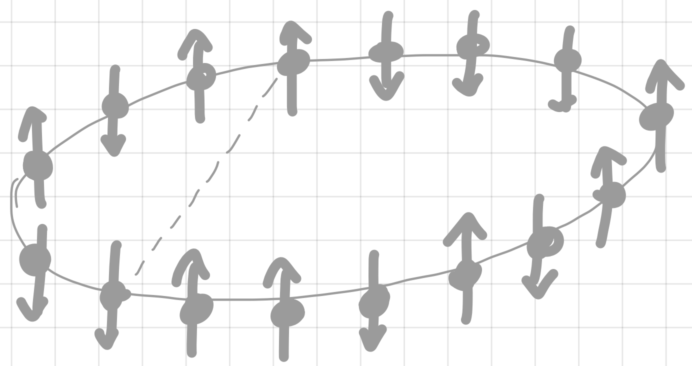
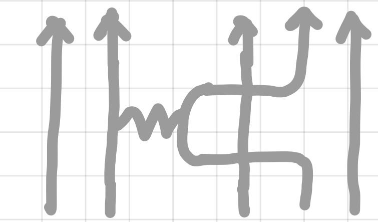


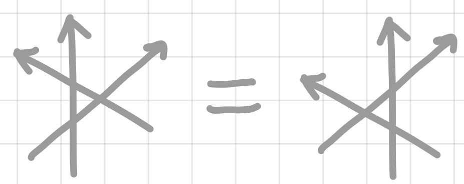


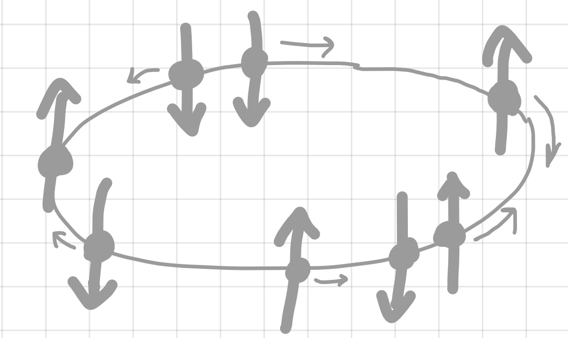
the q-deformed Inozemtsev chain

by

Jules Lamers

Institut de Physique Théorique





joint work with Rob Klabbers
arXiv:2306.13066 + in preparation

Non-perturbative long-range integrability

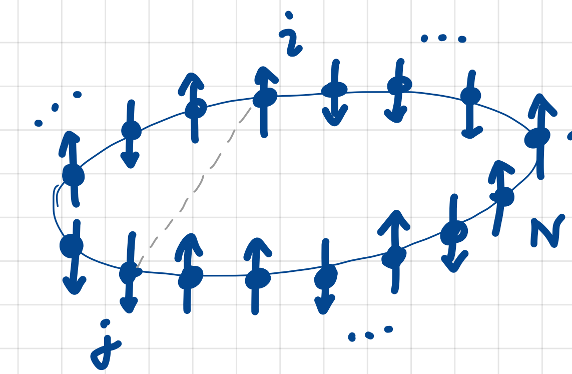
nature atomic, molecular and optical physics
naturally occur in AdS/CFT integrability
rich representation theory (affine Hecke algebras etc)

consider spin-$\frac{1}{2}$ chain $$H = \sum_{i<j}^{N} V(i-j)\,\overbrace{(1-P_{ij})}^{(1-\vec{\sigma}_i\cdot\vec{\sigma}_j)/2} \quad \text{on } (\mathbb{C}^2)^{\otimes N}$$

pairwise

pair potential

spin interaction



$$\delta_{|x| \bmod N,1} \xleftarrow{\kappa\to\infty} \sim \wp(x) \text{ periods } N, \tfrac{i\pi}{\kappa} \xrightarrow{\kappa\to 0} \frac{(\pi/N)^2}{\sin(\frac{\pi}{N}x)^2} = \frac{1}{\mathrm{crd}^2}$$

$\downarrow N\to\infty$

$\kappa^2/\sinh^2(\kappa x)$

guest role in AdS/CFT
Serban Staudacher 04

"su(2)$_1$ WZW on the lattice"
Ha et al 92
Bernard et al 94
Bouwknegt et al 94 96

Heisenberg XXX ⟵ **Inozemtsev** ⟶ **Haldane-Shastry**

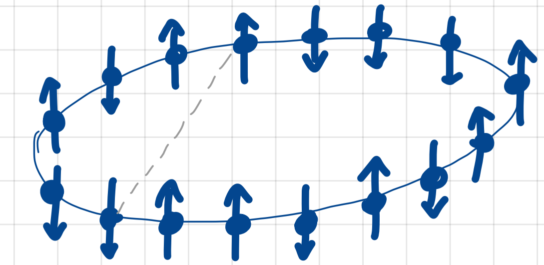
Heisenberg 28
Bethe 31

Inozemtsev 90
Inozemtsev 90 95 00
Klabbers JL 22

Haldane 88 Shastry 88
Haldane 91 Bernard et al 93

note: all wrapping corrections are automatically included

Non-perturbative long-range integrability

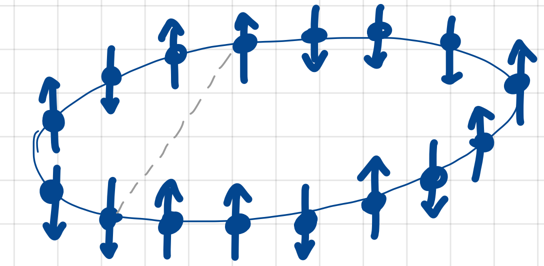


nature atomic, molecular and optical physics
naturally occur in AdS/CFT integrability
rich representation theory (affine Hecke algebras etc)

	paradigm for q integrable models	challenges our understanding of q integrability	paradigm for long-range q integrab
Yangian	✓ Faddeev et al late 70s	? unknown	✓ different from Heis Ha et al 92 Bernard et al 93
commuting hamiltonians	✓ Sutherland 70	✓ conjectured partial proof Dittrich Inozemtsev 08	✓ Inozemtsev 90 Ha et al 92 Talstra Haldane 95
exactly solvable	✓ up to solving BAE Bethe 31	✓ up to solving BAE Inozemtsev 90 95 00 Klabbers JL 22	✓ explicit Haldane 91 Bernard et al 93
	Heisenberg XXX ← Heisenberg 28 Bethe 31	**Inozemtsev** Inozemtsev 90 Inozemtsev 90 95 00 Klabbers JL 22	→ **Haldane-Shastry** Haldane 88 Shastry 88 Haldane 91 Bernard et al 93

note: all wrapping corrections are automatically included

Lessons from controlled symmetry breaking



$$H_{Heis} = \frac{1}{2}\sum \left(\Delta - \sigma^x_i \sigma^x_{i+1} - \Gamma\, \sigma^y_i \sigma^y_{i+1} - \Delta\, \sigma^z_i \sigma^z_{i+1}\right)$$

studying **deformations**
sharpens understanding
even for undeformed case
e.g. new methods

