New invariants in birational geometry

with Chambert-Loir, Cheltsov, Hassett, Kontsevich, Kresch, K. Yang, Zh. Zhang

Function fields

Let X be a smooth projective algebraic variety over k, a field of characteristic zero.

Function fields

Let X be a smooth projective algebraic variety over k, a field of characteristic zero.

Basic questions:

- Classify $K = k(X)$ up to isomorphism.

Function fields

Let X be a smooth projective algebraic variety over k, a field of characteristic zero.

Basic questions:

- Classify $K = k(X)$ up to isomorphism. In particular, when is X rational, i.e.,

$$K \simeq k(\mathbb{P}^n) = k(x_1, \ldots, x_n).$$

Function fields

Let X be a smooth projective algebraic variety over k, a field of characteristic zero.

Basic questions:

- Classify $K = k(X)$ up to isomorphism. In particular, when is X rational, i.e.,

$$K \simeq k(\mathbb{P}^n) = k(x_1, \ldots, x_n).$$

- Classify finite groups

$$G \subset \mathrm{Aut}(K/k) = \mathrm{BirAut}(X),$$

up to conjugation.

Function fields

Let X be a smooth projective algebraic variety over k, a field of characteristic zero.

Basic questions:

- Classify $K = k(X)$ up to isomorphism. In particular, when is X rational, i.e.,

$$K \simeq k(\mathbb{P}^n) = k(x_1, \ldots, x_n).$$

- Classify finite groups

$$G \subset \mathrm{Aut}(K/k) = \mathrm{BirAut}(X),$$

up to conjugation. In particular, when does a finite group

$$G \subset \mathrm{Cr}_n := \mathrm{BirAut}(\mathbb{P}^n)$$

arise from a linear action on $\mathbb{P}^n$?

Cremona group

Recall that

$$\mathrm{Cr}_2 = \mathrm{BirAut}(\mathbb{P}^2)$$

is generated by elements in $\mathrm{PGL}_3 = \mathrm{Aut}(\mathbb{P}^2)$

Recall that

$$\mathrm{Cr}_2 = \mathrm{BirAut}(\mathbb{P}^2)$$

is generated by elements in $\mathrm{PGL}_3 = \mathrm{Aut}(\mathbb{P}^2)$ and the Cremona involution

$$(x : y : z) \mapsto (\frac{1}{x} : \frac{1}{y} : \frac{1}{z}).$$

Cremona group

Recall that

$$\mathrm{Cr}_2 = \mathrm{BirAut}(\mathbb{P}^2)$$

is generated by elements in $\mathrm{PGL}_3 = \mathrm{Aut}(\mathbb{P}^2)$ and the Cremona involution

$$(x : y : z) \mapsto (\frac{1}{x} : \frac{1}{y} : \frac{1}{z}).$$

Conjugacy classes of finite $G \subset \mathrm{PGL}_3$ are known.

Recall that

$$\mathrm{Cr}_2 = \mathrm{BirAut}(\mathbb{P}^2)$$

is generated by elements in $\mathrm{PGL}_3 = \mathrm{Aut}(\mathbb{P}^2)$ and the Cremona involution

$$(x : y : z) \mapsto (\frac{1}{x} : \frac{1}{y} : \frac{1}{z}).$$

Conjugacy classes of finite $G \subset \mathrm{PGL}_3$ are known.

Open problem: Classify $G \subset \mathrm{PGL}_3$ up to conjugation in Cr_2.

Recall that

$$\mathrm{Cr}_2 = \mathrm{BirAut}(\mathbb{P}^2)$$

is generated by elements in $\mathrm{PGL}_3 = \mathrm{Aut}(\mathbb{P}^2)$ and the Cremona involution

$$(x : y : z) \mapsto (\frac{1}{x} : \frac{1}{y} : \frac{1}{z}).$$

Conjugacy classes of finite $G \subset \mathrm{PGL}_3$ are known.

Open problem: Classify $G \subset \mathrm{PGL}_3$ up to conjugation in Cr_2.

The classification of abelian $G \subset \mathrm{Cr}_2$ has been completed in 2006 (Blanc).

Recall that

$$\mathrm{Cr}_2 = \mathrm{BirAut}(\mathbb{P}^2)$$

is generated by elements in $\mathrm{PGL}_3 = \mathrm{Aut}(\mathbb{P}^2)$ and the Cremona involution

$$(x : y : z) \mapsto (\frac{1}{x} : \frac{1}{y} : \frac{1}{z}).$$

Conjugacy classes of finite $G \subset \mathrm{PGL}_3$ are known.

Open problem: Classify $G \subset \mathrm{PGL}_3$ up to conjugation in Cr_2.

The classification of abelian $G \subset \mathrm{Cr}_2$ has been completed in 2006 (Blanc). Even the classification of involutions in Cr_3 is still open.

Flavors of birationality

Birationality:

- varieties
- varieties with additional structures, e.g.,
  - G-varieties
  - varieties with logarithmic volume forms
  - varieties with Azumaya algebras ...

