Quantum Symmetries, Subfactors and Conformal Field Theory

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20 July 2023

Overview

- subfactors as quantum symmetries beyond groups and quantum groups
- Haagerup subfactor and its double – and related familes
- construction of a CFT from a subfactor or a quantum symmetry

Operator Algebras

$\bigotimes^n M_2 \simeq M_{2^n} \simeq End(\bigotimes^n \mathbb{C}^2)$

$$x \to x \otimes 1$$

$\bigotimes^\infty M_2 \simeq M_{2^\infty}$

$e = e^* = e^2$:

$\mathrm{dim(e)} = \mathrm{trace(e)} \in \{0, \frac{1}{2^n}, \frac{2}{2^n}, \frac{3}{2^n} \dots , 1\}$

$$\to \mathbb{N}[1/2]$$

$$K_0(\otimes_{\mathbb{N}} M_2) = \mathbb{Z}[1/2]$$

Factors

$$\otimes^n M_2 \subset End(\otimes^n H_2) \qquad H_2 = M_2\,, \quad \langle x, y\rangle = \mathrm{tr} y^* x$$

$$R = \otimes^\infty M_2 \subset End(\otimes^\infty H_2)$$

$$K_0(R) = \mathbb{R}$$

factor $R' \cap R = \mathbb{C}$

- I $\quad M_n \quad B(\mathcal{H})$
- II $\quad R, \quad R \otimes B(\mathcal{H})$
- III $\quad \lambda \in [0,1]$

$$\langle x, y\rangle = \mathrm{tr} e^{-H} y^* x$$

$$\mathrm{III}_\lambda \qquad e^{-H} = \begin{pmatrix} \frac{1}{1+\lambda} & 0 \\ 0 & \frac{\lambda}{1+\lambda} \end{pmatrix} \qquad 0 < \lambda < 1$$

low index subfactors

$N = M^G \subset M \subset M \rtimes G$,

index $[M, N]$
$= |G| \in 1, 2, 3, 4 \cdots$

1234 Jones St, San Francisco

index $= \mathbf{4}$

$$\begin{array}{ccc} R = \otimes^\infty M_2 & \subset & M_2 \otimes R \\ \cup & & \cup \\ R^G & \subset & (M_2 \otimes R)^G \end{array}$$

$G \subset SU(2)$ affine ADE classification + cohom. obstruction

index $< \mathbf{4}$ ADE classification

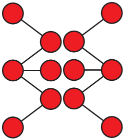
beyond 4: Haagerup subfactor at index $(5 + \sqrt{13})/2$

< 4	4	$\frac{1}{2}(5 + \sqrt{13})$	$4.3772\ldots$	$\frac{1}{2}(5 + \sqrt{17})$	$3 + \sqrt{3}$	$\frac{1}{2}(5 + \sqrt{21})$	5
ADE	affine ADE	**Haagerup**	extended Haagerup	Asaeda-Haagerup	$SU(2) \to E_6$	$G_2 \to E_6$	$[G; H]$

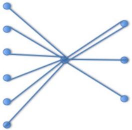
principal graphs

$\kappa : M \to N, \quad \bar{\kappa} : N \to M \qquad \kappa\bar{\kappa} \succeq id_N \qquad Bimod_\lambda\, a.x.b = ax\lambda(b)$

κ generates N-N, N-M, M-N, M-M sectors via $\kappa\bar{\kappa}$, $\kappa\bar{\kappa}\kappa$, $\bar{\kappa}\kappa\bar{\kappa}$ etc

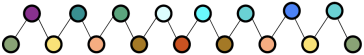


N-N N-M M-M

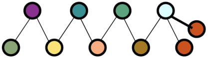


$g \in G \qquad \pi \in \hat{G}$

A series for SU(2)



orbifold of A_{4n+1} $N^{\mathbb{Z}_2} \subset M^{\mathbb{Z}_2}$ is D_{2n}



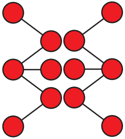
tunnel, tower and commuting squares

$$\kappa : M \to N, \quad \bar\kappa : N \to M \qquad \kappa\bar\kappa \succeq id_M$$

$$\cdots \subset \bar\kappa\kappa M \subset \kappa M \subset M \subset \langle M, e\rangle = M \otimes_N M \subset M \otimes_N M \otimes_N M \subset \dots$$

