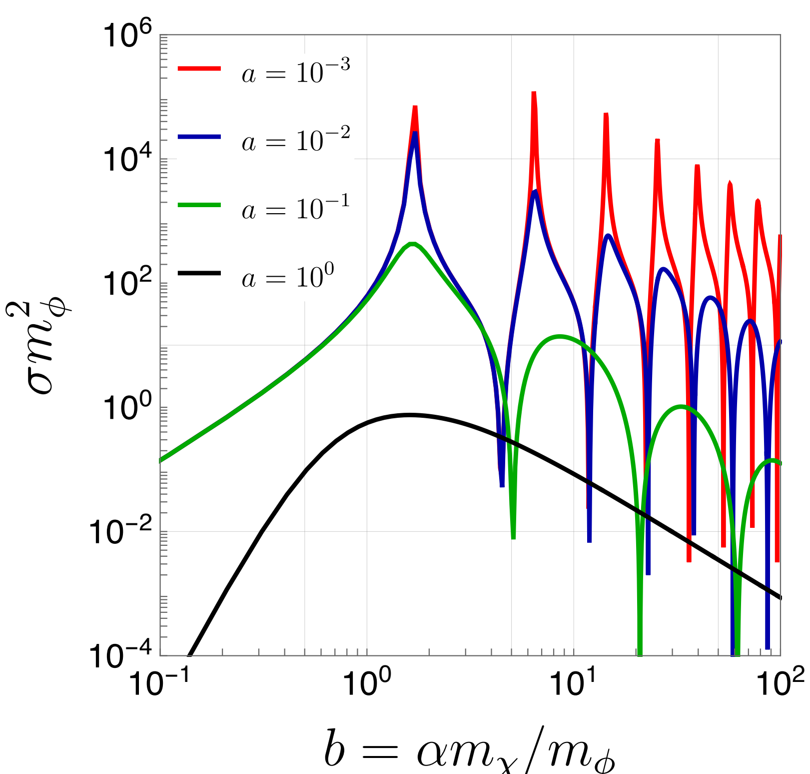
- some correlation is known
  - main obstacle in SIDM model building
  - resonant enhancement occurs at the same parameter point

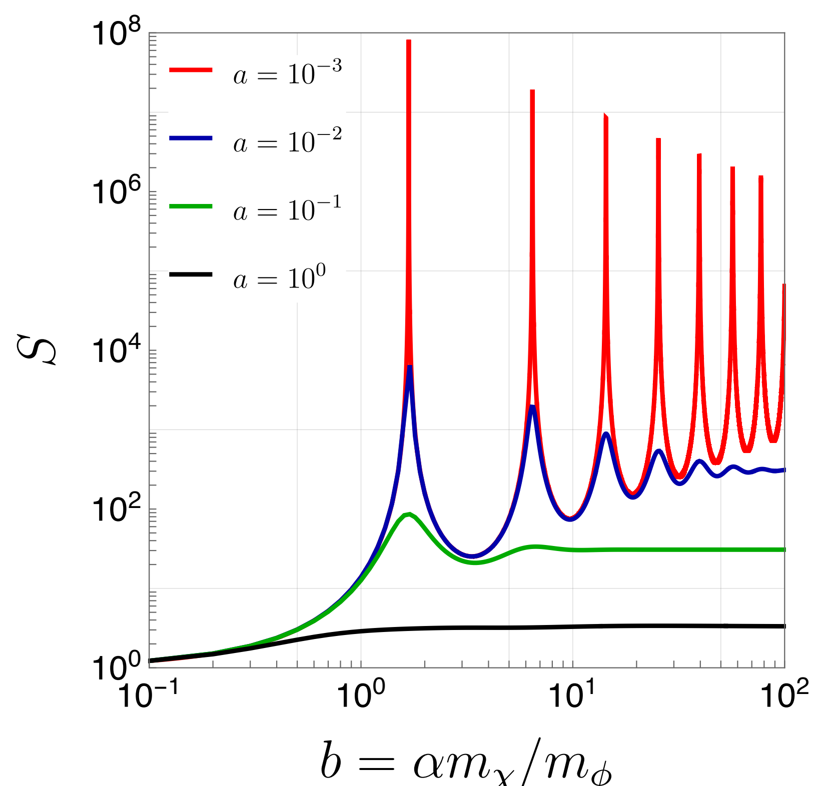


- dark photon

$$a = \frac{v_{\rm rel}}{2\alpha_\chi}$$

$$b = \frac{\alpha_\chi m_\chi}{m_\phi}$$





- Yukawa potential

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Scattering in quantum mechanics

Schrödinger equation

Weinberg, "Lectures on Quantum Mechanics"

$$\left[-\frac{1}{2\mu}\nabla^2 + V(r)\right]\psi_k(\vec{x}) = E\psi_k(\vec{x}) \qquad E = \frac{k^2}{2\mu} \qquad k = \mu v_{\rm rel}$$

- potential from long-range force

- reduced mass ($\mu = m/2$ for identical particle)

- scattering state (energy-eigenstate of Schrödinger equation)

$$\psi_k(\vec{x}) \to e^{ikz} + f(k,\theta)\frac{e^{ikr}}{r} \qquad r \to \infty$$

- (in-coming) plane wave
- scattering amplitude
- out-going spherical wave

Partial-wave decomposition

$$\psi_k(\vec{x}) = \sum_{\ell=0}^{\infty} R_{k,\ell}(r)P_\ell(\cos\theta)$$

- radial Schrödinger equation

$$\left[\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} + k^2 - \frac{\ell(\ell+1)}{r^2} - 2\mu V(r)\right]R_{k,\ell}(r) = 0$$

Sommerfeld enhancement and self-scattering

Scattering phase

- radial wave function at infinity

$$R_{k,\ell}(r) \to \frac{\sin(kr - \frac{1}{2}\ell\pi + \delta_\ell)}{r} \qquad r \to \infty$$

$$f(k,\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell(k) P_\ell(\cos\theta) \qquad f_\ell(k) = \frac{e^{2i\delta_\ell} - 1}{2ik}$$

$$\sigma = \sum_{\ell=0}^{\infty} \sigma_\ell \qquad \sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\sin^2\delta_\ell(k)$$

- diagonalized S-matrix $S_\ell = e^{2i\delta_\ell}$

Sommerfeld enhancement

Iengo, JHEP, 2009

Cassel, J.Phys.G, 2010

- radial wave function around the origin

- annihilation through the contact interaction (delta function potential)

$$S_{k,\ell} = \left| \frac{R_{k,\ell}(r)}{R^{(0)}_{k,\ell}(r)} \right|^2 \qquad r \to 0$$

- without potential

Correlation

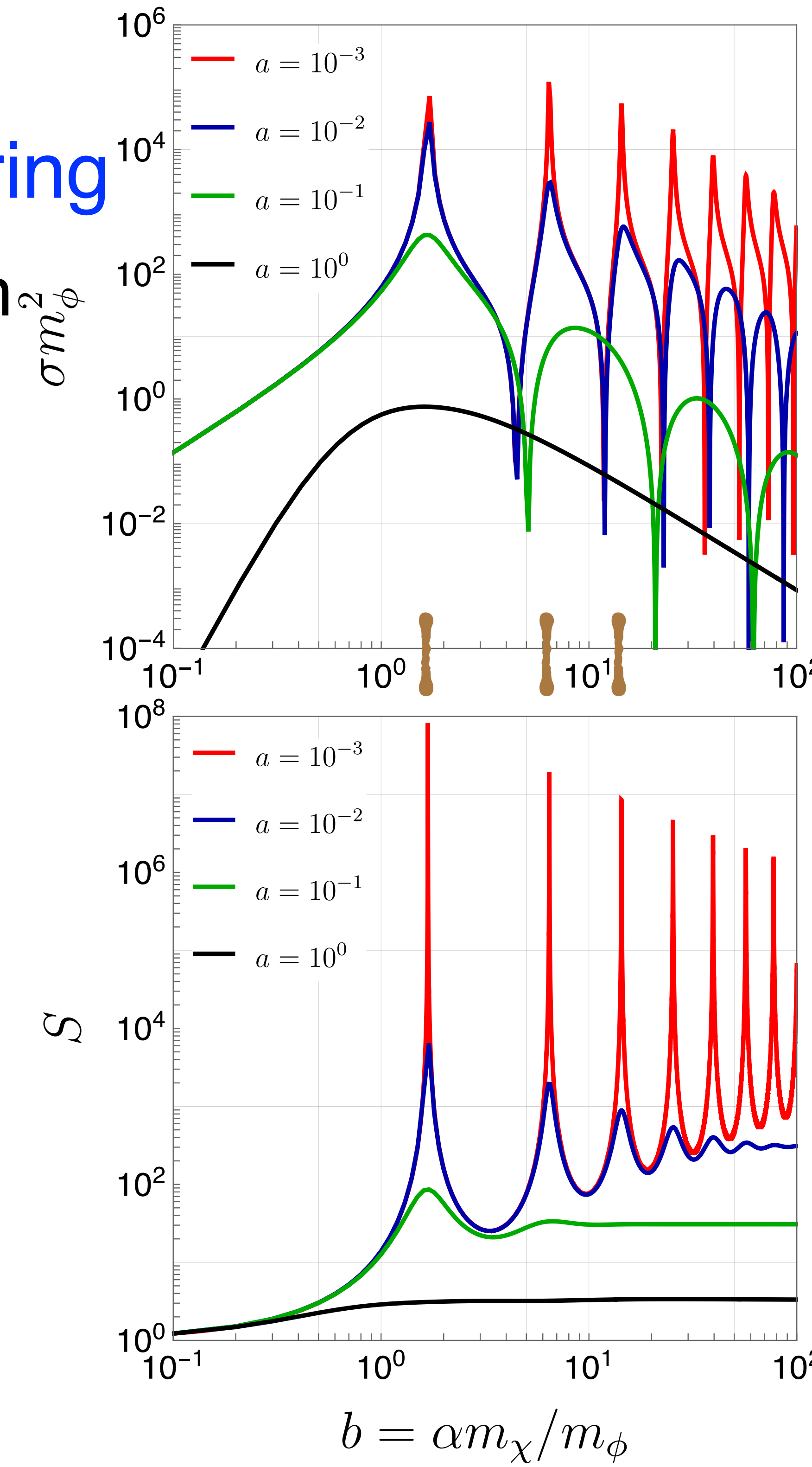
Sommerfeld enhancement and self-scattering

- determined by a single radial wave function

- not surprising that we see a correlation

    - resonances for the same parameter

- **still we want to formulate the direct relation**



Remarks

- hereafter ignore a contact interaction in the Schrödinger equation

    - not a problem unless the wave function is localized around the origin

    - at resonances (later) and quite small velocities, we need to take it into account; otherwise Unitarity is violated **Blum, Sato and Slatyer JHEP, 2016**

Effective range theory

Analyticity of scattering amplitude

Chu, Garcia-Cely and Murayama, JCAP, 2020

$$f_\ell = \frac{1}{k\cot\delta_\ell - ik}$$

- effective range theory

$$k \to 0 \quad k^{2\ell+1}\cot\delta_\ell \to -\frac{1}{a_\ell^{2\ell+1}} + \frac{1}{2r_{e,\ell}^{2\ell-1}}k^2$$

- scattering length
- effective range

- $f_\ell \propto k^\ell k^\ell$ to make $f(\vec{k})$ an analytic function around $k=0$
  - initial $\quad \ell = 1 \quad k\cos\theta = k_z$
  - final
- higher partial-wave is suppressed at low energy