↓ degree of
spin symmetry

anisotropic — **Heisenberg XYZ**
Sutherland 70
Baxter 73

face/vertex transformation

Q-operator

$\downarrow$ $\Gamma \to 1$

partially (an)isotr — **Heisenberg XXZ**
S^z symm
= weight-preserving

Orbach 58
Yang Yang 66

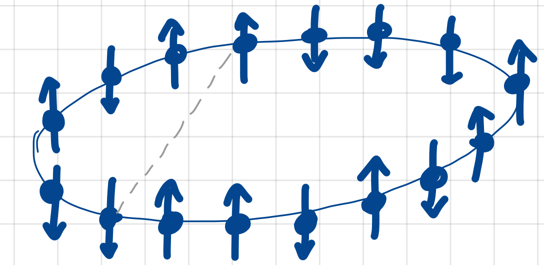
$\downarrow$ $\Delta \to 1$

isotropic — **Heisenberg XXX**
SU(2) symm
= $\mathfrak{sl}_2$ invariance

Heisenberg 28
Bethe 31

Bethe ansatz

New integrable unification of Inozemtsev and qHS

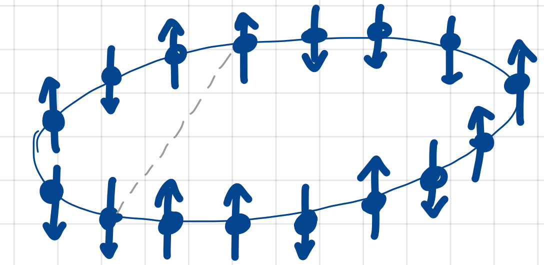


interaction range

degree of spin symmetry

	nearest neighbour	intermediate range	long range
anisotropic	Heisenberg XYZ Sutherland 70 Baxter 73		
partially (an)isotr S^z symm = weight-preserving	Heisenberg XXZ Orbach 58 Yang Yang 66		
isotropic SU(2) symm = $\mathfrak{sl}_2$ invariance	Heisenberg XXX Heisenberg 28 Bethe 31	Inozemtsev Inozemtsev 90 Inozemtsev 90 95 00 Klabbers JL 22	Haldane-Shastry Haldane 88 Shastry 88 Haldane 91 Bernard et al 93

New integrable unification of Inozemtsev and *q*HS



interaction range →

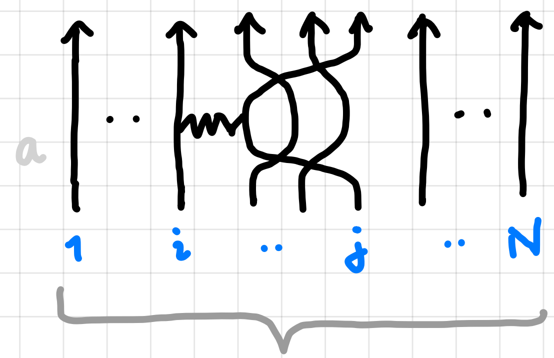
↓ degree of spin symmetry

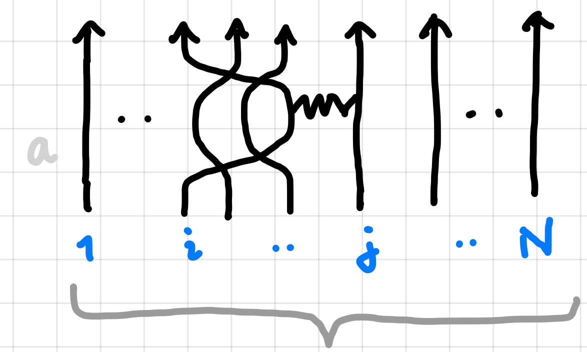
	nearest neighbour	intermediate range	long range
anisotropic	Heisenberg XYZ Sutherland 70 Baxter 73	?	?
partially (an)isotr S^z symm = weight-preserving	Heisenberg XXZ Orbach 58 Yang Yang 66	q-deformed Inozemtsev Klabbers JL 2306.13066	q-deformed HS Uglov 95 JL 18 JL Pasquier Serban 22
isotropic SU(2) symm = $\mathfrak{sl}_2$ invariance	Heisenberg XXX Heisenberg 28 Bethe 31	Inozemtsev Inozemtsev 90 Inozemtsev 90 95 00 Klabbers JL 22	Haldane-Shastry Haldane 88 Shastry 88 Haldane 91 Bernard et al 93

The q-deformed Inozemtsev chain: anatomy

Klabbers JL 23

'chiral' hamiltonians

$$H^L = \sum_{i<j}^{N} V(i-j) \times$$ 

$$H^R = \sum_{i<j}^{N} V(i-j) \times$$ 

cf JL 18
JL Pasquier Serban 22

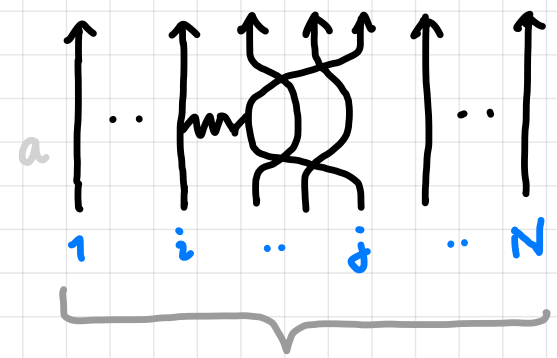
isotropic limit $\quad P_{j-1,j}\cdots P_{i+1,i+2}(1-P_{i,i+1})P_{i+1,i+2}\cdots P_{j-1,j} = 1-P_{ij} = P_{i\,i+1}\cdots P_{j-2,j-1}(1-P_{j-1,j})P_{j-2,j-1}\cdots P_{i,i+1}$