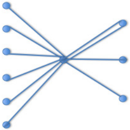
$\leftarrow$ tunnel tower $\rightarrow$

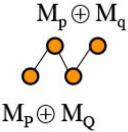
$M_k \subset M_l\,, \quad k \le l \qquad (M_k)' \cap M_l$ finite dimensional



$$\begin{array}{ccccc} N' \cap M_k & \subset & N' \cap M_{k+1} & \to & B \\ \cup & & \cup & & \cup \\ M' \cap M_k & \subset & M' \cap M_{k+1} & \to & A \end{array}$$

N-N N-M M-M



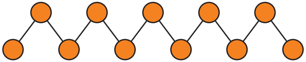


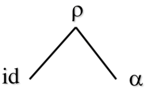
$$a \oplus b \to$$

$$a \bigoplus \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

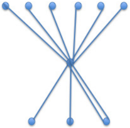
$g \in G \qquad \pi \in \hat{G}$

CFT - the search for the exotic





$$\rho^2 = id + \alpha \qquad \alpha\rho = \rho$$
$$\rho^2 = 1 + \alpha + 0.\rho \qquad \mathbb{Z}_2$$

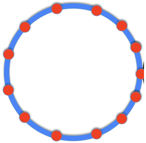


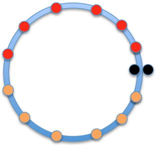
$$\rho^2 = \sum_g g$$
$$g\rho = \rho$$

Tambara-Yamagami

Orbifold

$g \leftrightarrow -g$



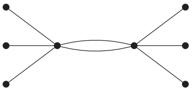


near-group systems $\rho^2 = n'\rho + \sum_{g\in G} g$

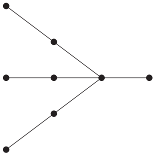
$\rho = \bar{\rho} \qquad g\rho = \rho = \rho g$

$\rho^2 = n'\rho + \sum_{g\in G} g \qquad d_\rho = \frac{n'+\sqrt{n'^2+4n}}{2} \qquad d_\lambda = [M, \lambda M]^{1/2}$

- $n' = 0$ Tambara-Yamagami
- $n' = |G| - 1$



- $n' = k|G|$
- $n' = |G|$



DE-Gannon, Izumi

near group $\rho^2 = |G|\rho + \sum_{g \in G} g$

$\mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2$ $\quad$ $2, 2, 1$ $\quad$ Izumi

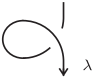
$\mathbb{Z}_5, \mathbb{Z}_6, \mathbb{Z}_7, \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_9, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_{10}, \mathbb{Z}_{11}, \mathbb{Z}_{12}, \mathbb{Z}_2 \times \mathbb{Z}_6, \mathbb{Z}_{13}$

3, 4, 2, 8, 4, 2, 1, 4, 4, 4, 2, 4

DE-Gannon

$\lambda\mu = \sum_\nu N^\nu_{\lambda\mu}\nu, \quad \lambda\mu = \mathrm{Ad}u(\lambda,\mu)\mu\lambda, \qquad u(\lambda,\mu) =$ 

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \to [S_{\lambda\mu}]$ 

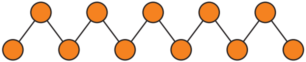
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \to [T_{\lambda\mu}]$ 

$N^k_{i,j} = \sum_l \frac{S_{i,l}}{S_{0,l}} S_{j,l} S^*_{k,l}$

$S_{i,j} = \overline{T}_{i,i}\overline{T}_{j,j} T_{0,0} \sum_k T_{k,k} N^k_{i,j} d_k / S_{0,0}$

- cyclic group $\mathbb{Z}_n$, $T_{gg} = x\, e^{\pi i a g^2/n}$,
  $S_{gh} = e^{\pi i a((g+h)^2 - g^2 - h^2))/n} / \sqrt{n}$
- Quadratic form Q on abelian group G
  $\langle g, h\rangle_Q = e^{\pi i\,(Q(g+h) - Q(g) - Q(h))}$
  $T_{gg} = x e^{\pi i Q(g)}$, $S_{gh} = \langle g, h\rangle_Q / \sqrt{|G|}$
  $x^{-3} = \Sigma_{k\in G} e^{\pi i Q(k)} / \sqrt{|G|}$

loop group subfactors



$\mathrm{Ver}_k(SU(2)) \cong \mathbb{Z}[\rho]/\langle S_{k+1}\rangle$

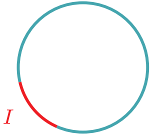
- $\lambda \in$ pos energy rep of $LSU(n) = Map(S^1, SU(n))$
- $N(I) = \pi_0(L_I SU(n))''$
- $\pi_\lambda(L_I SU(n))'' \subset \pi_\lambda(L_{I'} SU(n))'$