Effective range theory

Yukawa potential

AK, Kuwahara and Patel, in preparation

- s-wave

$$k \cot \delta_0 \to -\frac{1}{a_0} + \frac{r_{e,0}}{2} k^2$$

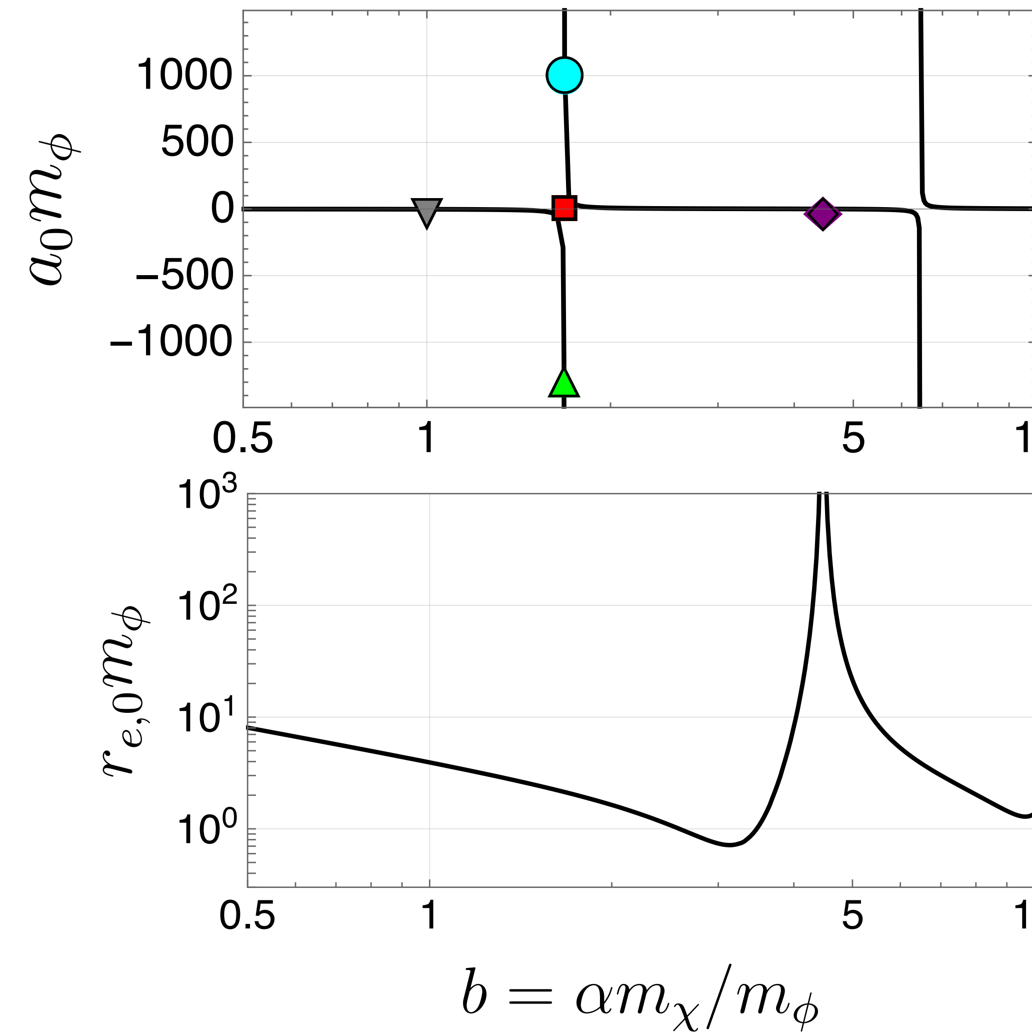
- on resonance $a_0 \to \infty$

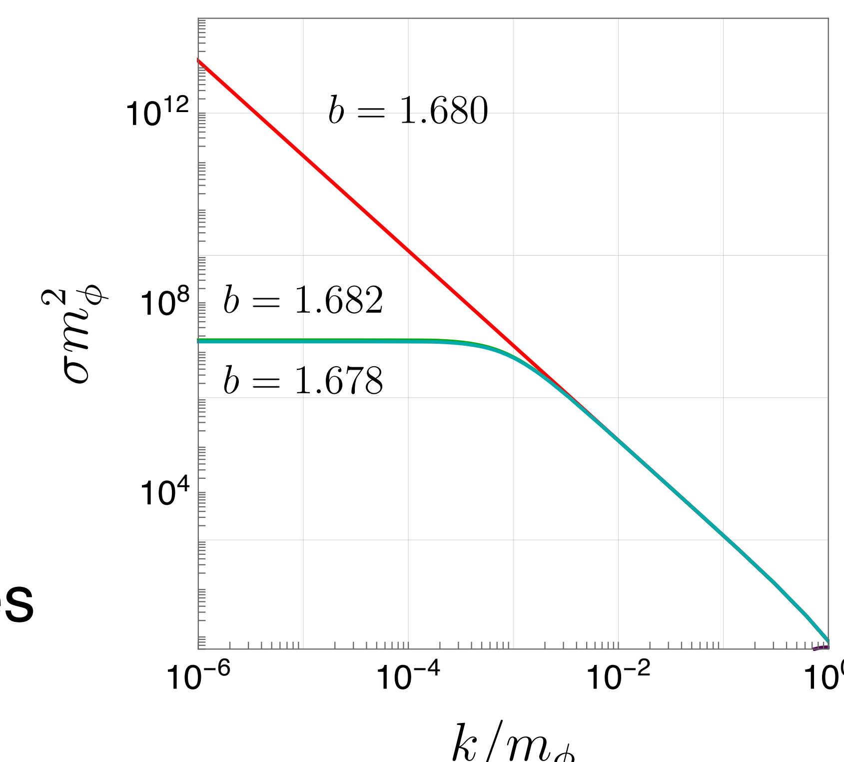
- around resonances

- shallow virtual state
  - non-normalizable $\kappa_b < 0$
- shallow bound state
  - pole of scattering amplitude $k = i\kappa_b$

$$\kappa_{b0} \approx \frac{1}{a_0}$$ - s-wave $\kappa_b > 0$

$$\kappa_{b\ell}^2 \approx -\frac{2 r_{e,\ell}^{2\ell-1}}{a_\ell^{2\ell+1}}$$ - higher-partial waves





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Watson theorem

Weinberg, "Lectures on Quantum Mechanics"

"In" and "out" states

- Lippmann-Schwinger equation $|\psi^{\pm}> = |\phi> + \dfrac{1}{E - H_0 \pm i\epsilon} V |\psi^{\pm}>$

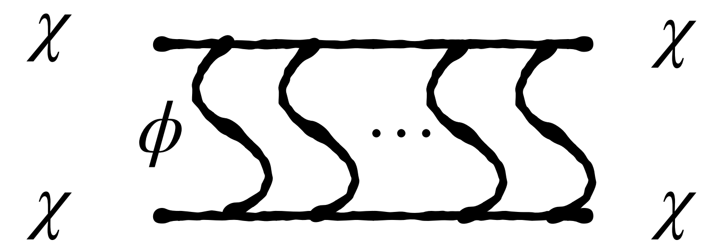
  - free $\quad \epsilon > 0$

  - "+": in (far future)
    "-": out (far past)

- S-matrix relates these two $|\psi^{+}_{\alpha}> = \int d\beta S_{\beta\alpha} |\psi^{-}_{\beta}>$

$$S_{\beta\alpha} = \langle \psi^{-}_{\beta} | \psi^{+}_{\alpha} \rangle = \delta(\beta - \alpha) - 2i\pi\delta(E_\alpha - E_\beta) T_{\beta\alpha}$$

$$V|\psi^{+}_{\alpha}> = \int d\beta T_{\beta\alpha} |\phi_\beta>$$



- in-state as a whole

- for partial waves $\quad S_{\ell,\beta\alpha} = I_{\beta\alpha} + 2i\sqrt{\sigma_\alpha \sigma_\beta} T_{\ell,\beta\alpha} \qquad \sigma_\alpha = \beta_\alpha \theta(k^2 - k_\alpha^2)$

  - velocity

- $\psi^{+}_{k}(\vec{x}) \to e^{ikz} + f(k,\theta)\dfrac{e^{ikr}}{r}$ is actually in-state

  $r \to \infty$

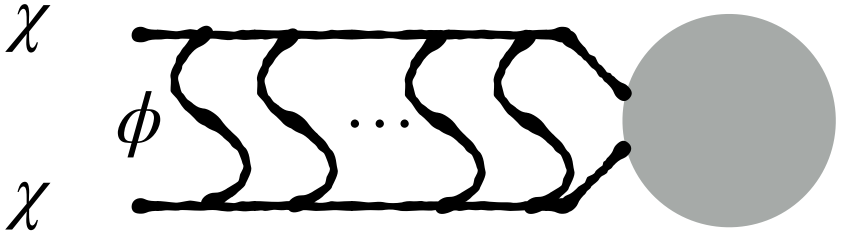
- out-state $\quad \psi^{-}_{k}(\vec{x}) \to e^{ikz} + f(k,\theta)^{*}\dfrac{e^{-ikr}}{r} \qquad r \to \infty$

Watson theorem

Oller, “A Brief Introduction to Dispersion Relations”

Annihilation matrix element

$$\Gamma_\alpha(k^2+i\epsilon) = \langle 0 | \Theta_\chi | \psi^+_{\alpha,k} \rangle$$



- in-state as a whole

- inserting out states

$$\Gamma_\alpha(k^2+i\epsilon) = \sum_\beta S_{\beta\alpha}(k) \langle 0 | \Theta_\chi | \psi^-_{\beta,k} \rangle = \sum_\beta S_{\beta\alpha}(k) \Gamma_\beta(k^2 - i\epsilon)$$

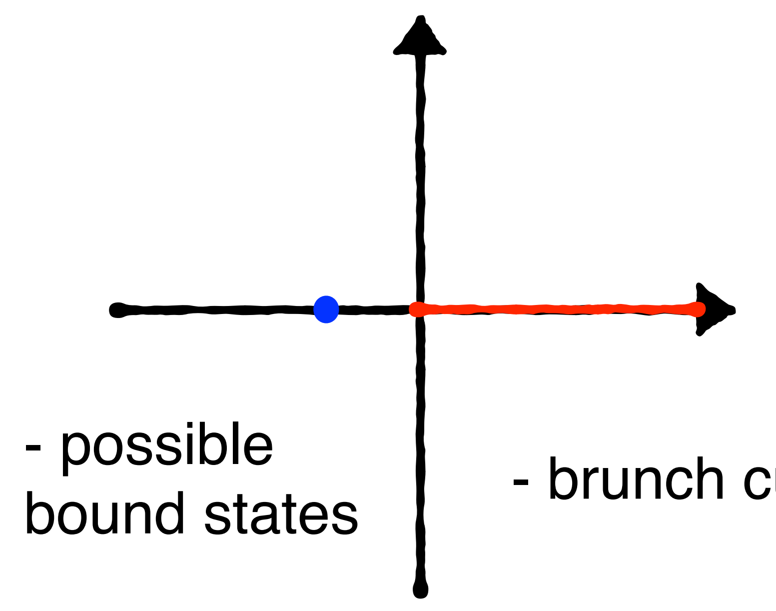
- assuming the real matrix element (T-invariance)

$$\Gamma_\alpha(k^2+i\epsilon) = \sum_\beta S_{\beta\alpha}(k) \Gamma_\beta(k^2+i\epsilon)^*$$

- for a partial wave

$$\Gamma_\ell(k^2+i\epsilon) = e^{2i\delta_\ell} \Gamma_\ell(k^2+i\epsilon)^*$$

- complex k^2 plane



Omnès solution

Omnès function

$$\Omega_\ell(k^2) = \exp[\omega_\ell(k^2)] \qquad \omega(k^2) = \frac{1}{\pi}\int_0^\infty dq^2 \frac{\delta_\ell(q)}{q^2 - k^2}$$

- principal value

- computed by phase shift and reproduce the brunch cut

$$F_\ell(k^2) = \prod_{b_\ell} \frac{k^2}{k^2 + \kappa_{b,\ell}^2}$$

- rational function reproducing bound-state poles (need to know somehow)

$$\Gamma_\ell(k^2) = \Omega_\ell(k^2) F_\ell(k^2)$$

- from Liouville theorem

- we normalize $\delta_\ell(k) \to 0 \quad \Gamma_\ell(k^2) \to 1 \; k^2 \to \infty$