The q-deformed Inozemtsev chain: anatomy

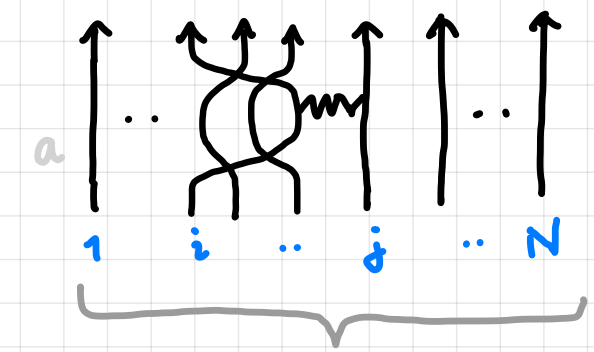
Klabbers JL 23

'chiral' hamiltonians

$$H^L = \sum_{i<j}^{N} V(i-j) \times$$



$$H^R = \sum_{i<j}^{N} V(i-j) \times$$



cf JL 18
JL Pasquier Serban 22

isotropic limit $\quad P_{j-1,j}\cdots P_{i+1,i+2}(1-P_{i,i+1})P_{i+1,i+2}\cdots P_{j-1,j} = 1-P_{ij} = P_{i,i+1}\cdots P_{j-2,j-1}(1-P_{j-1,j})P_{j-2,j-1}\cdots P_{i,i+1}$

both with **potential** periods $N, i\pi/\kappa$

$$V(x) = -\frac{\rho(x+\eta)-\rho(x-\eta)}{\theta(2\eta)} \sim \frac{1}{\mathrm{sn}(x+\eta)\,\mathrm{sn}(x-\eta)}$$

η anisotropy $\quad q = e^{i\eta}$

Klabbers JL 22

isotropic limit $\quad \xrightarrow{\eta\to 0} -\rho'(x) = \wp(x) + cst = V^{\mathrm{Ino}}(x)$

$\kappa > 0$ ~ interaction range

pre-pot $\quad \rho(x) = \frac{\theta'(x)}{\theta(x)} = \zeta(x) + cst\cdot x \xrightarrow{N\to\infty} \pi\cot(\pi x)$

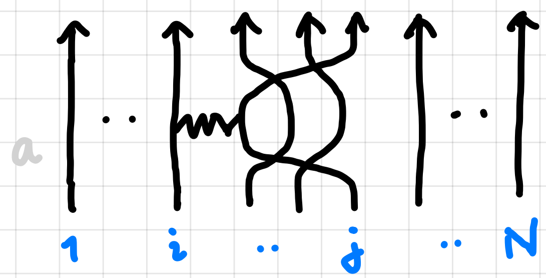
odd Jacobi $\theta(x)$ entire, odd, $\theta'(0)=1$, $\begin{cases}\theta(x+i\pi/\kappa) = -\theta(x)\\ \theta(x+N) = -e^{\kappa(N+2x)}\theta(x)\end{cases}$

$\xrightarrow{N\to\infty} \kappa^{-1}\sinh(\kappa x)$

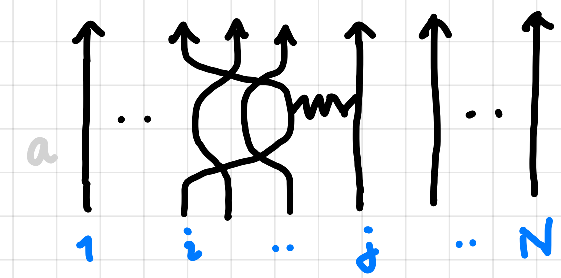
The q-deformed Inozemtsev chain: anatomy

Klabbers JL 23

'chiral' hamiltonians

$$H^L = \sum_{i<j}^{N} V(i-j) \times$$ 

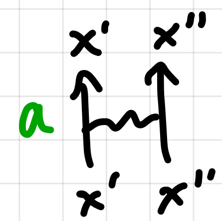
$$= \prod_{j>k>i}^{\leftarrow} P_{k\,k+1}(j-k) \cdot E_{i\,i+1}(i-j) \cdot \prod_{i<k<j}^{\rightarrow} P_{k\,k+1}(k-j)$$

$$H^R = \sum_{i<j}^{N} V(i-j) \times$$ 

$$= \prod_{i<k<j}^{\rightarrow} P_{k\,k+1}(i-k) \cdot E_{i\,i+1}(i-j) \cdot \prod_{j>k>i}^{\leftarrow} P_{k\,k+1}(k-i)$$

cf JL 18
JL Pasquier Serban 22

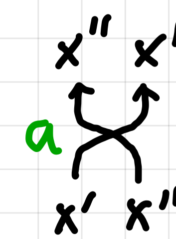
with

$$E_{i\,i+1}(x) = \frac{P_{i\,i+1}(-x)\,P'_{i\,i+1}(x)}{\theta(\eta)\,V(x)} = a$$ 

$$P_{i\,i+1}(x) = \check{R}_{i\,i+1}(x; a-(\sigma^z_1+\dots+\sigma^z_{i-1}))$$

YBE $\quad P_{i\,i+1}(x-y)\,P_{i+1\,i+2}(x)\,P_{i\,i+1}(y) = P_{i+1\,i+2}(y)\,P_{i\,i+1}(x)\,P_{i+1\,i+2}(x-y)$ (braid) & unitarity

elliptic dynamical R-matrix

$$\check{R}(x;a) = \begin{pmatrix} 1 & & & \\ & f(x,\eta;\eta a) & f(x,\eta;\eta a) & \\ & f(\eta,x;-\eta a) & f(\eta,x;-\eta a) & \\ & & & 1 \end{pmatrix} = a$$ 

$x = x' - x''$

$$f(x,\eta;a) = \frac{\theta(\eta+a)\,\theta(x)}{\theta(a)\,\theta(x+\eta)}$$

Felder 94

(face / IRF / SOS type)

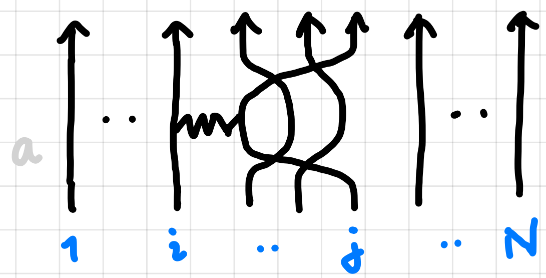
and e.g. $P_{23}(x)\,|s_1 s_2 s_3\rangle = |s_1\rangle \otimes \check{R}(x, a-s_1)\,|s_2 s_3\rangle \qquad s_i = \pm 1 = \uparrow,\downarrow$