  $\lambda = 0 : \quad N = N \qquad \lambda N \subset N$ Jones-Wassermann, Wassermann

λ endomorphism, N type III_1 factor: $\lambda\mu = \sum_\nu N^{\nu}_{\lambda\mu}\nu$

representations
$R(N)$ of
conformal net of
factors $N(I)$

$I \subset S^1$



$$\chi_\lambda = \mathrm{trace}_{\mathcal{H}_\lambda}\, q^{L_0 - c/24} \qquad q = e^{2\pi i \tau}$$

quantum double

$\mathcal{X}$ on N system of endomorphisms not necessarily braided

Take $\mathcal{X} \times \mathcal{X}^{opp}$ on $N \otimes N^{opp}$.

Construct

$\iota : A \subset N \otimes N^{opp}$ such that:

- $\iota\bar{\iota} = \Sigma_{\nu \in \mathcal{X}} \nu \otimes \nu^{opp}$
- A-A system is non-degenerately braided

Ocneanu, Longo-Rehren, Izumi, Popa

double of $\rho^2 = |G|\rho + \sum_{g \in G} g$ system

primaries $|G|(|G|+3) \;\; = |G| + |G| + |G|(|G|-1)/2 + |G|(|G|+3)/2$

$$T^{Q,Q'} = \mathrm{diag}(\langle g,g\rangle; \langle h,h\rangle; \langle k,l\rangle; \langle m,m\rangle \, \langle \gamma,\gamma\rangle'),$$

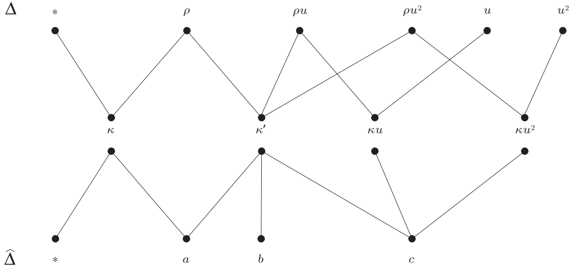
$$S^{Q,Q'} = \frac{1}{\lambda}\begin{pmatrix} \overline{\langle g,g'\rangle}^2 & (\delta+1)\overline{\langle g,h'\rangle}^2 & (\delta+2)\overline{\langle g,k'+l'\rangle} & \delta\overline{\langle g,m'\rangle}^2 \\ (\delta+1)\overline{\langle h,g'\rangle}^2 & \overline{\langle h,h'\rangle}^2 & (\delta+2)\overline{\langle h,k'+l'\rangle} & -\delta\overline{\langle h,m'\rangle}^2 \\ (\delta+2)\overline{\langle k+l,g'\rangle} & (\delta+2)\overline{\langle k+l,h'\rangle} & (\delta+2)\left(\overline{\langle k,k'\rangle\langle l,l'\rangle} + \overline{\langle k,l'\rangle\langle l,k'\rangle}\right) & 0 \\ \delta\overline{\langle m,g'\rangle}^2 & -\delta\overline{\langle m,h'\rangle}^2 & 0 & -\delta\overline{\langle m,m'\rangle}\left(\langle\gamma,\gamma'\rangle' + \overline{\langle\gamma,\gamma'\rangle'}\right) \end{pmatrix}$$

Q, Q' quadratic forms for groups G and G' where $|G'| = |G| + 4$

$\langle g, h\rangle_Q = e^{2\pi i\,(Q(g+h)-Q(g)-Q(h))}$

- $d_\rho^2 = |G|d_\rho + |G| \qquad d_\rho = \frac{|G|+\sqrt{|G|^2+4|G|}}{2}$
- $G = H \times H, \quad |G| = \nu^2$ interesting case,
  $d_\rho = \nu(\frac{\nu+\sqrt{\nu^2+4}}{2})$ related to Haagerup $\rho^2 = 1 + \sum_{h\in H} \rho h$