- scattering phase and Sommerfeld enhancement are negligible at high velocity

Sommerfeld enhancement

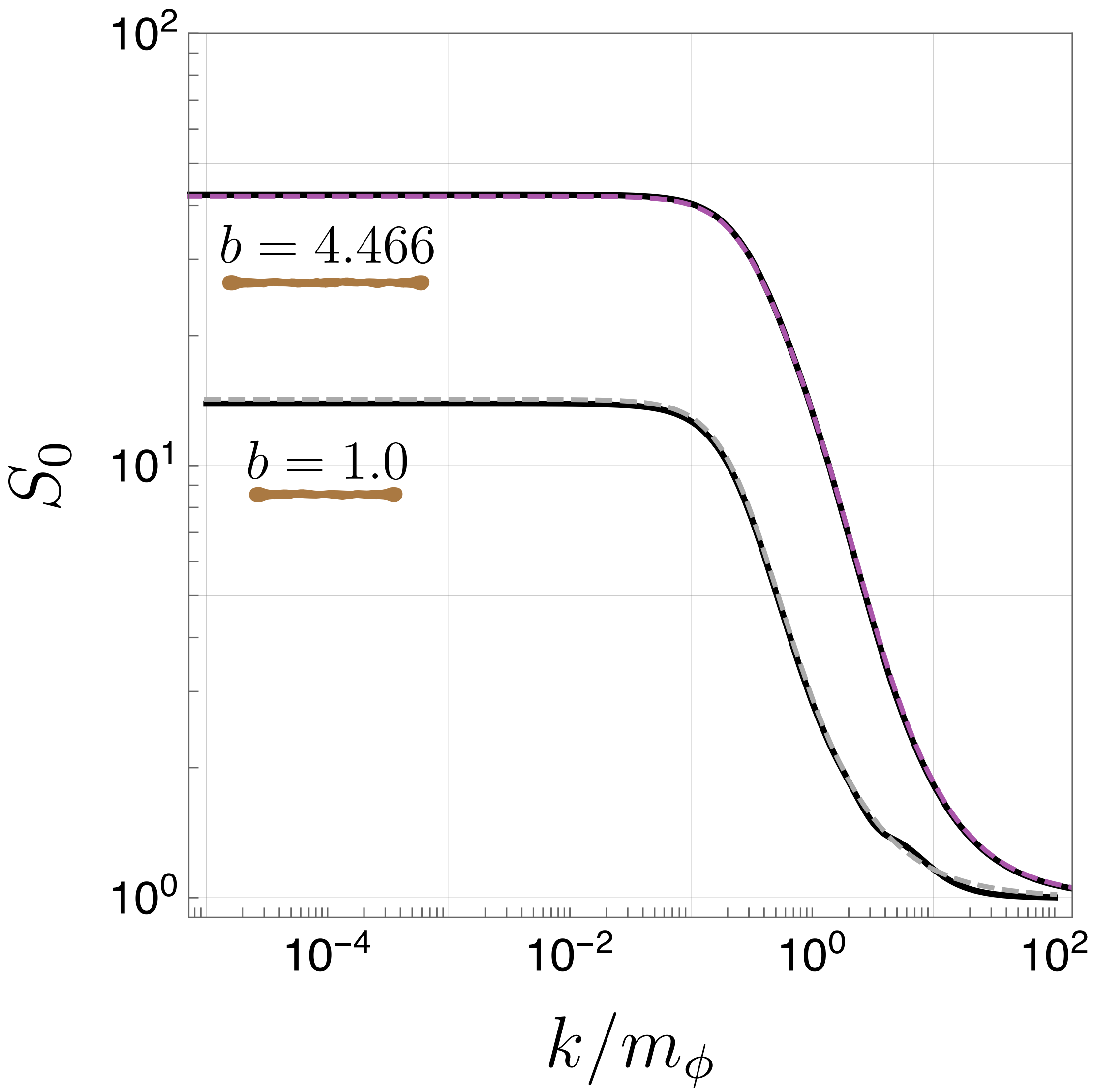
$$S_\ell = |\Gamma_\ell(k^2)|^2$$

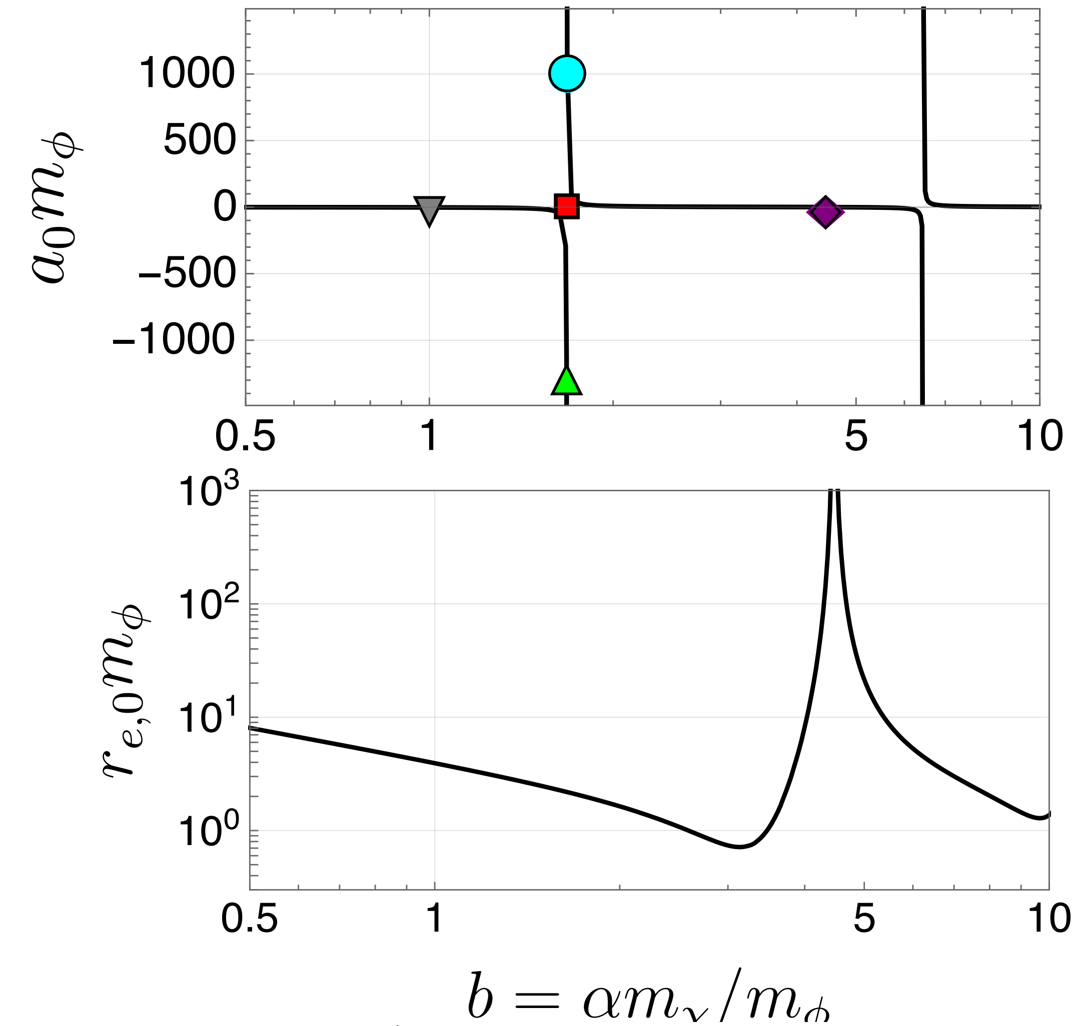
Omnès solution

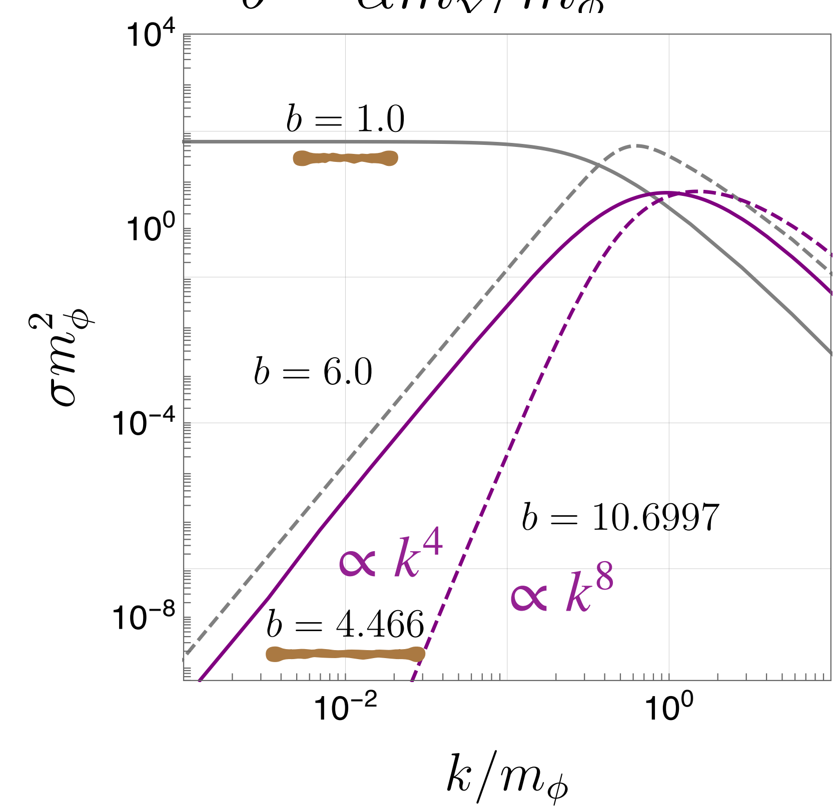
Yukawa potential

AK, Kuwahara and Patel, in preparation

- s-wave







- Omnès solution agrees with direct computation from scattering state

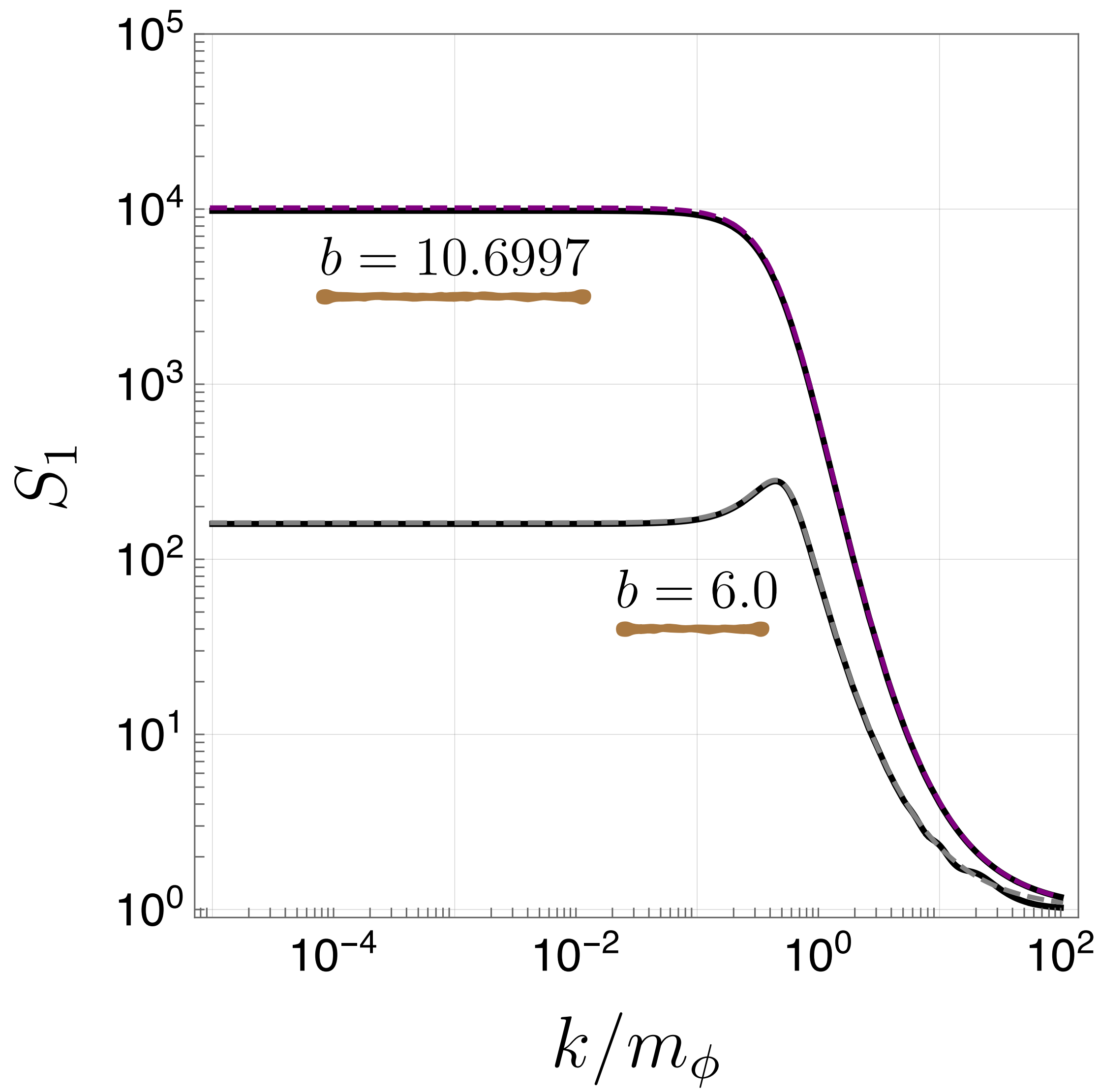
- with proper $F_0(k^2)$ (later)