Flavors of rationality

- $X \sim \mathbb{P}^n$ – rationality

Flavors of rationality

- $X \sim \mathbb{P}^n$ – rationality
- $X \times \mathbb{P}^n \sim \mathbb{P}^m$ – stable rationality

Flavors of rationality

- $X \sim \mathbb{P}^n$ – rationality
- $X \times \mathbb{P}^n \sim \mathbb{P}^m$ – stable rationality
- $X \sim_G \mathbb{P}^n$ – G finite group, X a G-variety, action $\mathbb{P}^n$ is linear

Flavors of rationality

- $X \sim \mathbb{P}^n$ – rationality
- $X \times \mathbb{P}^n \sim \mathbb{P}^m$ – stable rationality
- $X \sim_G \mathbb{P}^n$ – G finite group, X a G-variety, action $\mathbb{P}^n$ is linear
- $X \times \mathbb{P}^m \sim_G \mathbb{P}^n$ – the action $\mathbb{P}^n$ is linear and on $\mathbb{P}^m$ is trivial

Flavors of rationality

- $X \sim \mathbb{P}^n$ – rationality
- $X \times \mathbb{P}^n \sim \mathbb{P}^m$ – stable rationality
- $X \sim_G \mathbb{P}^n$ – G finite group, X a G-variety, action $\mathbb{P}^n$ is linear
- $X \times \mathbb{P}^m \sim_G \mathbb{P}^n$ – the action $\mathbb{P}^n$ is linear and on $\mathbb{P}^m$ is trivial
- $(X, \omega_X) \sim (\mathbb{P}^n, \omega_n)$, where

$$\omega_n = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$$

is the standard volume form with logarithmic poles on $\mathbb{P}^n$.

Burnside groups

- $\mathrm{Burn}_n(k)$ – free abelian group on isomorphism classes of finitely generated extensions of k of transcendence degree n, i.e., birational equivalence classes of n-dimensional algebraic varieties over k

Burnside groups

- $\mathrm{Burn}_n(k)$ – free abelian group on isomorphism classes of finitely generated extensions of k of transcendence degree n, i.e., birational equivalence classes of n-dimensional algebraic varieties over k
- $\mathrm{Burn}_n(G, k)$ – free abelian group on birational equivalence classes of n-dimensional algebraic G-varieties over k

Burnside groups

- $\mathrm{Burn}_n(k)$ – free abelian group on isomorphism classes of finitely generated extensions of k of transcendence degree n, i.e., birational equivalence classes of n-dimensional algebraic varieties over k
- $\mathrm{Burn}_n(G, k)$ – free abelian group on birational equivalence classes of n-dimensional algebraic G-varieties over k
- $\mathbf{Burn}_n(k)$ – free abelian group on birational equivalence classes of n-dimensional algebraic varieties over k equipped with a volume form with logarithmic poles

Burnside groups

- $\mathrm{Burn}_n(k)$ – free abelian group on isomorphism classes of finitely generated extensions of k of transcendence degree n, i.e., birational equivalence classes of n-dimensional algebraic varieties over k
- $\mathrm{Burn}_n(G, k)$ – free abelian group on birational equivalence classes of n-dimensional algebraic G-varieties over k
- $\mathbf{Burn}_n(k)$ – free abelian group on birational equivalence classes of n-dimensional algebraic varieties over k equipped with a volume form with logarithmic poles
- ...

Why are these interesting?

Why are these interesting?

There is a ring structure on (all variants of)

$$\mathrm{Burn}(k) := \oplus_{n\geq 0} \mathrm{Burn}_n(k).$$

Why are these interesting?

There is a ring structure on (all variants of)

$$\mathrm{Burn}(k) := \oplus_{n\geq 0} \mathrm{Burn}_n(k).$$

These rings have an intricate internal structure, reflecting, e.g., nontrivial stable birationalities.

$\mathbf{Burn}_n(k)$ (with Chambert-Loir and Kontsevich, 2023)

Let

$$[X, \omega_X] \in \mathbf{Burn}(k)$$

be the class of the pair (X, ω_X) in this ring.

$\mathbf{Burn}_n(k)$ (with Chambert-Loir and Kontsevich, 2023)

Let

$$[X, \omega_X] \in \mathbf{Burn}(k)$$

be the class of the pair (X, ω_X) in this ring.

Example

In dimension 0, we have

$$\epsilon = [\mathrm{Spec}(k), -1] \in \mathbf{Burn}_0(k).$$

$\mathbf{Burn}_n(k)$ (with Chambert-Loir and Kontsevich, 2023)

Let

$$[X, \omega_X] \in \mathbf{Burn}(k)$$

be the class of the pair (X, ω_X) in this ring.

Example

In dimension 0, we have

$$\epsilon = [\mathrm{Spec}(k), -1] \in \mathbf{Burn}_0(k).$$

In dimension 1,

$$\mathbf{T} := [\mathbb{P}^1, \mathrm{d}t/t] \in \mathbf{Burn}_1(k).$$

Let X be a model of a function field $K = k(X)$ such that the polar divisor of ω_X is

$$D = \cup_{\alpha \in \mathcal{A}} D_\alpha,$$

a divisor with normal crossings.