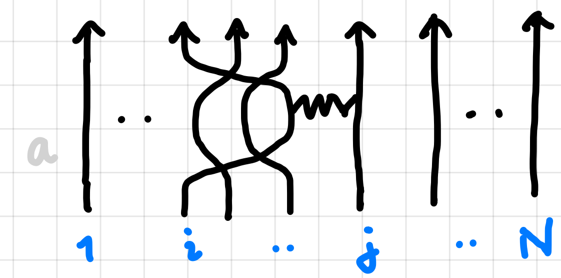
note: 'inhomogeneities' $x_j = j$

The q-deformed Inozemtsev chain: properties

Klabbers JL 23

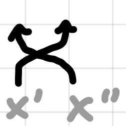
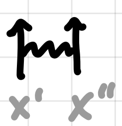
'chiral' hamiltonians

$$H^L = \sum_{i<j}^{N} V(i-j) \times$$ 

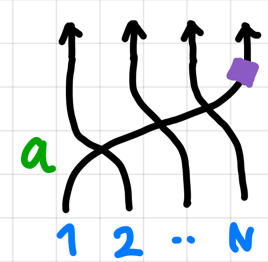
$$H^R = \sum_{i<j}^{N} V(i-j) \times$$ 

cf JL 18
JL Pasquier Serban 22

key limits

	Inozemtsev	q-def Haldane-Shastry
	$\eta \to 0$ & $a \to -i\infty$	$\kappa \to 0^+$ & $a \to -i\infty$
$V(x) = -\dfrac{\rho(x+\eta)-\rho(x-\eta)}{\theta(2\eta)}$	$\wp(x) + \text{cst}$	$\dfrac{1}{\sin[\frac{\pi}{N}(x+\eta)]\sin[\frac{\pi}{N}(x-\eta)]}$
 $= P_{i\,i+1}(x) = \check{R}_{i\,i+1}(x; a-(\sigma^z_1+\cdots+\sigma^z_{i-1}))$	$P_{i\,i+1}$	$\check{R}_{i\,i+1}(x)$ Jimbo $U_q\widehat{\mathfrak{sl}}_2$
 $= E_{i\,i+1}(x) = \dfrac{P_{i\,i+1}(-x)P'_{i\,i+1}(x)}{\theta(\eta)\,V(x)}$	$1-P_{i\,i+1}$	$e_{i\,i+1}$ Temperley-Lieb

$q = e^{i\eta/N}$

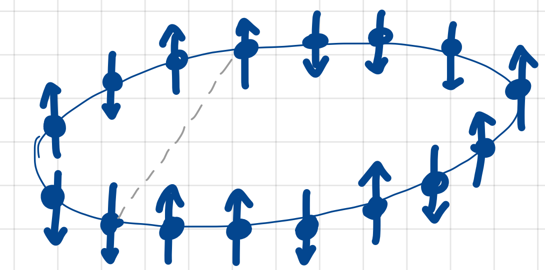
integrability belong to hierarchy of commuting ham's
that also includes
twisted translation 

$= K_N^{-1} P_{N-1,1}(1-N)\cdots P_{12}(1-2)$
(diagonal) twist

new limits short range: dynamical twisted Heisenberg XXZ
related to affine Temperley-Lieb