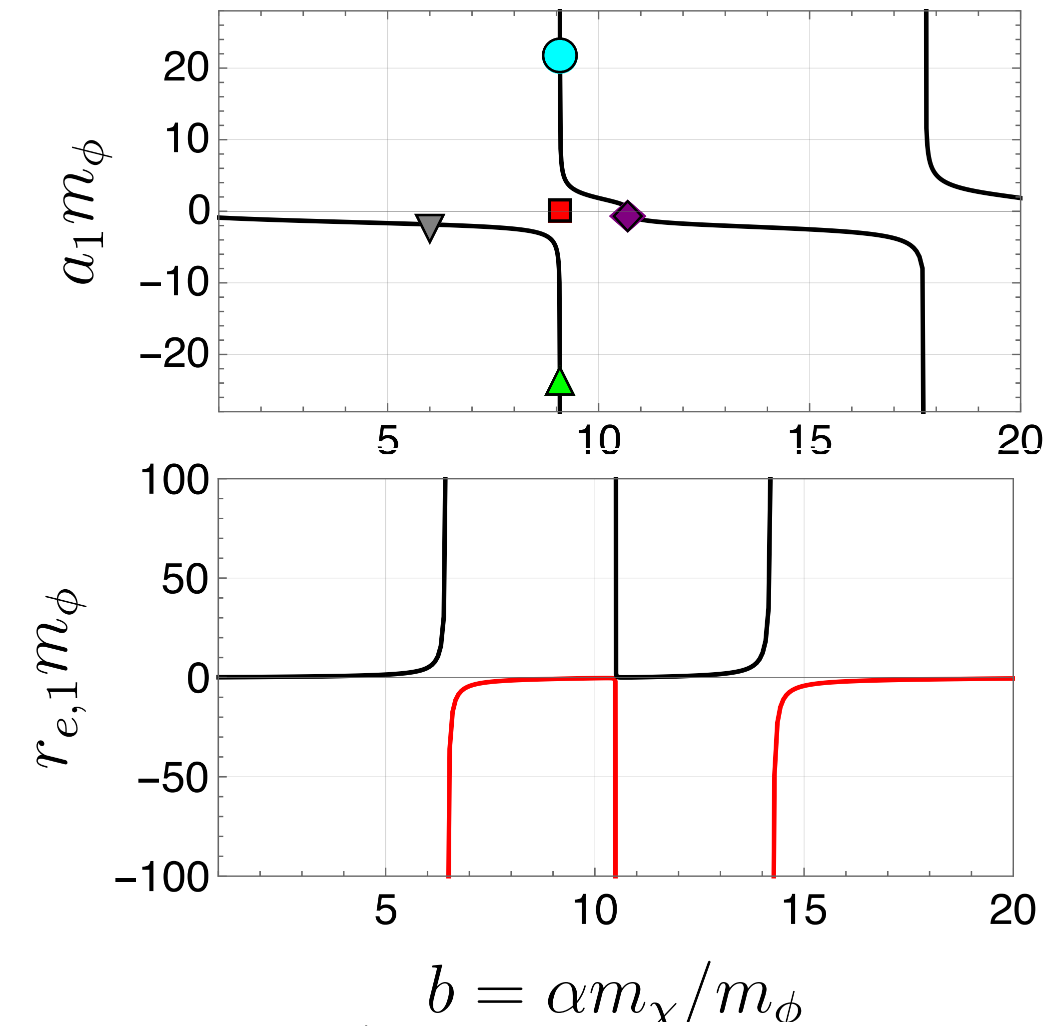
Omnès solution

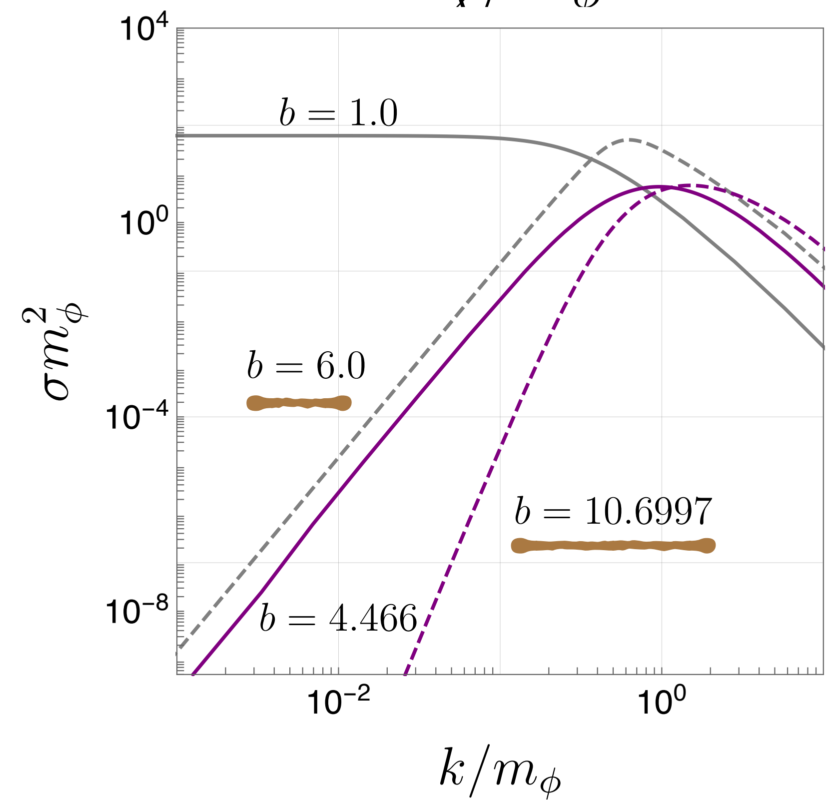
Yukawa potential

AK, Kuwahara and Patel, in preparation

- p-wave







Around resonances

AK, Kuwahara and Patel, in preparation

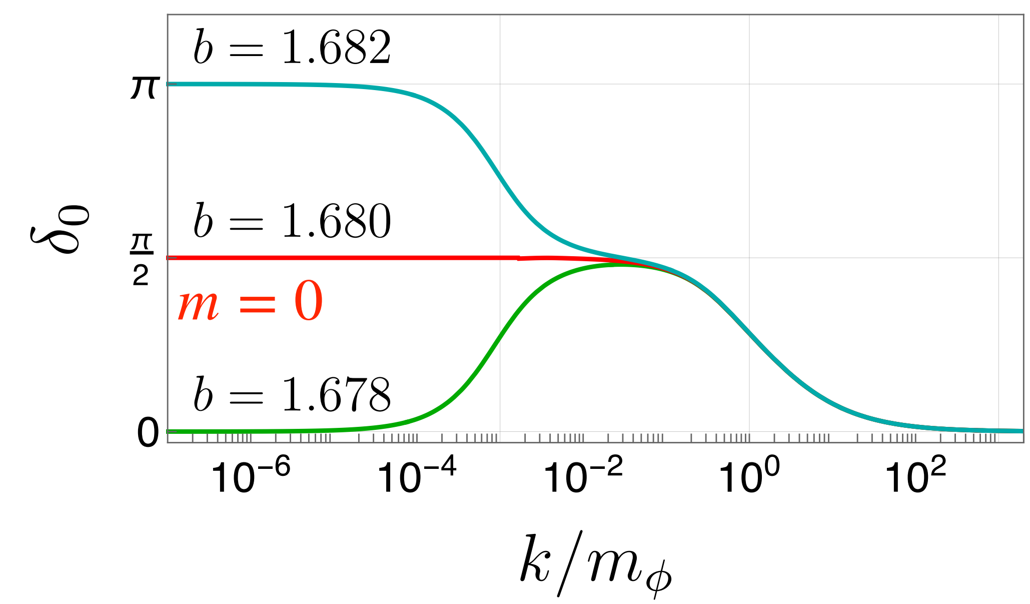
Effective range theory

- s-wave resonances

$$k \to 0 \quad k\cot\delta_0 \to -\frac{1}{a_0} + \frac{r_{e0}}{2}k^2$$

$$a_0 \to \infty$$

$$k \to 0 \quad \delta_0 \to \left(\frac{1}{2}+m\right)\pi \quad m = 0, 1, 2\ldots$$



- Omnès function

for m=0 (later)

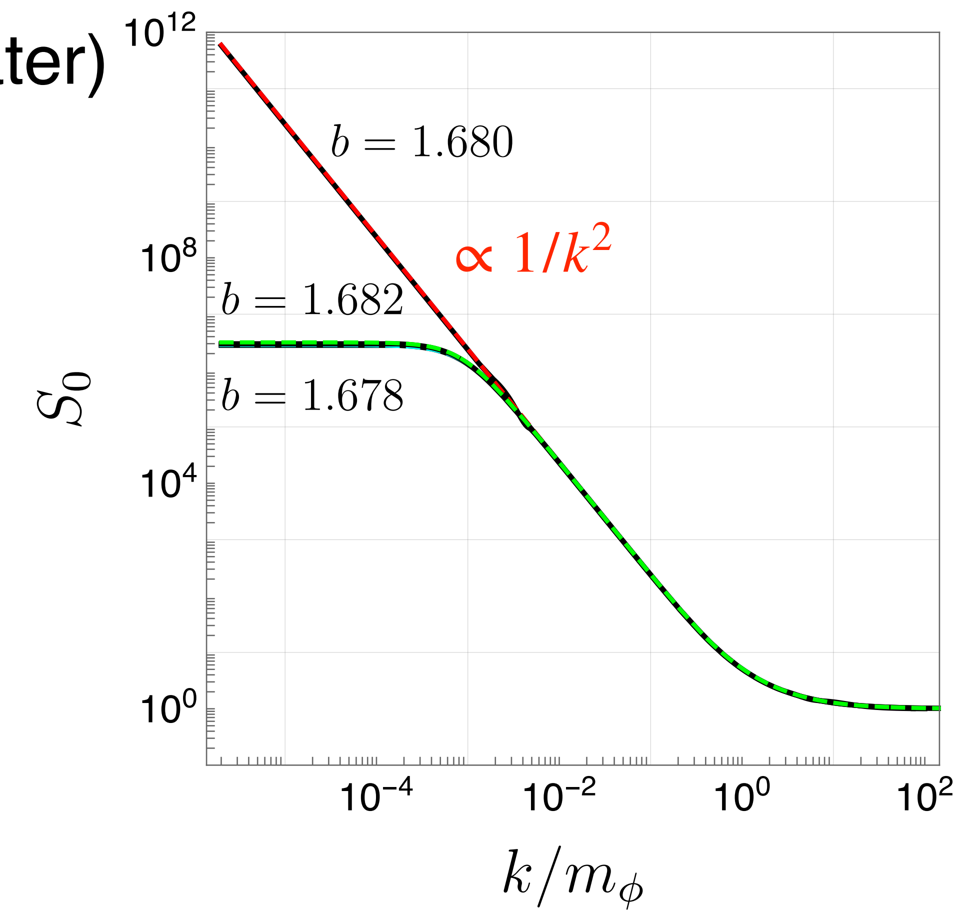
$$\omega_0(k^2) = \frac{1}{\pi}\int_0^\infty dq^2 \frac{\delta_0(q)}{q^2-k^2}$$

$$F_0(k^2) = 1$$

$$k \to 0 \quad \to -\left(\frac{1}{2}+m\right)\ln(r_{e,0}^2 k^2)$$

$$\Gamma_0(k^2) = \exp[\omega_\ell(k^2)]F_0(k^2) \qquad S_0 = |\Gamma_0(k^2)|^2$$

$$k \to 0 \quad \to \frac{F_0(k^2)}{k^{1+2m}}$$



Around resonances

Levison theorem

Weinberg, “Lectures on Quantum Mechanics”