Let X be a model of a function field $K = k(X)$ such that the polar divisor of ω_X is

$$D = \cup_{\alpha \in \mathcal{A}} D_\alpha,$$

a divisor with normal crossings. For each $A \subseteq \mathcal{A}$, let $D_A := \cap_{\alpha \in A} D_\alpha$ and ω_A be the iterated residue of ω_X along D_A.

$\mathbf{Burn}_n(k)$

There is a (well-defined) derivation:

$$\partial : \mathbf{Burn}_n(k) \to \mathbf{Burn}_{n-1}(k),$$

given by

$$\partial([X, \omega]) = \sum_{\emptyset \neq A \subset \mathscr{A}} (-1)^{\mathrm{Card}(A)-1} [D_A, \omega_A] \cdot \mathbf{T}^{\mathrm{Card}(A)-1},$$

$\mathbf{Burn}_n(k)$

There is a (well-defined) derivation:

$$\partial : \mathbf{Burn}_n(k) \to \mathbf{Burn}_{n-1}(k),$$

given by

$$\partial([X, \omega]) = \sum_{\emptyset \neq A \subset \mathscr{A}} (-1)^{\mathrm{Card}(A)-1} [D_A, \omega_A] \cdot \mathbf{T}^{\mathrm{Card}(A)-1},$$

which satisfies

$$\partial \circ \partial = 0.$$

There is a (well-defined) derivation:

$$\partial : \mathbf{Burn}_n(k) \to \mathbf{Burn}_{n-1}(k),$$

given by

$$\partial([X, \omega]) = \sum_{\emptyset \neq A \subset \mathscr{A}} (-1)^{\mathrm{Card}(A)-1} [D_A, \omega_A] \cdot \mathbf{T}^{\mathrm{Card}(A)-1},$$

which satisfies

$$\partial \circ \partial = 0.$$

This was inspired by polar homology of Khesin-Rosly (2003), except that

- we record contributions from strata of all codimensions, rather than only from those of codimension one,

$\mathbf{Burn}_n(k)$

There is a (well-defined) derivation:

$$\partial : \mathbf{Burn}_n(k) \to \mathbf{Burn}_{n-1}(k),$$

given by

$$\partial([X,\omega]) = \sum_{\emptyset \neq A \subset \mathscr{A}} (-1)^{\mathrm{Card}(A)-1} [D_A, \omega_A] \cdot \mathbf{T}^{\mathrm{Card}(A)-1},$$

which satisfies

$$\partial \circ \partial = 0.$$

This was inspired by polar homology of Khesin-Rosly (2003), except that

- we record contributions from strata of all codimensions, rather than only from those of codimension one,
- we record birational types of strata, rather than the strata themselves.

$\mathbf{Burn}_n(k)$

Moreover,

$$\partial(a \cdot b) = \epsilon^n \cdot \partial(a) \cdot b + a \cdot \partial(b) - \mathbf{T} \cdot \partial(a) \cdot \partial(b),$$

when $a \in \mathrm{Burn}_m(k)$ and $b \in \mathrm{Burn}_n(k)$.

Applications: invariants of birational maps

Now consider $\mathrm{Burn}(k)$ – free abelian group on birational equivalence classes of algebraic varieties over k.

Applications: invariants of birational maps

Now consider $\mathrm{Burn}(k)$ – free abelian group on birational equivalence classes of algebraic varieties over k.

Let

$$\phi : X \dashrightarrow Y$$

be a birational map between smooth projective varieties over k.

Now consider $\mathrm{Burn}(k)$ – free abelian group on birational equivalence classes of algebraic varieties over k.

Let

$$\phi : X \dashrightarrow Y$$

be a birational map between smooth projective varieties over k. Let $\mathrm{Ex}(\phi)$ and $\mathrm{Ex}(\phi^{-1})$ be the sets of divisorial components of the exceptional locus of ϕ, respectively ϕ^{-1}.

Applications: invariants of birational maps

Now consider $\mathrm{Burn}(k)$ – free abelian group on birational equivalence classes of algebraic varieties over k.

Let

$$\phi : X \dashrightarrow Y$$

be a birational map between smooth projective varieties over k. Let $\mathrm{Ex}(\phi)$ and $\mathrm{Ex}(\phi^{-1})$ be the sets of divisorial components of the exceptional locus of ϕ, respectively ϕ^{-1}. Put

$$c(\phi) := \sum_{E \in \mathrm{Ex}(\phi^{-1})} [k(E)] - \sum_{D \in \mathrm{Ex}(\phi)} [k(D)].$$

Applications: invariants of birational maps

Theorem (Lin-Shinder 2022)

This assignment respects compositions of birational maps of n-dimensional varieties over k,

$$\mathrm{c}(\phi \circ \psi) := \mathrm{c}(\phi) + \mathrm{c}(\psi) \in \mathrm{Burn}_{n-1}(k).$$

Theorem (Lin-Shinder 2022)