intermediate isotropic: a-dependent generalisation of Inozemtsev

Summary: integrable long-range spin chains



interaction range →

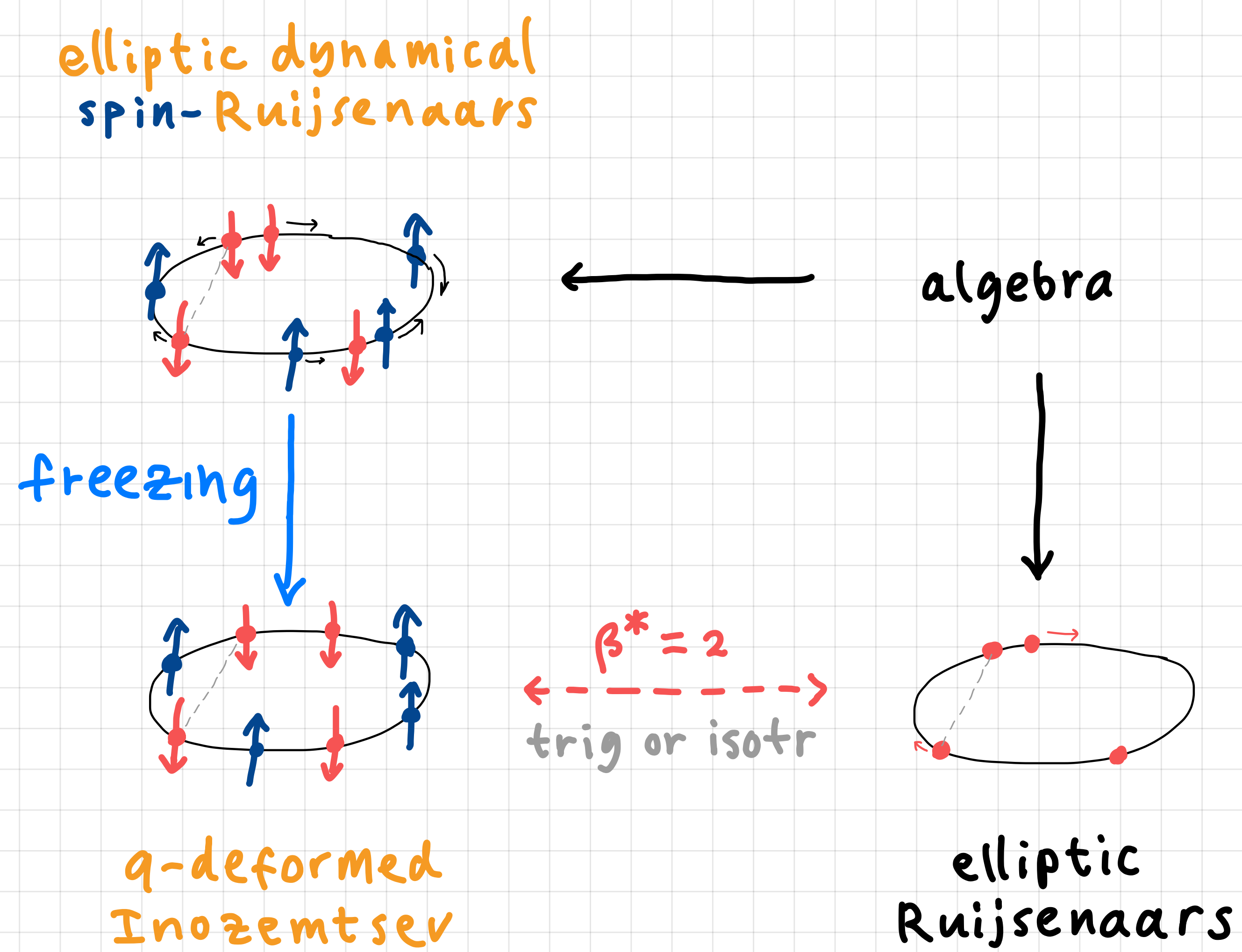
degree of spin symmetry ↓

	nearest neighbour		intermediate range	long range
anisotropic	Heisenberg XYZ Sutherland 70 Baxter 73		?	?
partially (an)isotr S^z symm	Heis XXZ Orbach 58 Yang Yang 66	dynamical twisted XXZ	q-deformed Inozemtsev Klabbers JL 2306.13066	q-deformed HS Uglov 95 JL 18 JL Pasquier Serban 22
			a-dept Inozemtsev	
isotropic SU(2) symm	Heisenberg XXX Heisenberg 28 Bethe 31		Inozemtsev Inozemtsev 90 Inozemtsev 90 95 00 Klabbers JL 22	Haldane-Shastry Haldane 88 Shastry 88 Haldane 91 Bernard et al 93

Behind the scenes: quantum many-body systems

Polychronakos 92
Bernard et al 93
Talstra Haldane 94
Uglov 95
Inozemtsev 95
Klabbers JL 22
JL Pasquier Serban 22
Matushko Zotov 23
Klabbers JL 23

Key to long-range integrable spin chains:
connections to QMBS

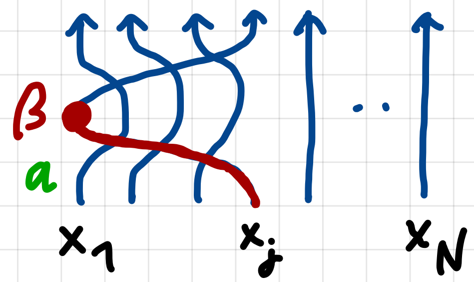


Behind the scenes: dynamical spin-Ruijsenaars and freezing

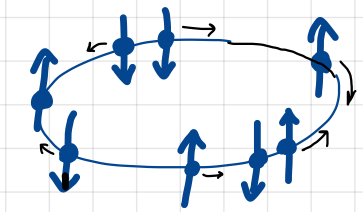
hierarchy of commuting difference operators, including

Klabbers JL 23

$$\widetilde{D}_1 = \sum_{j=1}^{N} A_j \times$$



$$= A_1 \Gamma_1 + A_2\, P_{12}(x_2-x_1)\, \Gamma_2\, P_{12}(x_1-x_2) + A_3\, P_{23}(x_3-x_2)\, P_{12}(x_3-x_1)\, \Gamma_3\, P_{12}(x_1-x_3)\, P_{23}(x_2-x_3) + \cdots$$

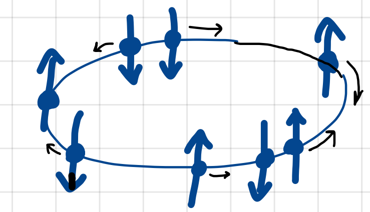


$$A_j = \prod_{k(\neq j)}^{N} \frac{\theta(x_k - x_j + \eta)}{\theta(x_k - x_j)} \qquad \Gamma_j : x_i \mapsto x_i - \delta_{ij}\, \beta$$

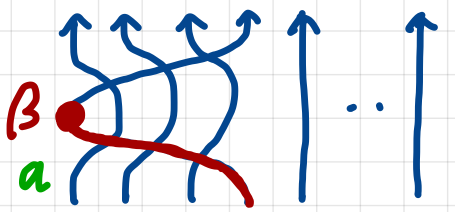
Behind the scenes: dynamical spin-Ruijsenaars and freezing

hierarchy of commuting difference operators, including

Klabbers JL 23



$$\tilde{D}_1 = \sum_{j=1}^{N} A_j \times$$ β, a



$x_1 \quad x_j \quad x_N$

$$= A_1 \Gamma_1 + A_2\, P_{12}(x_2-x_1)\, \Gamma_2\, P_{12}(x_1-x_2) + A_3\, P_{23}(x_3-x_2) P_{12}(x_3-x_1)\, \Gamma_3\, P_{12}(x_1-x_3) P_{23}(x_2-x_3) + \cdots$$

$$A_j = \prod_{k(\neq j)}^{N} \frac{\theta(x_k - x_j + \eta)}{\theta(x_k - x_j)} \qquad \Gamma_j : x_i \mapsto x_i - \delta_{ij}\,\beta$$

semiclass limit $\beta \to 0$

Polychronakos 92
Talstra Haldane 94
Uglov 95
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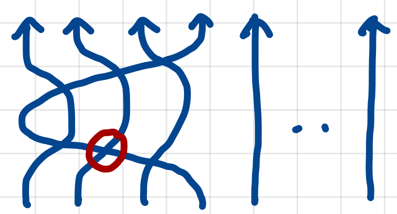
Matushko Zotov 23
Klabbers JL 23

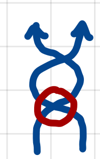
$$\Gamma_j = 1 + \beta\, i\, \partial_{x_j} + O(\beta)^2$$

$$\tilde{D}_1 = 1 + \beta\, i \sum_{j=1}^{N} A_j \Big(\underbrace{\partial_{x_j}}_{\text{remove consistently}} + \underbrace{\sum_{i(<j)} a \ \ldots}_{\sim \text{spin chain}} \Big) + O(\beta)^2$$