Principal graphs of the Haagerup $(5+\sqrt{13})/2$ subfactor



$$\alpha^3 = 1\,, \quad \rho\alpha = \alpha^2\rho\,, \quad \rho^2 = 1 + \rho + \rho\alpha + \rho\alpha^{\mathbf{2}}$$

$$d_\lambda = [M, \lambda M]^{1/2} \qquad d_\rho^2 = 1 + 3d_\rho; \quad d_\rho = (3+\sqrt{13})/2$$

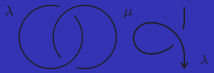
generalised Haagerup $\quad \rho^2 = 1 + \sum_{g\in G} \rho g\,, \quad g\rho = \rho g^{-1}$

zumi: $\quad \mathbb{Z}_3, \mathbb{Z}_5 \quad |G|^2 + 4 = 13, 29$

DE-Gannon: $\quad \mathbb{Z}_7, \mathbb{Z}_{\mathbf{9}}, \mathbb{Z}_{\mathbf{11}}, \mathbb{Z}_{13}, \mathbb{Z}_{15}, \mathbb{Z}_{17}, \mathbb{Z}_{\mathbf{19}}$

$|G|^2 + 4 \quad$ 53, **85**, **125**, 173, 229, 293, **365**

Modular data for Haagerup $\mathcal{D}$Hg



$$S = \frac{1}{3}\begin{pmatrix}
x & 1-x & 1 & 1 & 1 & 1 & y & y & y & y & y & y \\
1-x & x & 1 & 1 & 1 & 1 & -y & -y & -y & -y & -y & -y \\
1 & 1 & 2 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
y & -y & 0 & 0 & 0 & 0 & c(1) & c(2) & c(3) & c(4) & c(5) & c(6) \\
y & -y & 0 & 0 & 0 & 0 & c(2) & c(4) & c(6) & c(5) & c(3) & c(1) \\
y & -y & 0 & 0 & 0 & 0 & c(3) & c(6) & c(4) & c(1) & c(2) & c(5) \\
y & -y & 0 & 0 & 0 & 0 & c(4) & c(5) & c(1) & c(3) & c(6) & c(2) \\
y & -y & 0 & 0 & 0 & 0 & c(5) & c(3) & c(2) & c(6) & c(1) & c(4) \\
y & -y & 0 & 0 & 0 & 0 & c(6) & c(1) & c(5) & c(2) & c(4) & c(3)
\end{pmatrix}$$

$$T = \mathrm{diag}(1,1,1,1,\xi_3,\overline{\xi_3},\xi_{13}^{6},\xi_{13}^{-2},\xi_{13}^{2},\xi_{13}^{5},\xi_{13}^{-6},\xi_{13}^{-5})$$

$x = (13 - 3\sqrt{13})/26$ $\quad$ $y = 3/\sqrt{13}$ $\quad$ $c(j) = -2y\,\cos(2\pi j/13)$ $\quad$ $\xi = e^{2\pi i/13}$

$$S_{jj'} = (-2y/3)\cos(2\pi jj'/13)$$

DE-Gannon

$N_{i,j}^{k} = \sum_l \frac{S_{i,l}}{S_{0,l}} S_{j,l} S_{k,l}^{*}$ $\qquad$ $S_{i,j} = \overline{T}_{i,i}\overline{T}_{j,j} T_{0,0} \sum_k T_{k,k} S_{k,0} N_{i,j}^{k}$

Modular data for $SO(13)_2$

$$S = \frac{1}{3}\begin{pmatrix} y/2 & y/2 & 3/2 & 3/2 & y & y & y & y & y & y \\ y/2 & y/2 & -3/2 & -3/2 & y & y & y & y & y & y \\ 3/2 & -3/2 & 3/2 & -3/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3/2 & -3/2 & -3/2 & 3/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ y & y & 0 & 0 & -c(1) & -c(2) & -c(3) & -c(4) & -c(5) & -c(6) \\ y & y & 0 & 0 & -c(2) & -c(4) & -c(6) & -c(5) & -c(3) & -c(1) \\ y & y & 0 & 0 & -c(3) & -c(6) & -c(4) & -c(1) & -c(2) & -c(5) \\ y & y & 0 & 0 & -c(4) & -c(5) & -c(1) & -c(3) & -c(6) & -c(2) \\ y & y & 0 & 0 & -c(5) & -c(3) & -c(2) & -c(6) & -c(1) & -c(4) \\ y & y & 0 & 0 & -c(6) & -c(1) & -c(5) & -c(2) & -c(4) & -c(3) \end{pmatrix}$$