- # of bound states is given by phase shift

$$\delta_\ell(k \to 0) - \delta_\ell(k \to \infty) = \left[\#b_\ell \left(+\frac{1}{2}\right)\right] \pi$$

- excluding virtual states

- zero in our normalization

- only for s-wave resonances

- underlying idea

  - consider the system confined in a large sphere

$$R_{k\ell}(r) \to \frac{\sin(kr - \frac{1}{2}\ell\pi + \delta_\ell)}{r}$$

$$r \to \infty$$

$$k > 0$$

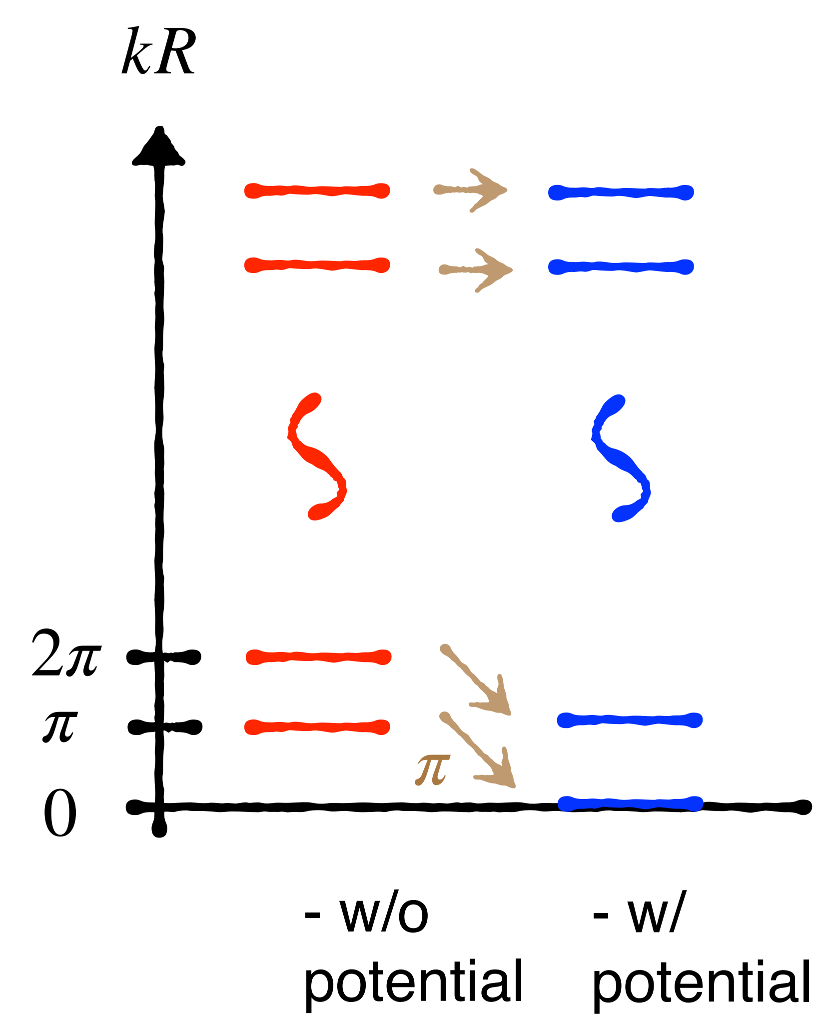
$$kR - \frac{1}{2}\ell\pi + \delta_\ell = n\pi$$

$$n = 0, \pm 1, \pm 2 \ldots$$

  - scattering states are discretized (countable infinity)

  - decrease in # of scattering states = # of bound states

  - total number does not change



Around resonances

AK, Kuwahara and Patel, in preparation

Bound states

- s-wave

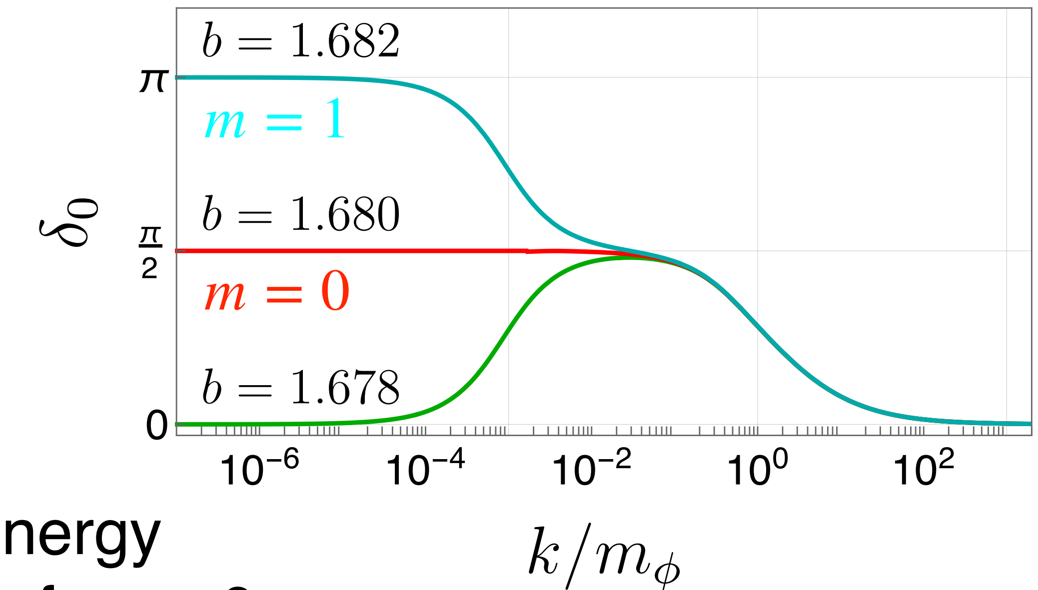
$$\delta_0(k \to 0) = \left[\#b_0 \left(+\frac{1}{2}\right)\right]\pi$$

- resonances

$$k \to 0 \quad \delta_0 \to \left(\frac{1}{2} + m\right)\pi \quad \#b_0 = m$$

$$F_0(k^2) = \prod_{b_0=1}^{m} \frac{k^2}{k^2 + \kappa_{b,0}^2}$$

- only zero energy "virtual" state for m=0



- slightly below the 1st resonance

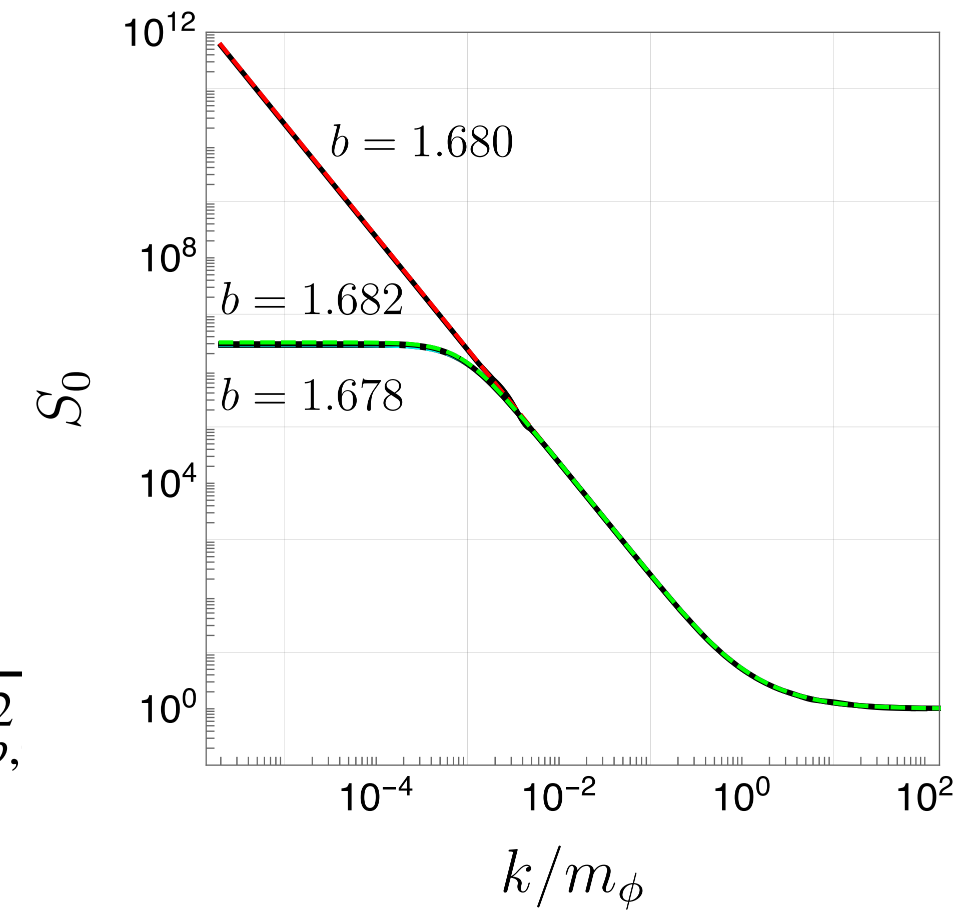
$$k \to 0 \quad \delta_0 \to 0$$ - no bound state $F_0(k^2) = 1$

- slightly above the 1st resonance

$$k \to 0 \quad \delta_0 \to \pi \quad \omega_0(k^2) \to -\ln(r_{e,0}^2 k^2)$$

- single bound state $F_0(k^2) = \frac{k^2}{k^2 + \kappa_{b,}^2}$

$$\Gamma_0(k^2) \to \frac{1}{k^2 + \kappa_{b,0}^2}$$ - saturates at low k



Around resonances

AK, Kuwahara and Patel, in preparation

Bound states

- p-wave

$$\delta_1(k \to 0) = \# b_1 \pi$$

- resonances

$$k \to 0 \quad k^3 \cot \delta_1 \to -\frac{1}{a_1^3} + \frac{1}{2r_{e,1}} k^2$$