This assignment respects compositions of birational maps of n-dimensional varieties over k,

$$\mathrm{c}(\phi \circ \psi) := \mathrm{c}(\phi) + \mathrm{c}(\psi) \in \mathrm{Burn}_{n-1}(k).$$

This yields a homomorphism:

$$c : \mathrm{BirAut}(X) \to \mathrm{Burn}_{n-1}(k).$$

Theorem (Lin-Shinder 2022)

This assignment respects compositions of birational maps of n-dimensional varieties over k,

$$\mathrm{c}(\phi \circ \psi) := \mathrm{c}(\phi) + \mathrm{c}(\psi) \in \mathrm{Burn}_{n-1}(k).$$

This yields a homomorphism:

$$c : \mathrm{BirAut}(X) \to \mathrm{Burn}_{n-1}(k).$$

Corollary: Cr_n is not generated by regularizable maps, for $n \geq 4$, (disproving a conjecture from 2004). A map $\phi \in \mathrm{Cr}_n$ is regularizable if there exists a birational $\alpha : \mathbb{P}^n \dashrightarrow X$ such that $\alpha \circ \phi \circ \alpha^{-1} \in \mathrm{Aut}(X)$.

Applications: invariants of birational maps

Theorem (Lin-Shinder 2022)

This assignment respects compositions of birational maps of n-dimensional varieties over k,

$$c(\phi \circ \psi) := c(\phi) + c(\psi) \in \mathrm{Burn}_{n-1}(k).$$

This yields a homomorphism:

$$c : \mathrm{BirAut}(X) \to \mathrm{Burn}_{n-1}(k).$$

Corollary: Cr_n is not generated by regularizable maps, for $n \geq 4$, (disproving a conjecture from 2004). A map $\phi \in \mathrm{Cr}_n$ is regularizable if there exists a birational $\alpha : \mathbb{P}^n \dashrightarrow X$ such that $\alpha \circ \phi \circ \alpha^{-1} \in \mathrm{Aut}(X)$.

Proof: It suffices to present one nonregularizable map; done by Hassett-Lai (2018).

This formalism extends to the equivariant, orbifold, and logarithmic volume forms context (Kresch-T. 2022, Chambert-Loir–Kontsevich-T. 2023).

Applications: invariants of birational maps

This formalism extends to the equivariant, orbifold, and logarithmic volume forms context (Kresch-T. 2022, Chambert-Loir–Kontsevich-T. 2023).

This yields new structural information about the Cremona group

$$\mathrm{Cr}_n(k).$$

Applications: Failure of (stable) rationality via specialization

- Voisin (2013): integral decomposition of Δ (Bloch-Srinivas)
- Colliot-Thélène–Pirutka (2015): universal CH_0-triviality
- Nicaise–Shinder (2017): $\mathrm{K}_0(\mathit{Var}_k)/\mathbb{L}$, $\mathrm{char}(k) = 0$
- Kontsevich–T. (2017): $\mathrm{Burn}(k)$, $\mathrm{char}(k) = 0$

Applications: Failure of (stable) rationality via specialization

- Voisin (2013): integral decomposition of Δ (Bloch-Srinivas)
- Colliot-Thélène–Pirutka (2015): universal CH_0-triviality
- Nicaise–Shinder (2017): $\mathrm{K}_0(\mathit{Var}_k)/\mathbb{L}$, $\mathrm{char}(k) = 0$
- Kontsevich–T. (2017): $\mathrm{Burn}(k)$, $\mathrm{char}(k) = 0$

These developments led to a wealth of new results in birational geometry,

Applications: Failure of (stable) rationality via specialization

- Voisin (2013): integral decomposition of Δ (Bloch-Srinivas)
- Colliot-Thélène–Pirutka (2015): universal CH_0-triviality
- Nicaise–Shinder (2017): $\mathrm{K}_0(\mathit{Var}_k)/\mathbb{L}$, $\mathrm{char}(k) = 0$
- Kontsevich–T. (2017): $\mathrm{Burn}(k)$, $\mathrm{char}(k) = 0$

These developments led to a wealth of new results in birational geometry, for the following reasons:

- new, computable, obstructions to (stable) rationality arise in singular fibers,

Applications: Failure of (stable) rationality via specialization

- Voisin (2013): integral decomposition of Δ (Bloch-Srinivas)
- Colliot-Thélène–Pirutka (2015): universal CH_0-triviality
- Nicaise–Shinder (2017): $\mathrm{K}_0(\mathit{Var}_k)/\mathbb{L}$, $\mathrm{char}(k) = 0$
- Kontsevich–T. (2017): $\mathrm{Burn}(k)$, $\mathrm{char}(k) = 0$

These developments led to a wealth of new results in birational geometry, for the following reasons:

- new, computable, obstructions to (stable) rationality arise in singular fibers,
- one can use general position arguments to establish rationality.

Specialization

Let $\mathfrak{o}$ be a DVR, k its residue field and K the function field. Let X be a smooth projective variety over K of relative dimension n and $\mathcal{X}$ a proper model over $\mathfrak{o}$, with special fiber $\cup_{\alpha\in\mathcal{A}} D_\alpha$.