$$T = \mathrm{diag}(-1, -1; -i, i; -\xi_{13}^{6l^2})$$

$$y = 3/\sqrt{13} \qquad c(j) = -2y\cos(2\pi j/13) \qquad \xi = e^{2\pi i/13}$$

$$S_{jj'} = (+2y/3)\cos(2\pi jj'/13)$$

Characters for $\mathcal{D}$Hg, $c = 8$, $\gamma = 0, 1$

$$\begin{pmatrix} ch_0(\tau) \\ ch_{\mathfrak{b}}(\tau) \\ ch_{\mathfrak{a}}(\tau) = ch_{\mathfrak{c}_0}(\tau) \\ ch_{\mathfrak{c}_1}(\tau) \\ ch_{\mathfrak{c}_2}(\tau) \\ ch_{\mathfrak{d}_1}(\tau) \\ ch_{\mathfrak{d}_2}(\tau) \\ ch_{\mathfrak{d}_3}(\tau) \\ ch_{\mathfrak{d}_4}(\tau) \\ ch_{\mathfrak{d}_5}(\tau) \\ ch_{\mathfrak{d}_6}(\tau) \end{pmatrix} = \begin{pmatrix} q^{2/3}\left(q^{-1} + (6+13\gamma) + (120+78\gamma)q + (956+351\gamma)q^2 + (6010+1235\gamma)q^3 + \cdots\right) \\ q^{2/3}\left((80-13\gamma) + (1250-78\gamma)q + (10630-351\gamma)q^2 + (65042-1235\gamma)q^3 + \cdots\right) \\ q^{2/3}\left(81 + 1377q + 11583q^2 + 71037q^3 + \cdots\right) \\ 3 + 243q + 2916q^2 + 21870q^3 + \cdots \\ q^{1/3}\left(27 + 594q + 5967q^2 + 39852q^3 + \cdots\right) \\ q^{5/39}\left((7-\gamma) + (292-6\gamma)q + (3204-43\gamma)q^2 + (23010-146\gamma)q^3 + \cdots\right) \\ q^{20/39}\left((42+16\gamma) + (777+121\gamma)q + (7147+547\gamma)q^2 + (45367+2000\gamma)q^3 + \cdots\right) \\ q^{32/39}\left(\gamma q^{-1} + (11\gamma+119) + (73\gamma+1623)q + (300\gamma+12996)q^2 + (76429+1063\gamma)q^3 + \cdots \right. \\ q^{2/39}\left((5-3\gamma) + (229-50\gamma)q + (2738-252\gamma)q^2 + (19942-1032\gamma)q^3 + \cdots\right) \\ q^{8/39}\left((13-5\gamma) + (347-37\gamma)q + (3804-212\gamma)q^2 + (26390-794\gamma)q^3 + \cdots\right) \\ q^{11/39}\left((14+7\gamma) + (441+61\gamma)q + (4445+303\gamma)q^2 + (30329+1167\gamma)q^3 + \cdots\right) \end{pmatrix}$$

DE-Gannon

Potts model and orbifold $H(\sigma) = -\sum_{i,j\ n\cdot n} J\delta(\sigma_i,\sigma_j)$

$W_j W_{j+1} = e^{2\pi i/Q} W_{j+1} W_j\,,\ W_i W_j = W_j W_i\,, |i-j|>1$ in $M_Q \otimes M_Q \otimes ..$

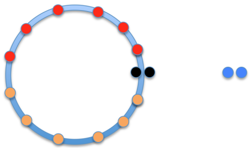
$e_i =$Spectral$(W_i, 1)$ $\qquad e_i e_{i\pm1} e_i = e_i/Q$

$V = \exp L \sum e_{2i+1}\,, \quad W = \exp L^* \sum e_{2i} \quad (e^L - 1)(e^{L^*} - 1) = Q$