$$a_1 \to \infty$$

$$k \to 0 \quad \delta_1 \to m\pi \qquad \# b = m$$

- including zero energy bound state

$$\omega_1(k^2) \to -m \ln(r_{e,1}^2 k^2)$$

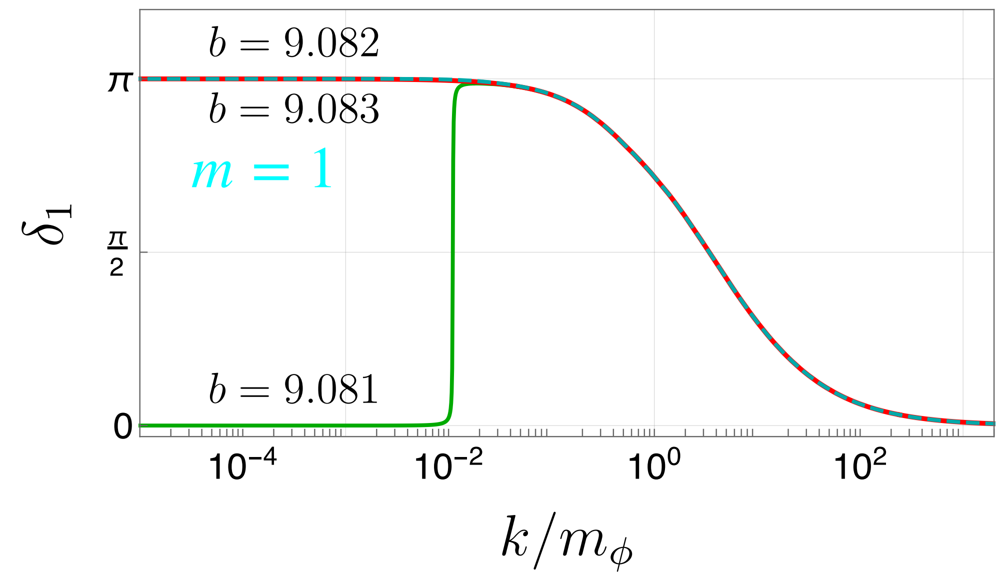
for m=1

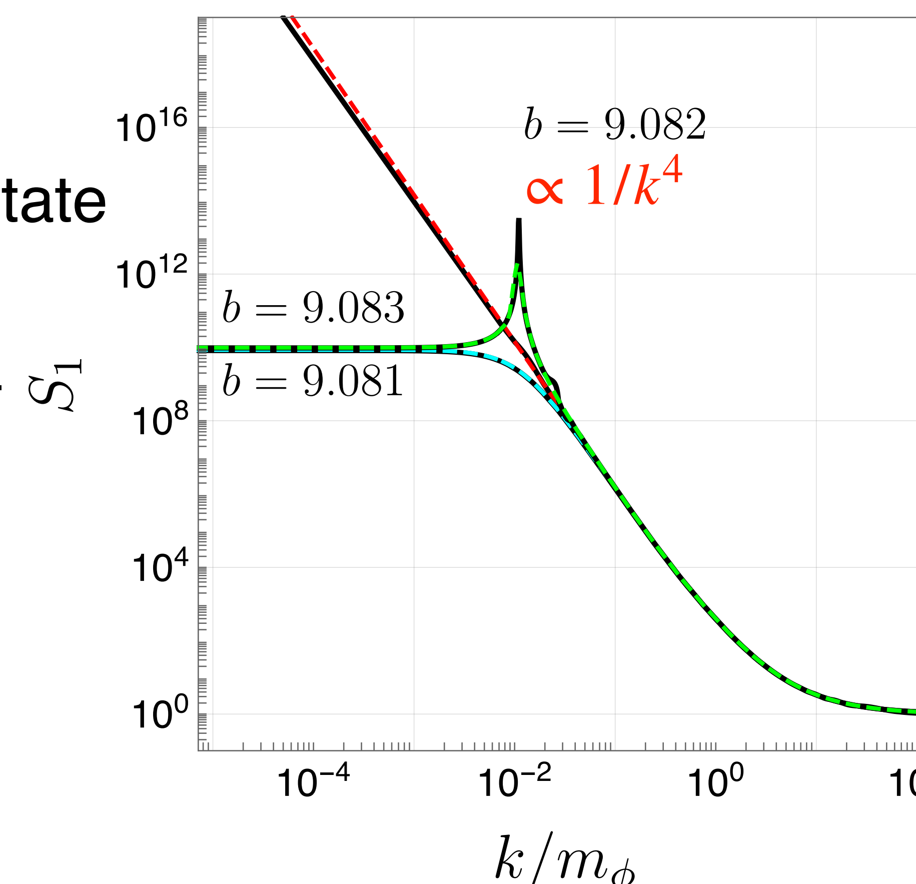
$$F_1(k^2) = \prod_{b_1=1}^{\cancel{m-1}} \frac{k^2}{k^2 + \kappa_{b,1}^2}$$

$$\Gamma_1(k^2) \to \frac{1}{k^2}$$

- slightly below/above the 1st resonance

  - similar to s-wave





Summary

Long-range force of dark matter

- Sommerfeld enhancement and self-scattering cross section
  - indirect detection and structure formation
- the two are known to be correlated
  - they are determined by a single wave function

This talk

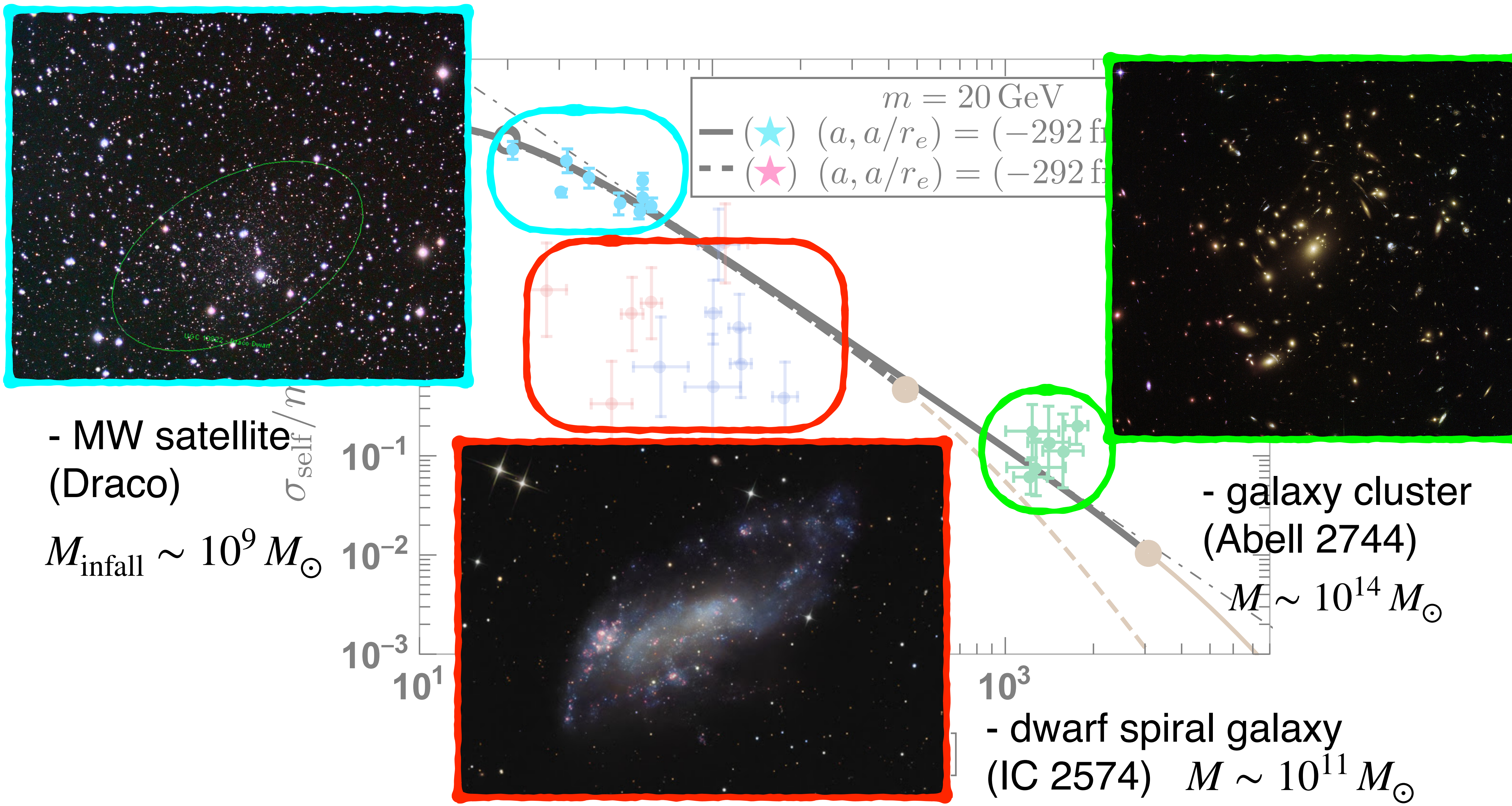
- we formulate the direct relation between the two
  - Watson theorem and Omnès solution
- we discuss how we can understand the velocity dependence around the resonances by using our formulation
  - effective range theory and Levinson theorem

Thank you

Data points

Overview

- cores in various-size halos



- MW satellite (Draco)

$M_{\rm infall} \sim 10^9\, M_\odot$

- galaxy cluster (Abell 2744)

$M \sim 10^{14}\, M_\odot$

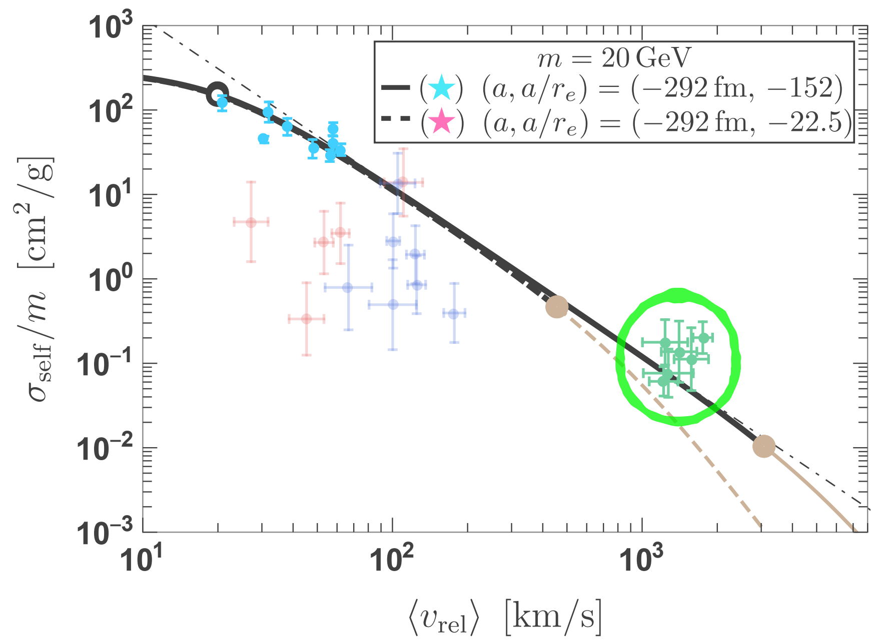
- dwarf spiral galaxy (IC 2574) $M \sim 10^{11}\, M_\odot$

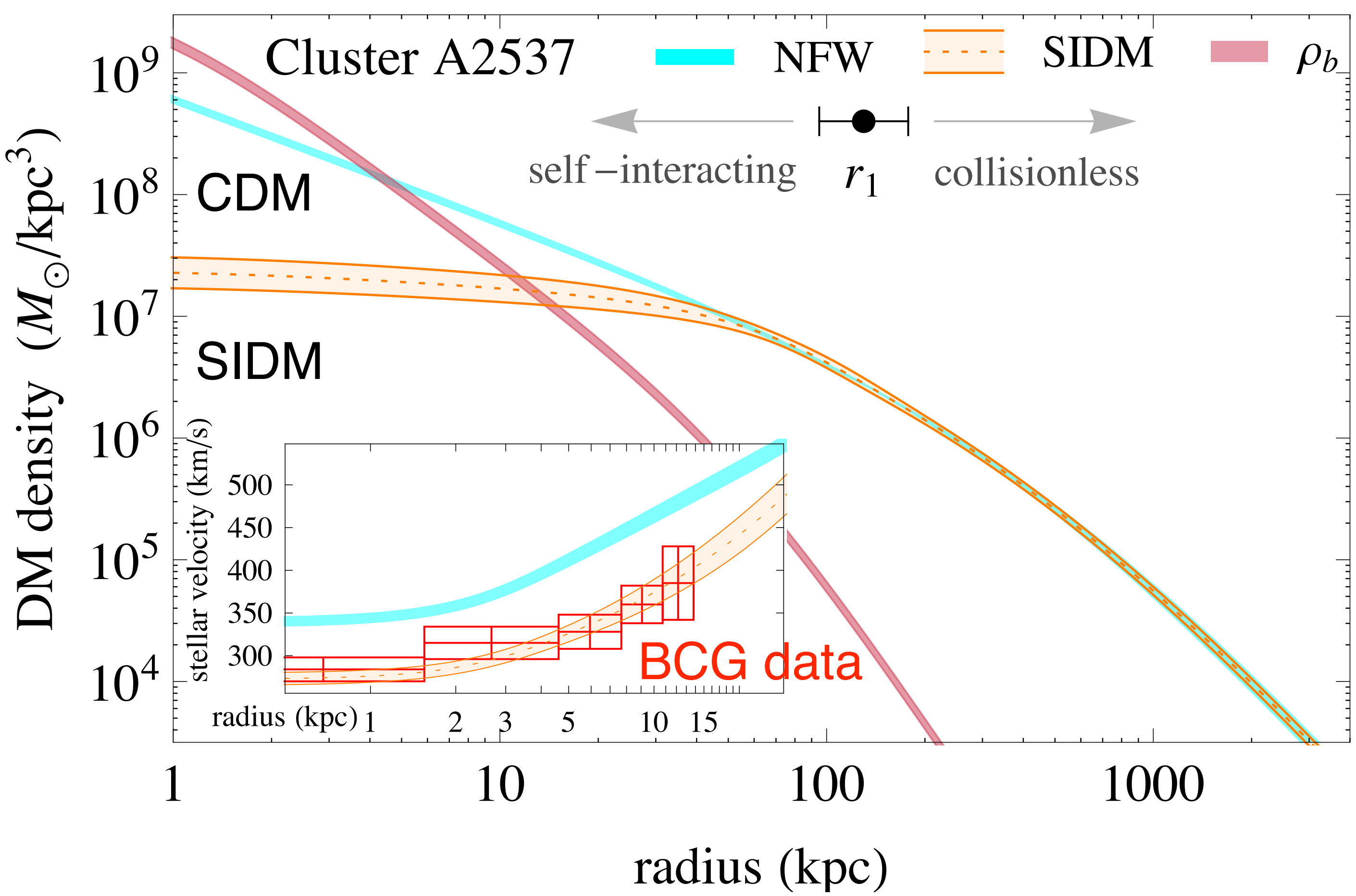
Data points

Galaxy clusters

- mass distribution in the outer region is determined by strong/weak gravitational lensing