Specialization

Let $\mathfrak{o}$ be a DVR, k its residue field and K the function field. Let X be a smooth projective variety over K of relative dimension n and $\mathcal{X}$ a proper model over $\mathfrak{o}$, with special fiber $\cup_{\alpha \in \mathcal{A}} D_\alpha$. (For some of the constructions, one may ignore multiplicities.)

Specialization

Let $\mathfrak{o}$ be a DVR, k its residue field and K the function field. Let X be a smooth projective variety over K of relative dimension n and $\mathcal{X}$ a proper model over $\mathfrak{o}$, with special fiber $\cup_{\alpha \in \mathcal{A}} D_\alpha$. (For some of the constructions, one may ignore multiplicities.)

Put

$$\rho(\mathcal{X}_K) := \sum_{A \subseteq \mathcal{A}} (-1)^{\mathrm{Card}(A)-1} [k(D_A)] \mathbf{L}^{\mathrm{Card}(A)-1}.$$

Theorem (Kontsevich-T.)

This gives a well-defined homomorphism of abelian groups

$$\rho : \mathrm{Burn}_n(K) \to \mathrm{Burn}_n(k).$$

Specialization

Let $\mathfrak{o}$ be a DVR, k its residue field and K the function field. Let X be a smooth projective variety over K of relative dimension n and $\mathcal{X}$ a proper model over $\mathfrak{o}$, with special fiber $\cup_{\alpha \in \mathcal{A}} D_\alpha$. (For some of the constructions, one may ignore multiplicities.)

Put

$$\rho(\mathcal{X}_K) := \sum_{A \subseteq \mathcal{A}} (-1)^{\mathrm{Card}(A)-1} [k(D_A)] \mathbf{L}^{\mathrm{Card}(A)-1}.$$

Theorem (Kontsevich-T.)

This gives a well-defined homomorphism of abelian groups

$$\rho : \mathrm{Burn}_n(K) \to \mathrm{Burn}_n(k).$$

This is essentially the same formula as the one for

$$\partial : \mathbf{Burn}_n(k) \to \mathbf{Burn}_{n-1}(k).$$

Specialization

Theorem (Kontsevich-T.)

Rationality specializes in smooth families.

Specialization

Theorem (Kontsevich-T.)

Rationality specializes in smooth families.

Similar specialization results hold for

- equivariant birational types (Kresch-T. 2022)

Theorem (Kontsevich-T.)

Rationality specializes in smooth families.

Similar specialization results hold for

- equivariant birational types (Kresch-T. 2022)
- birational types of varieties with logarithmic volume forms (Chambert-Loir, Kontsevich, T. 2023)

Theorem (Kontsevich-T.)

Rationality specializes in smooth families.

Similar specialization results hold for

- equivariant birational types (Kresch-T. 2022)
- birational types of varieties with logarithmic volume forms (Chambert-Loir, Kontsevich, T. 2023)
- birational types of orbifolds (Kresch-T. 2023)

Let G be a finite group. We had introduced

$$\mathrm{Burn}_n(G, k)$$

as the free abelian group on birational equivalence classes of algebraic G-varieties over k.

Equivariant Burnside groups (Kresch-T. 2020)

Let G be a finite group. We had introduced

$$\mathrm{Burn}_n(G, k)$$

as the free abelian group on birational equivalence classes of algebraic G-varieties over k.

To distinguish such classes, we would like to have an analog of ∂, extracting invariants from information about subvarieties.

Problem: How to distinguish equivariant birational types of linear actions?

Basics

Problem: How to distinguish equivariant birational types of linear actions? How to distinguish linear actions from nonlinear actions?

Basics

Problem: How to distinguish equivariant birational types of linear actions? How to distinguish linear actions from nonlinear actions?

Basic facts:

- If X is rational and G is cyclic, then $X^G \neq \emptyset$.

Problem: How to distinguish equivariant birational types of linear actions? How to distinguish linear actions from nonlinear actions?

Basic facts:

- If X is rational and G is cyclic, then $X^G \neq \emptyset$.
- If $Y \dashrightarrow X$ is a G-birational map between smooth projective G-varieties, and G is abelian, then

$$Y^G \neq \emptyset \Leftrightarrow X^G \neq \emptyset.$$

Basic facts

More precisely, let X be smooth projective of dimension n, G abelian, and let $\mathfrak{p} \in X^G$. Let $\{a_1, \ldots, a_n\}$ be the characters (weights) of G in the tangent space to X at $\mathfrak{p}$.

Basic facts

More precisely, let X be smooth projective of dimension n, G abelian, and let $\mathfrak{p} \in X^G$. Let $\{a_1, \ldots, a_n\}$ be the characters (weights) of G in the tangent space to X at $\mathfrak{p}$. Let

$$\det(a_1, \ldots, a_n) = a_1 \wedge \ldots \wedge a_n \in \wedge^n (G^\vee)$$

be the determinant.