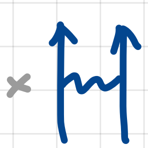


$x_i \quad x_j \quad x_N$

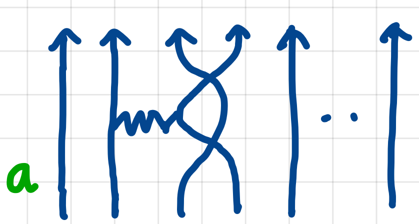
class equilib $x_k \mapsto k$



$= \theta(\eta)\, V \times$

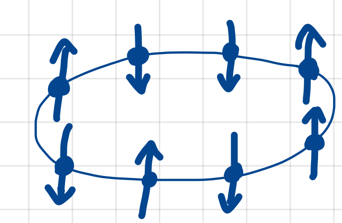


yields $$H^L = \sum_{i<j}^{N} V(i-j) \times a$$



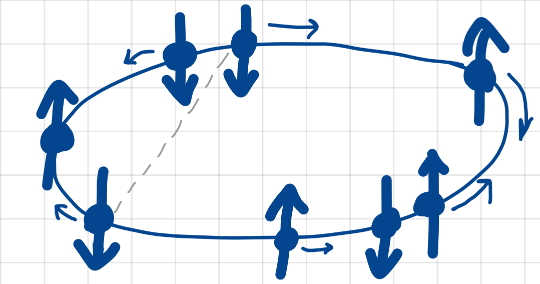
$i \quad j \quad N$

$$E_{i\,i+1}(x) = \frac{P_{i\,i+1}(-x)\, P'_{i\,i+1}(x)}{\theta(\eta)\, V(x)}$$



Klabbers JL

Outlook: integrable quantum many-body systems



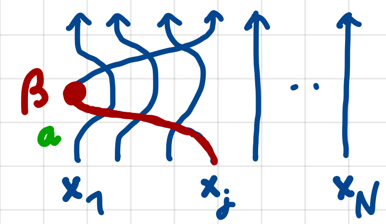
interaction range →

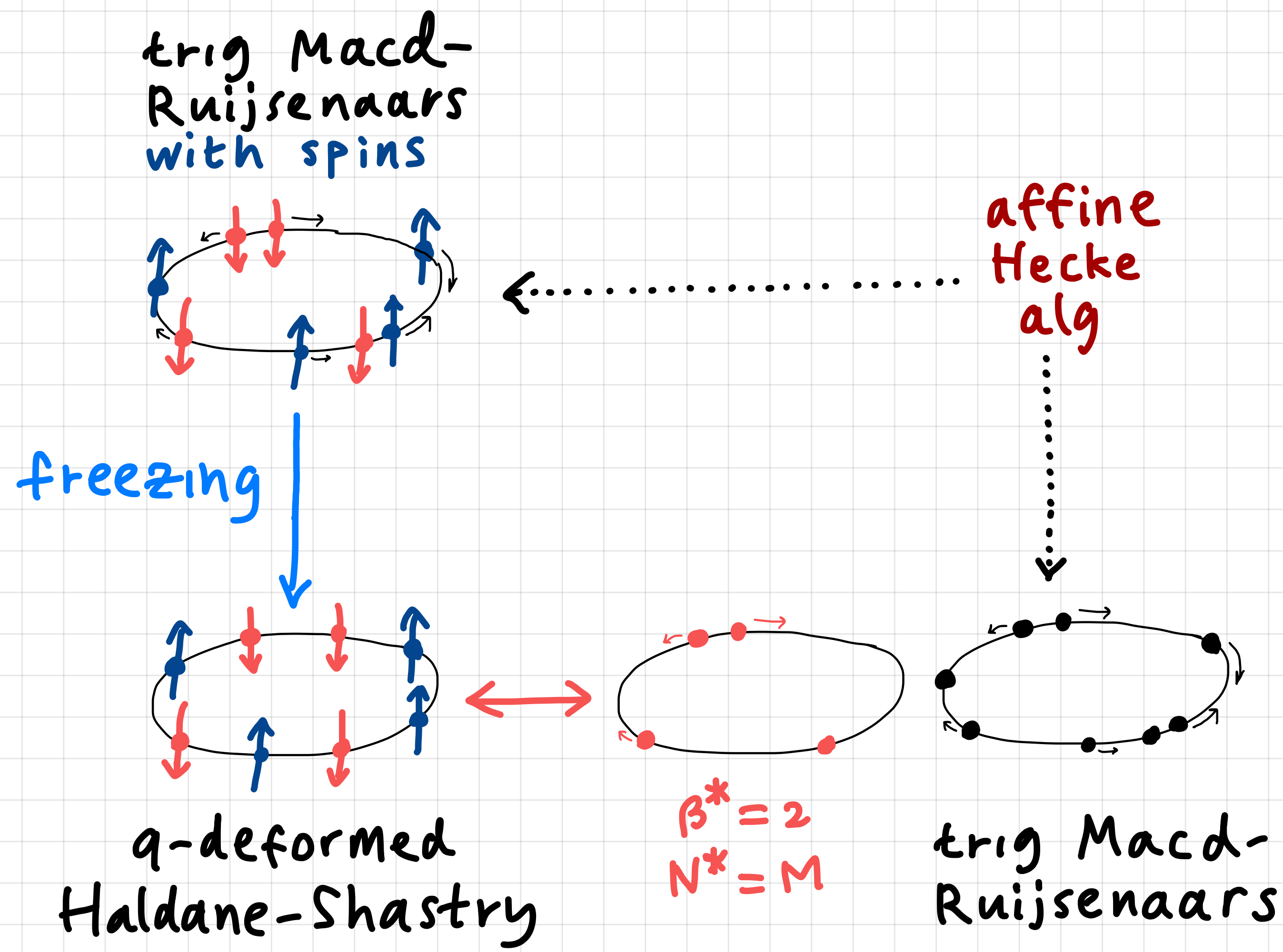
	nearest neighbour contact (positions)	**intermediate range** elliptic (positions)	**long range** trig (positions)
(?) elliptic (momenta)	?	'DELL'	?
relativistic trig (momenta) difference op's	?	**ell Ruijsenaars** dynamical spin Klabbers JL 2306.13066	**trig Ruijsenaars-Macdonald**
		a-dep ell Cal-Sut	
non-rlt rational (momenta) differential op's	~Lieb-Liniger?	**ell Cal-Sut**	**trig Cal-Sut**

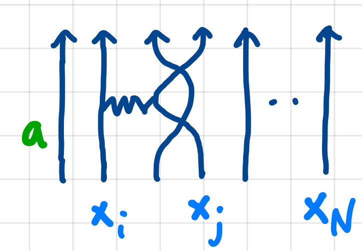
towards **general theory** for
long-range quantum integrability

Bonus: long-range integrability (trig case)

Polychronakos 92
Bernard et al 93
Talstra Haldane 94
Uglov 95
JL Pasquier Serban 22
JL Serban 22