- stellar kinematics in the central region (brightest cluster galaxies) prefer cored SIDM profile





$$\sigma_{\rm self}/m \sim 0.1\,\mathrm{cm^2/g}$$

$$\langle v_{\rm rel} \rangle \sim 10^3\,\mathrm{km/s}$$

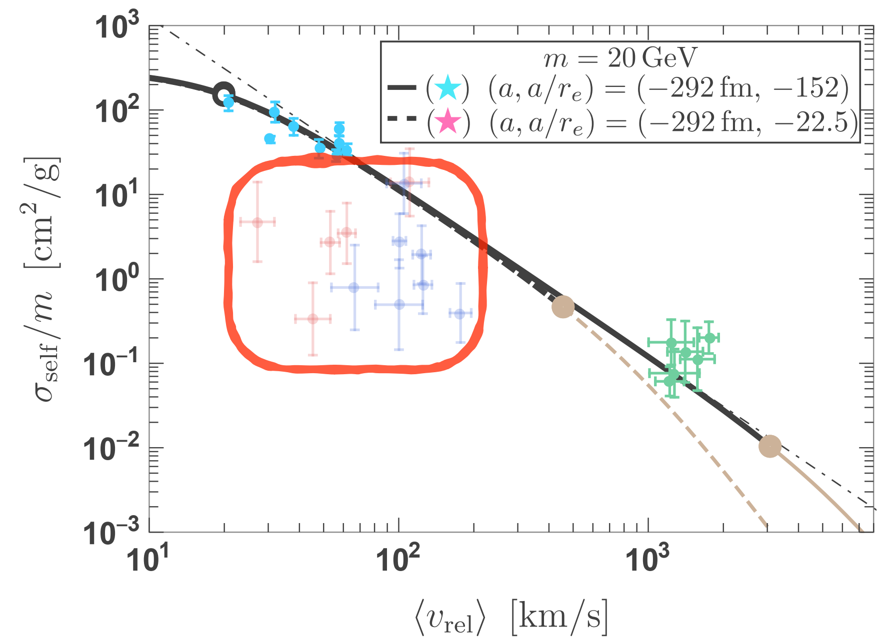
Kaplinghat, Tulin and Yu, PRL, 2016

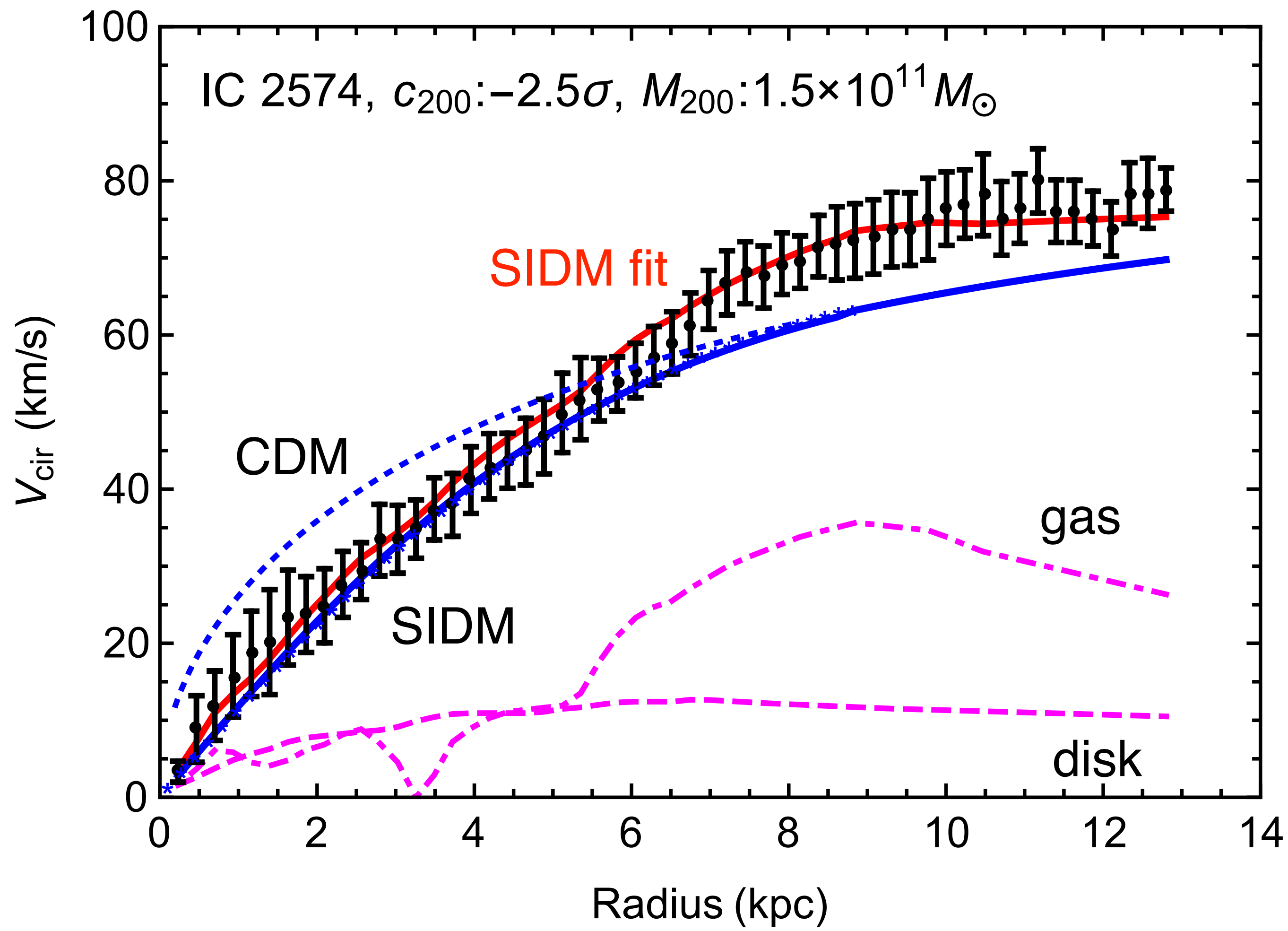
Data points

Dwarf spiral galaxies

- mass distribution is broadly determined by rotation curves

- rotation velocity in central region (of some galaxies) prefer cored SIDM profile





$$\sigma_{\rm self}/m \sim 1\,\mathrm{cm^2/g}$$

$$\langle v_{\rm rel}\rangle \sim 10^2\,\mathrm{km/s}$$

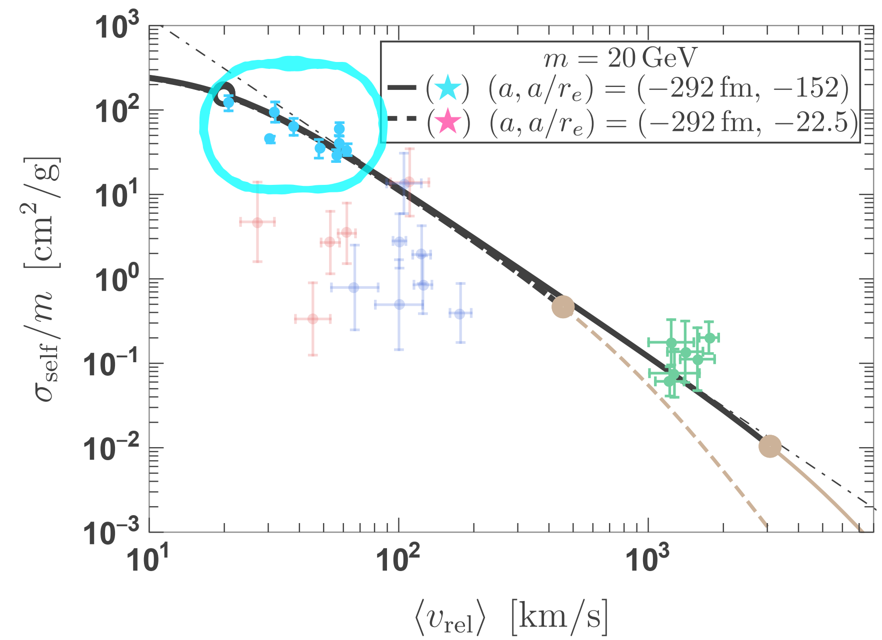
AK, Kaplinghat, Pace and Yu, PRL, 2017

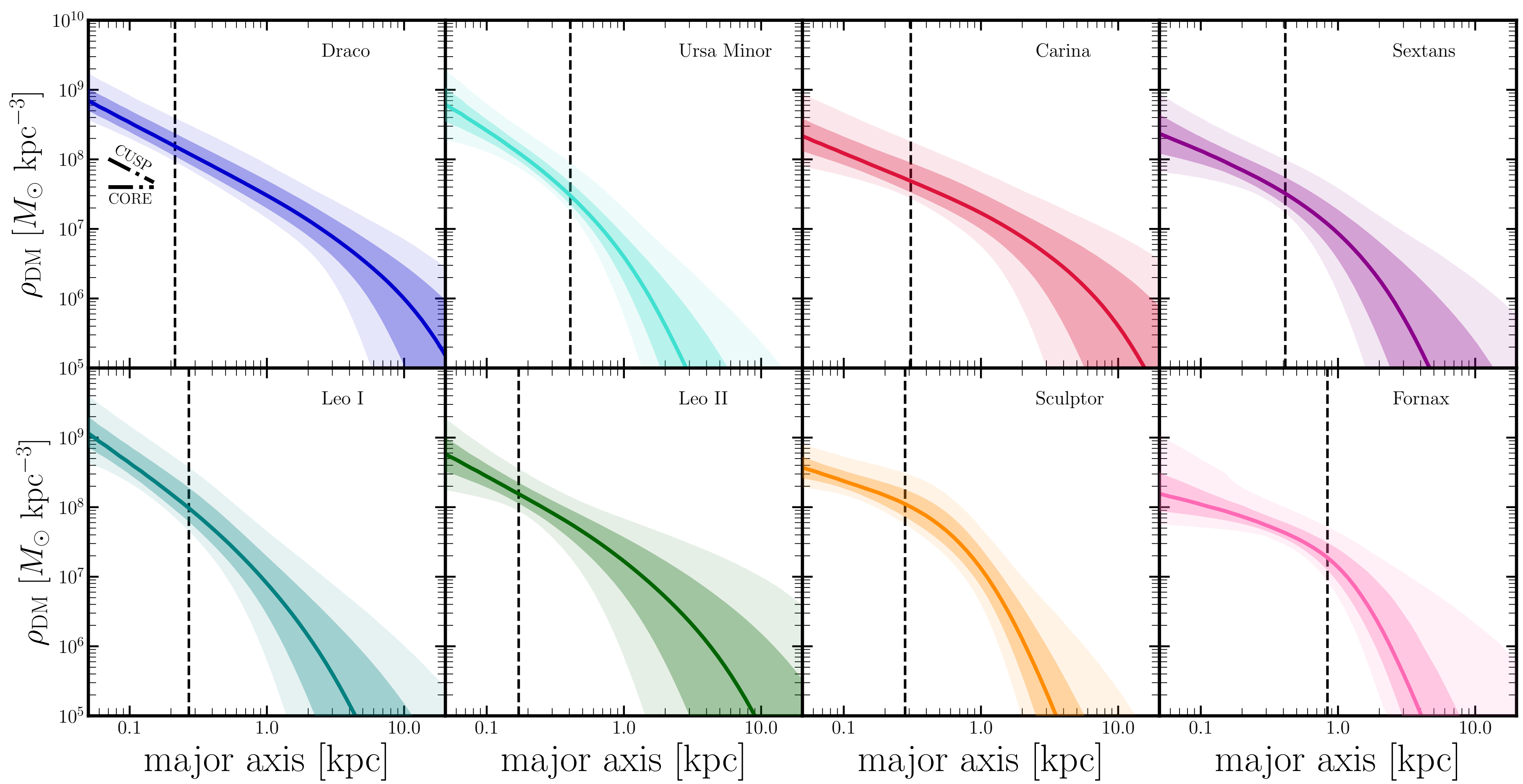
Data points

MW satellites

- mass distribution is determined by stellar kinematics

- stellar kinematics in the central region (of some satellites) prefer cuspy CDM profile