Basic facts

More precisely, let X be smooth projective of dimension n, G abelian, and let $\mathfrak{p} \in X^G$. Let $\{a_1, \ldots, a_n\}$ be the characters (weights) of G in the tangent space to X at $\mathfrak{p}$. Let

$$\det(a_1, \ldots, a_n) = a_1 \wedge \ldots \wedge a_n \in \wedge^n(G^\vee)$$

be the determinant.

Reichstein-Youssin (2002)

Let $Y \to X$ be a G-equivariant blowup. Then Y contains a point $\mathfrak{q} \in Y^G$ (in the preimage of $\mathfrak{p}$) with weights $\{b_1, \ldots, b_n\}$ in the tangent space, and such that

$$\det(b_1, \ldots, b_n) = \pm \det(a_1, \ldots, a_n),$$

i.e., this is a equivariant birational invariant.

Let V and W be n-dimensional faithful representations of an abelian group G of rank $r \leq n$, and

$$a_1, \ldots, a_n, \quad \text{respectively} \quad b_1, \ldots, b_n,$$

the characters of G appearing in V, respectively W. Then V and W are G-equivariantly birational if and only if

$$a_1 \wedge \cdots \wedge a_n = \pm\, b_1 \wedge \cdots \wedge b_n.$$

(This condition is meaningful only when $r = n$.)

Reichstein–Youssin (2002)

- Thus, cyclic linear actions on $\mathbb{P}^n$, with $n \geq 2$, of the same order, are equivariantly birational.

Reichstein–Youssin (2002)

- Thus, cyclic linear actions on $\mathbb{P}^n$, with $n \geq 2$, of the same order, are equivariantly birational.
- Note that any two faithful representations of G are equivariantly stably birational.

First examples: $\mathbb{P}^2$

Consider an action of $\mathbb{Z}/p\mathbb{Z}$ on $X = \mathbb{P}^2$ given by

$$(x : y : z) \mapsto (\zeta^a x : \zeta^b y : z),$$

$$\zeta = \zeta_p, \quad a, b \in \mathbb{Z}/p\mathbb{Z}, \quad \gcd(a, b, p) = 1, \quad a \neq b.$$

Fixed points are

$$(0 : 0 : 1), \quad (0 : 1 : 0), \quad (1 : 0 : 0).$$

First examples: $\mathbb{P}^2$

Consider an action of $\mathbb{Z}/p\mathbb{Z}$ on $X = \mathbb{P}^2$ given by

$$(x : y : z) \mapsto (\zeta^a x : \zeta^b y : z),$$

$$\zeta = \zeta_p, \quad a, b \in \mathbb{Z}/p\mathbb{Z}, \quad \gcd(a, b, p) = 1, \quad a \neq b.$$

Fixed points are

$$(0 : 0 : 1), \quad (0 : 1 : 0), \quad (1 : 0 : 0).$$

Record the weights in the tangent space at these points as a formal sum:

$$\beta(X) = [a, b] + [a - b, -b] + [b - a, -a].$$

First examples: $\mathbb{P}^2$

All such actions are equivalent. Declare $\beta(X) = 0$, i.e.,

$$[a, b] = -[b - a, -a] - [a - b, -b]$$

Allowing

$$[a, b] = -[a, -b]$$

we find

$$[a, b] = [a, b - a] + [a - b, b].$$

Birational types $\mathcal{B}_2(\mathbb{Z}/p\mathbb{Z})$

Generators: $[a, b]$, $a, b \in \mathbb{Z}/p\mathbb{Z}$, $\gcd(a, b, p) = 1$

Relations:

- $[a, b] = [b, a]$
- $[a, b] = [a, b - a] + [a - b, b]$ if $a \neq b$
- $[a, a] = [a, 0]$

Birational types $\mathcal{B}_2(\mathbb{Z}/p\mathbb{Z})$

Generators: $[a, b]$, $a, b \in \mathbb{Z}/p\mathbb{Z}$, $\gcd(a, b, p) = 1$

Relations:

- $[a, b] = [b, a]$
- $[a, b] = [a, b - a] + [a - b, b]$ if $a \neq b$
- $[a, a] = [a, 0]$

For $p \geq 5$, the $\mathbb{Q}$-rank of $\mathcal{B}_2(\mathbb{Z}/p\mathbb{Z})$ equals

$$\frac{p^2 - 1}{24} + 1$$

Birational types $\mathcal{B}_2(\mathbb{Z}/p\mathbb{Z})$

Generators: $[a, b]$, $a, b \in \mathbb{Z}/p\mathbb{Z}$, $\gcd(a, b, p) = 1$

Relations:

- $[a, b] = [b, a]$
- $[a, b] = [a, b - a] + [a - b, b]$ if $a \neq b$
- $[a, a] = [a, 0]$

For $p \geq 5$, the $\mathbb{Q}$-rank of $\mathcal{B}_2(\mathbb{Z}/p\mathbb{Z})$ equals

$$\frac{p^2 - 1}{24} + 1 = \dim(\mathrm{H}^1(X_1(p), \mathbb{Q}))$$

Birational types

Let G be a finite abelian group, and $A = G^\vee$ its group of characters.