$\tilde{D}_1 = \sum_{j=1}^{N} A_j \times$ 

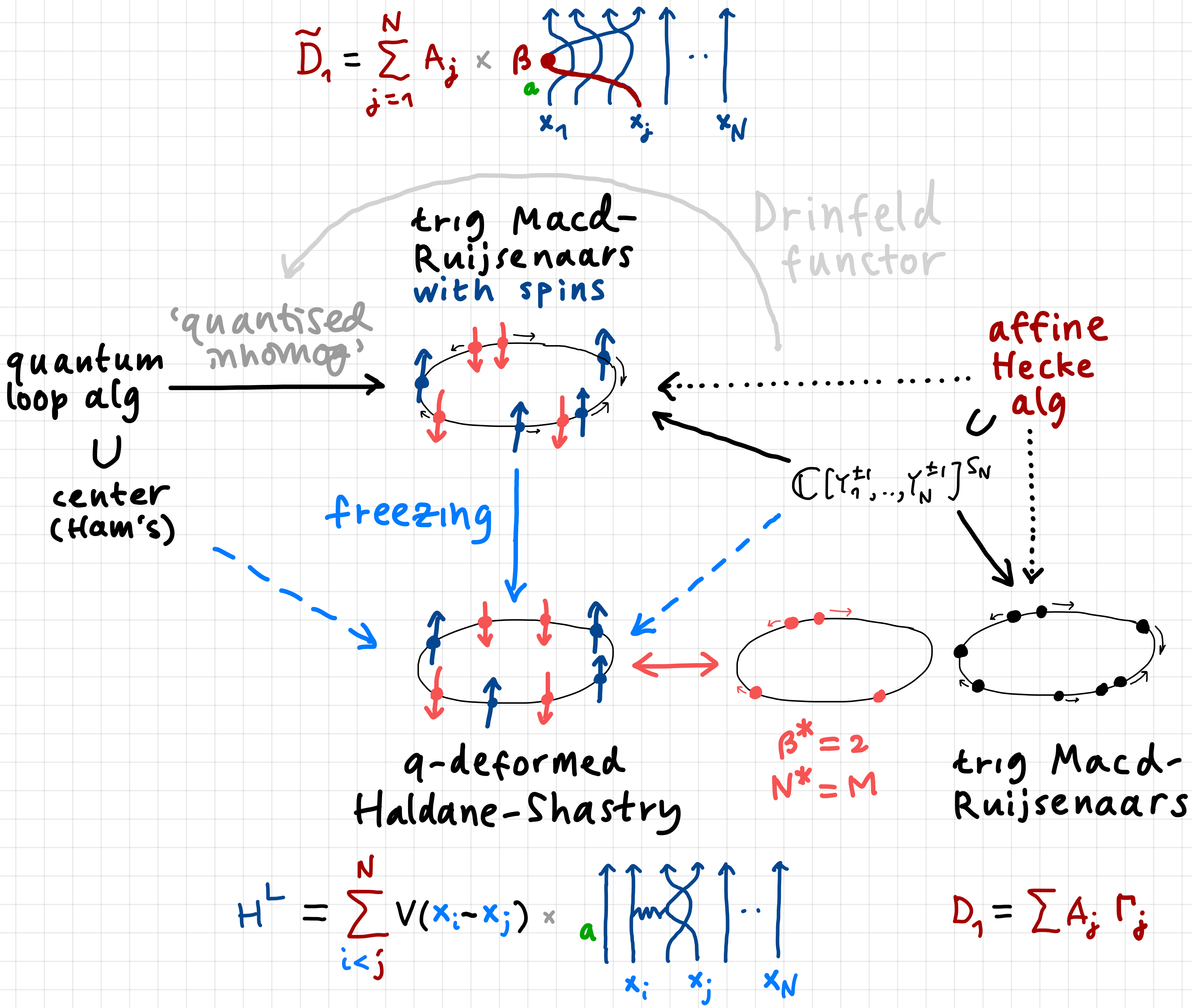


$H^L = \sum_{i<j}^{N} V(x_i - x_j) \times$ 

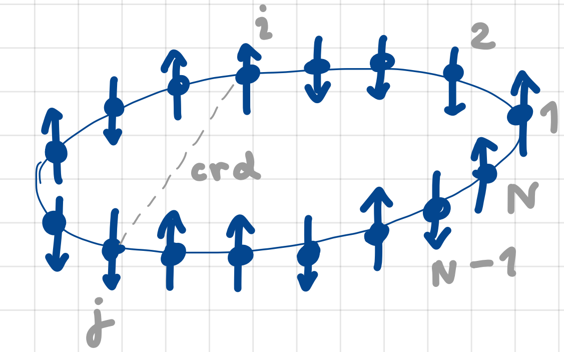
$D_1 = \sum A_j \, \Gamma_j$

Bonus: long-range integrability (trig case)

Polychronakos 92
Bernard et al 93
Talstra Haldane 94
Uglov 95
JL Pasquier Serban 22
JL Serban 22



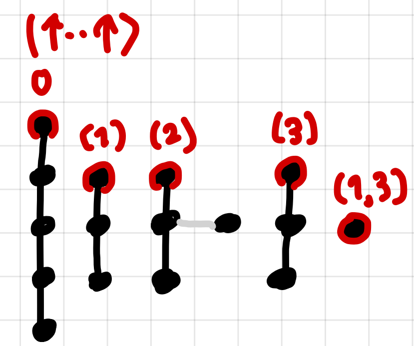
Bonus: more about Haldane–Shastry



Haldane 88 Shastry 88

on $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$ $\qquad H_{HS} = \sum_{i<j}^{N} \frac{1-P_{ij}}{4\sin^2\left(\frac{\pi}{N}(i-j)\right)}$

spectrum labelled by **motifs** $\mu = (\mu_1, \dots, \mu_M)$ s.t. $\mu_{m+1} > \mu_m + 1$



momentum $p(\mu) = \frac{2\pi}{N} \sum \mu_m$ $\qquad p_m = \frac{2\pi}{N}\mu_m$ quasimomenta

energy $E(\mu) = \frac{1}{2} \sum \mu_m (N-\mu_m)$ $\qquad \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ strictly additive

p:	0	$\frac{\pi}{2}$	π	$-\frac{\pi}{2}$	0
E:	0	$\frac{3}{2}$	2	$\frac{3}{2}$	3

Jack polyn

eigenvec $\qquad \sum_{i_1<\dots<i_M}^{N} \prod_{m<n}^{M} (z_{i_m} - z_{i_n})^2 \times P^{(1/2)}_{\nu(\mu)}(z_{i_1}, \dots, z_{i_M})\, \sigma^-_{i_1} \cdots \sigma^-_{i_M} |\uparrow \cdots \uparrow\rangle$ $\qquad z_j = e^{2\pi i j/N}$

partition s.t. $\ell(\nu) = M$, $\nu_1 \leq N - 2M + 1$

Haldane 91

Yangian highest weight
with know Drinfeld polyn $\quad P^{\mu}(u) = \prod_{\substack{n=1 \\ n \neq \mu, \mu+1}}^{N} \left(u - \frac{N-2n+1}{2}\right)$