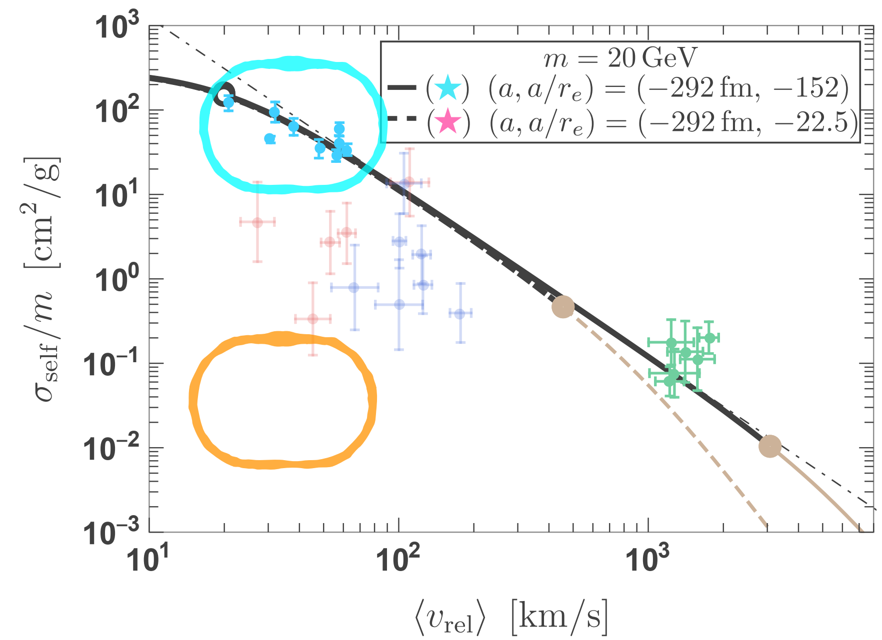




Hayashi, Chiba and Ishiyama, ApJ, 2020

Data points

MW satellites



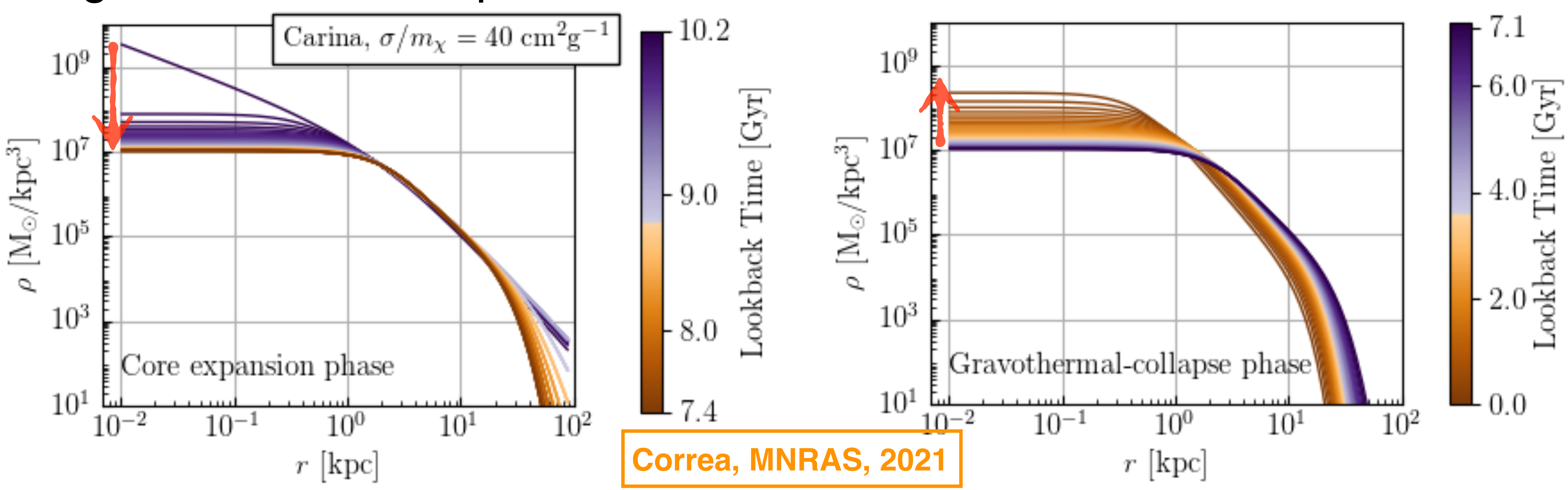
- one possibility is to take as a tiny cross section as $\sigma_{\rm self}/m \simeq 0.01\,\mathrm{cm^2/g}$

$$\langle v_{\rm rel}\rangle \sim 30\,\mathrm{km/s}$$

- resonance? Chu, Garcia-Cely and Murayama, PRL, 2019

- another possibility is to take as a large cross section as $\sigma_{\rm self}/m \sim 40\,\mathrm{cm^2/g}$ $\quad \langle v_{\rm rel}\rangle \sim 30\,\mathrm{km/s}$

- gravothermal collapse

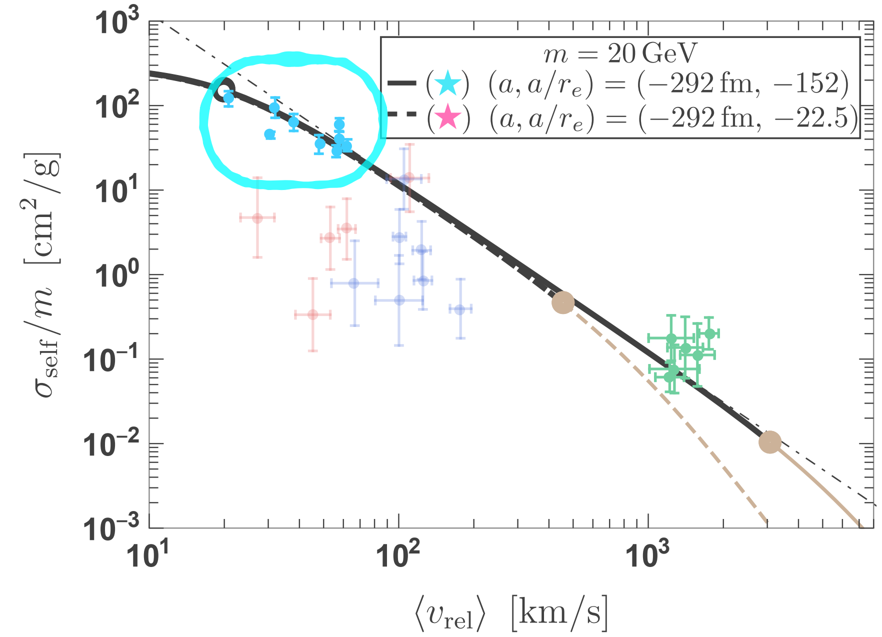


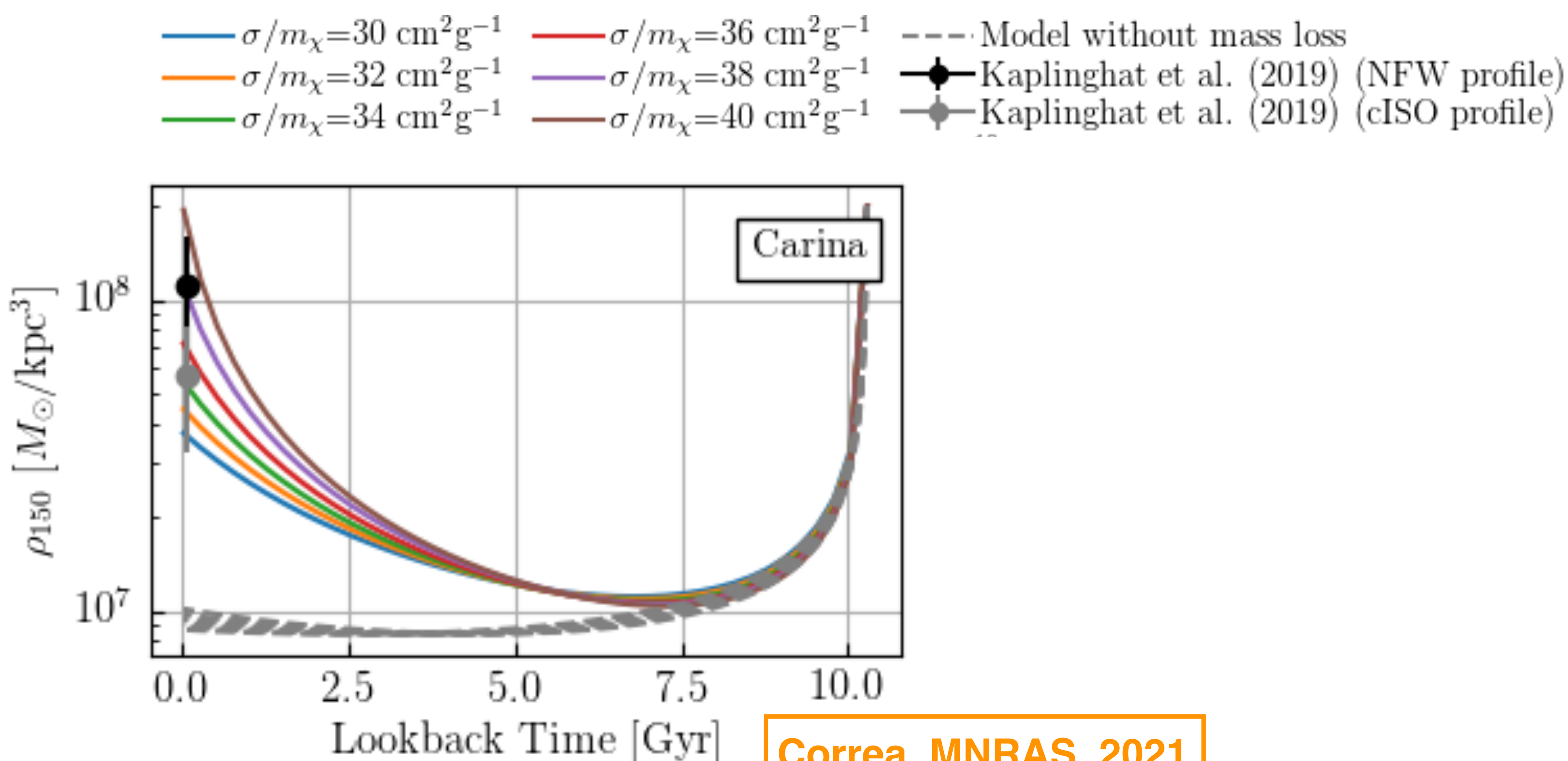
Correa, MNRAS, 2021

Data points

MW satellites

- gravothermal collapse
  - core shrinks and central density gets higher
  - central density at present is very sensitive to the cross section





Correa, MNRAS, 2021