Birational types

Let G be a finite abelian group, and $A = G^\vee$ its group of characters.

Let X be smooth projective, of dimension n, with regular G-action.
Consider $X^G = \sqcup F_\alpha$ and record eigenvalues of G

$$[a_{1,\alpha}, \ldots, a_{n,\alpha}]$$

in the tangent space $\mathcal{T}_{x_\alpha} X$, at some $x_\alpha \in F_\alpha$.

Birational types

Let G be a finite abelian group, and $A = G^\vee$ its group of characters.

Let X be smooth projective, of dimension n, with regular G-action.
Consider $X^G = \sqcup F_\alpha$ and record eigenvalues of G

$$[a_{1,\alpha}, \ldots, a_{n,\alpha}]$$

in the tangent space $\mathcal{T}_{x_\alpha} X$, at some $x_\alpha \in F_\alpha$. Put

$$\beta(X) := \sum_\alpha [a_{1,\alpha}, \ldots, a_{n,\alpha}]$$

Birational types

Let G be a finite abelian group, and $A = G^\vee$ its group of characters.

Let X be smooth projective, of dimension n, with regular G-action.
Consider $X^G = \sqcup F_\alpha$ and record eigenvalues of G

$$[a_{1,\alpha}, \dots, a_{n,\alpha}]$$

in the tangent space $\mathcal{T}_{x_\alpha}X$, at some $x_\alpha \in F_\alpha$. Put

$$\beta(X) := \sum_\alpha [a_{1,\alpha}, \dots, a_{n,\alpha}]$$

Here, we keep no information about F_α.

Birational types

Consider the free abelian group

$$\mathcal{S}_n(G)$$

spanned by unordered tupels

$$[a_1, \ldots, a_n], \quad a_i \in A,$$

Birational types

Consider the free abelian group

$$\mathcal{S}_n(G)$$

spanned by unordered tupels

$$[a_1, \ldots, a_n], \quad a_i \in A,$$

subject to condition:

(G) $\sum_i \mathbb{Z} a_i = A$,

Birational types

Consider the free abelian group

$$\mathcal{S}_n(G)$$

spanned by unordered tupels

$$[a_1, \ldots, a_n], \quad a_i \in A,$$

subject to condition:

(G) $\sum_i \mathbb{Z} a_i = A$,

We get a map

$$\begin{array}{rcl} \{ \ G\text{-varieties} \ \} & \to & \mathcal{S}_n(G) \\ X & \mapsto & \beta(X) \end{array}$$

Birational types $\mathcal{B}_n(G)$

Let $Y \to X$ be a G-equivariant blowup and impose relations:

$$\beta(Y) - \beta(X) = 0.$$

Birational types $\mathcal{B}_n(G)$

Let $Y \to X$ be a G-equivariant blowup and impose relations:

$$\beta(Y) - \beta(X) = 0.$$

All such relations can be encoded in a compact form:

Birational types $\mathcal{B}_n(G)$

Let $Y \to X$ be a G-equivariant blowup and impose relations:

$$\beta(Y) - \beta(X) = 0.$$

All such relations can be encoded in a compact form: Consider the quotient

$$\mathcal{S}_n(G) \to \mathcal{B}_n(G),$$

by relations

Birational types $\mathcal{B}_n(G)$

Let $Y \to X$ be a G-equivariant blowup and impose relations:

$$\beta(Y) - \beta(X) = 0.$$

All such relations can be encoded in a compact form: Consider the quotient

$$\mathcal{S}_n(G) \to \mathcal{B}_n(G),$$

by relations

(B) for all $a_1, a_2, b_3, \ldots, b_n \in A$ we have

$$[a_1, a_2, b_3, \ldots b_n] =$$

$$[a_1 - a_2, a_2, b_3, \ldots, b_n] + [a_1, a_2 - a_1, b_3, \ldots, b_n] \text{ if } a_1 \neq a_2,$$

$$[a_1, 0, b_3, \ldots, b_n] \qquad \text{if } a_1 = a_2.$$

Birational types

Kontsevich-T. 2019

The class

$$\beta(X) \in \mathcal{B}_n(G)$$

is a well-defined G-equivariant birational invariant.

Kontsevich-T. 2019

The class

$$\beta(X) \in \mathcal{B}_n(G)$$

is a well-defined G-equivariant birational invariant.

Proof: Equivariant Weak Factorization (Abramovich, Karu, Matsuki, Włodarczyk)

Birational types

For $G = \mathbb{Z}/p\mathbb{Z}$ and $n = 2$, we get $\binom{p}{2}$ linear equations in the same number of variables.

Birational types

For $G = \mathbb{Z}/p\mathbb{Z}$ and $n = 2$, we get $\binom{p}{2}$ linear equations in the same number of variables.

$$\mathrm{rk}_{\mathbb{Q}}(\mathcal{B}_2(G)) = \frac{p^2 - 1}{24} + 1$$

For $G = \mathbb{Z}/p\mathbb{Z}$ and $n = 2$, we get $\binom{p}{2}$ linear equations in the same number of variables.

$$\mathrm{rk}_{\mathbb{Q}}(\mathcal{B}_2(G)) = \frac{p^2 - 1}{24} + 1$$

For $n \geq 3$ the systems of equations are highly overdetermined.