Bernard et al 93

$Y\mathfrak{sl}_2$ acts by $\qquad S^{\pm} = \sum \sigma^{\pm}_i \qquad S^z = \frac{1}{2} \sum \sigma^z_i$

$$Q^{\pm} = \pm \frac{1}{2} \sum_{i<j}^{N} \cot \frac{\pi}{N}(i-j) \cdot (\sigma^z_i \sigma^{\pm}_j - \sigma^{\pm}_i \sigma^z_j)$$

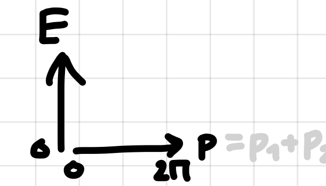
$$Q^z = \frac{1}{2} \sum_{i<j}^{N} \cot \frac{\pi}{N}(i-j) \cdot (\sigma^+_i \sigma^-_j - \sigma^-_i \sigma^+_j)$$

Ha et al 92
Bernard et al 93

...and
all of this nicely
q-deforms

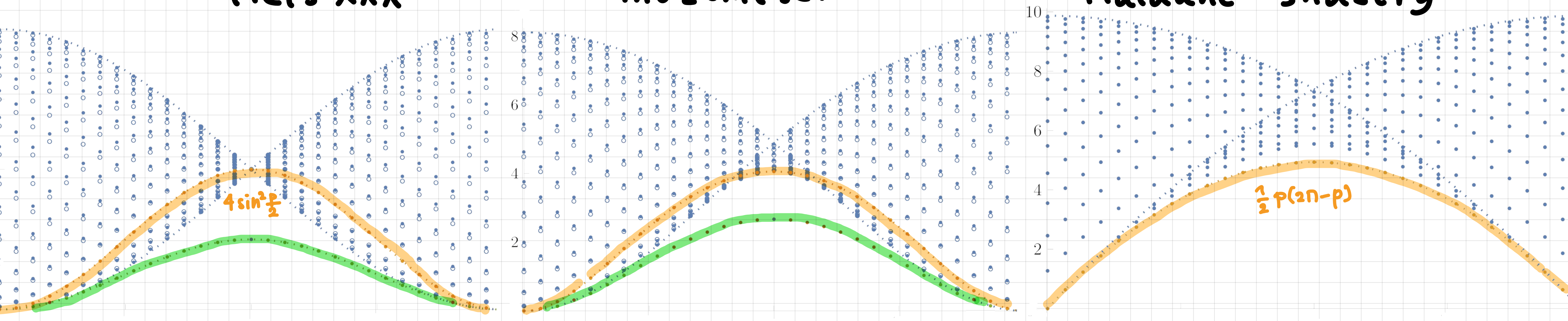
Bernard et al 93
Uglov 95
JL Pasquier Serban 22

Bonus: spectrum of Inozemtsev

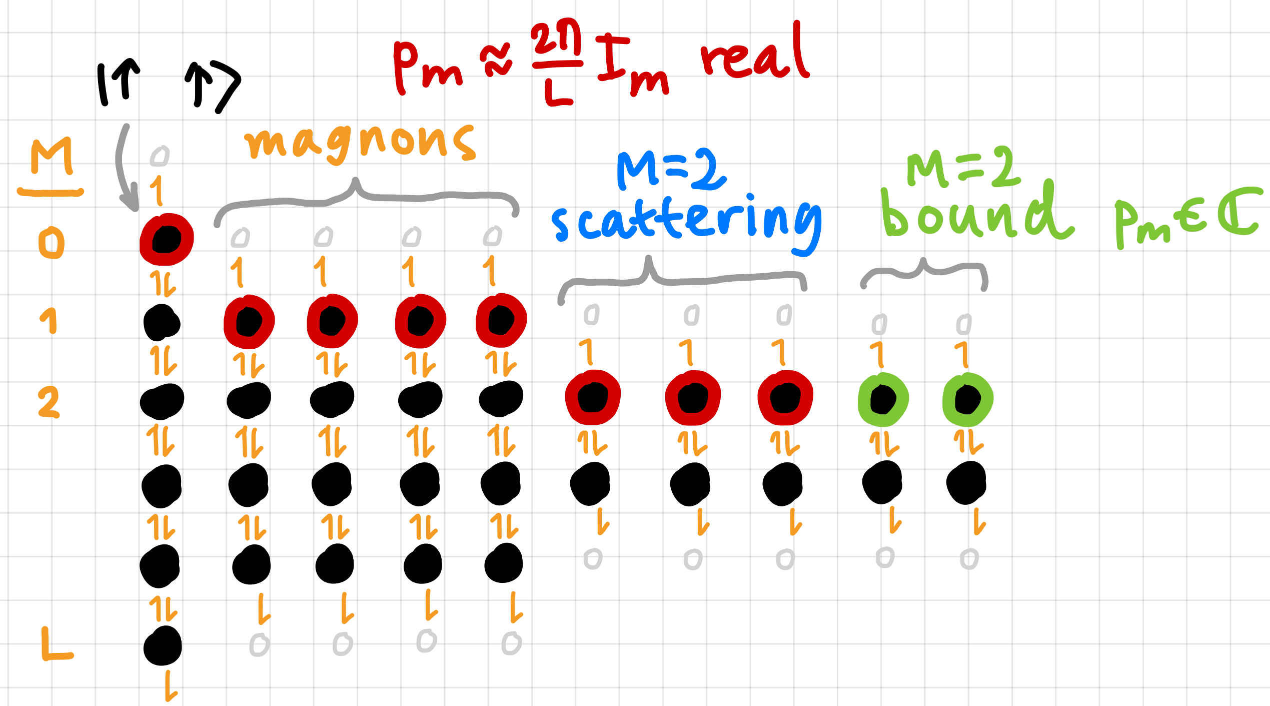


Bethe 31
Haldane 91–92
Inozemtsev 90
Klabbers JL 22

Heis XXX | Inozemtsev | Haldane–Shastry



usual isotropic spin chains



$2^{\otimes 5} = 6 \oplus 4^{\oplus 4} \oplus 2^{\oplus 5}$

hidden Yangian symmetry