μ on $\mathcal{O}_Q \qquad \mu^2 = \sum_g \alpha_g \qquad g \in \mathbb{Z}_Q$

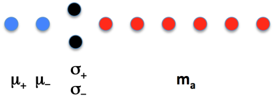
Orbifold $g \leftrightarrow -g$; $\qquad$ take $\rtimes \mathbb{Z}_2$

$\longrightarrow \hat{\mathbb{Z}}_2 = \sigma_\pm \qquad \mu_\pm \qquad m_g = \alpha_g \oplus \alpha_{-g}$

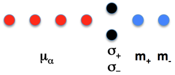


$m_a m_b = m_{a+b} + m_{a-b} \qquad m_0 = \sigma_+ + \sigma_-$
$\mu_\tau \mu_{\tau'} = \sigma_{\tau+\tau'} + \sum m_a \qquad a \sim -a \neq 0$
$\mu_\tau m_a = \mu_+ + \mu_- \qquad \sigma_\pm m_a = m_a$

fusion rules of double of Haagerup

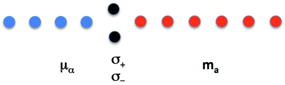


$\mathbb{Z}_{13}$



$\mathbb{Z}_3 \times \mathbb{Z}_3$

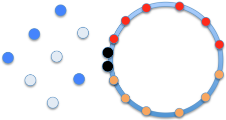
$$\sigma_\pm \simeq \mathbb{Z}_2 \quad m_a = e^{ia} + e^{-ia} \;:\; m_a m_b = m_{a+b} + m_{a-b}$$
$$\mu_\tau \mu_{\tau'} = \sigma_{\tau+\tau'} + \sum m_a \qquad a \sim -a \neq 0$$
$$\mu_\tau m_a = \mu_+ + \mu_- \qquad \sigma_\pm m_a = m_a$$



$$\sigma_+ = \text{identity}$$
$$\sigma_-^2 = \sigma_+ + \sigma_- + \sum \mu_\alpha + \sum m_a = R$$

$$\mu_\alpha m_b = R - \sigma_+ = R_-$$
$$\mu_\alpha \mu_\beta = R_- + \mu_{\alpha+\beta} + \mu_{\alpha-\beta}$$
$$m_a m_b = R_- - m_{a+b} - m_{a-b}$$
$$\sigma_- \mu_\alpha = R_- + \mu_\alpha$$

$$\mu_0 = \sigma_+ + \sigma_-$$
$$\sigma_- m_a = R_- - m_a$$
$$\sigma_- m_a = R_- - m_a$$



construction of CFT from subfactors

- for G finite abelian odd, $\exists$ conformal net $Rep(\mathcal{A}) = TY(G)^{\mathbb{Z}_2}$
- for G finite abelian, $\exists$ conformal net with $Rep(\mathcal{B}) = \mathcal{D}TY(G)$

Bischoff, DE-Gannon

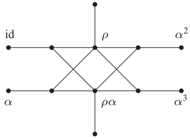
- for $\omega \in H^3(G, \mathbb{T})$, $\exists$ conformal net with $Rep(\mathcal{A}) = Rep\,\mathcal{D}^\omega(G)$
- If Rep$(\mathcal{A}) \simeq \mathcal{D}^\omega(G)$, then $\mathcal{A} \simeq \mathcal{V}^G$ for a holomorphic net $\mathcal{V}$
- $\mathcal{V}$ holomorphic conformal net, $G \subset S_k$ then
  Rep $(\mathcal{V}^{k\otimes})^G$ = Rep$\mathcal{D}^\omega(G)$ - with $\omega^3 = 1$

DE-Gannon

summary

- $\rho^2 = |G|\rho + \sum_{g \in G} g$ near group
- $\rho^2 = 1 + \sum_{g \in G} \rho g$ Haagerup
- modular data — grafting of two models that are understood
- related by orbifold in certain cases
- mixed systems e.g
  $(G_2)_4 \to (D_7)_1 \simeq \mathbb{Z}_4 = \langle \alpha \rangle$
  $\alpha\rho = \rho\alpha \neq \rho, \qquad \alpha^2\rho = \rho = \rho\alpha^2,$
  $\rho^2 = 2\rho + 2\rho\alpha + \mathrm{id} + \alpha^2$ DE-Pugh

$\rho M \rtimes_{\alpha^2} \mathbb{Z}_2 \subset M$



double of the Haagerup subfactor

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