Birational types

For $G = \mathbb{Z}/p\mathbb{Z}$ and $n = 2$, we get $\binom{p}{2}$ linear equations in the same number of variables.

$$\mathrm{rk}_{\mathbb{Q}}(\mathcal{B}_2(G)) = \frac{p^2 - 1}{24} + 1$$

For $n \geq 3$ the systems of equations are highly overdetermined.

$$\mathrm{rk}_{\mathbb{Q}}(\mathcal{B}_3(G)) \stackrel{?}{=} \frac{(p-5)(p-7)}{24} = \frac{p^2 - 1}{24} + 1 - \frac{p-1}{2}$$

Birational types

For $G = \mathbb{Z}/p\mathbb{Z}$ and $n = 2$, we get $\binom{p}{2}$ linear equations in the same number of variables.

$$\mathrm{rk}_{\mathbb{Q}}(\mathcal{B}_2(G)) = \frac{p^2-1}{24} + 1$$

For $n \geq 3$ the systems of equations are highly overdetermined.

$$\mathrm{rk}_{\mathbb{Q}}(\mathcal{B}_3(G)) \stackrel{?}{=} \frac{(p-5)(p-7)}{24} = \frac{p^2-1}{24} + 1 - \frac{p-1}{2}$$

Jumps at

$$p = 43, 59, 67, 83, \ldots$$

Birational types

For $G = \mathbb{Z}/p\mathbb{Z}$ and $n = 2$, we get $\binom{p}{2}$ linear equations in the same number of variables.

$$\mathrm{rk}_{\mathbb{Q}}(\mathcal{B}_2(G)) = \frac{p^2 - 1}{24} + 1$$

For $n \geq 3$ the systems of equations are highly overdetermined.

$$\mathrm{rk}_{\mathbb{Q}}(\mathcal{B}_3(G)) \stackrel{?}{=} \frac{(p-5)(p-7)}{24} = \frac{p^2-1}{24} + 1 - \frac{p-1}{2}$$

Jumps at

$$p = 43, 59, 67, 83, \ldots$$

These are interesting groups!

Birational types

Variant: introduce the quotient

$$\mu^- : \mathcal{B}_n(G) \to \mathcal{B}_n^-(G)$$

by an additional relation

$$[a_1, a_2, \ldots, a_n] = -[-a_1, a_2, \ldots, a_n].$$

Birational types

Variant: introduce the quotient

$$\mu^- : \mathcal{B}_n(G) \to \mathcal{B}_n^-(G)$$

by an additional relation

$$[a_1, a_2, \dots, a_n] = -[-a_1, a_2, \dots, a_n].$$

The class of $\mathbb{P}^n$, $n \geq 2$, with linear action of $G := \mathbb{Z}/N\mathbb{Z}$ is

- torsion in $\mathcal{B}_n(G)$ and
- trivial in $\mathcal{B}_n^-(G)$.

Cyclic action on $\mathbb{P}^n$, $n \geq 2$

Since all such actions are birationally equivalent, it suffices to consider one, with $G = \mathbb{Z}/N\mathbb{Z}$ acting by

$$(x_0, \ldots, x_n) \mapsto (\zeta_N x_0, x_1, \ldots, x_n).$$

Cyclic action on $\mathbb{P}^n$, $n \geq 2$

Since all such actions are birationally equivalent, it suffices to consider one, with $G = \mathbb{Z}/N\mathbb{Z}$ acting by

$$(x_0, \ldots, x_n) \mapsto (\zeta_N x_0, x_1, \ldots, x_n).$$

This action fixes the point $(1, 0, \ldots, 0)$ and the hyperplane $x_0 = 0$.

Cyclic action on $\mathbb{P}^n$, $n \geq 2$

Since all such actions are birationally equivalent, it suffices to consider one, with $G = \mathbb{Z}/N\mathbb{Z}$ acting by

$$(x_0, \ldots, x_n) \mapsto (\zeta_N x_0, x_1, \ldots, x_n).$$

This action fixes the point $(1, 0, \ldots, 0)$ and the hyperplane $x_0 = 0$. We have

$$\beta(\mathbb{P}^n) = [1, 0, \ldots, 0] + [0, -1, \ldots, -1] = [1, 0, \ldots] + [-1, 0, \ldots].$$

$[1,0] + [-1,0] \in \mathcal{B}_2(\mathbb{Z}/p\mathbb{Z})$

For $a, b \neq 0$, we have

$$\begin{aligned} [a,b] &= [a-b,b] + [a,b-a] \\ [a-b,a] &= [-b,a] + [a-b,b]. \end{aligned}$$

$[1, 0] + [-1, 0] \in \mathcal{B}_2(\mathbb{Z}/p\mathbb{Z})$

For $a, b \neq 0$, we have

$$\begin{aligned} [a, b] &= [a-b, b] + [a, b-a] \\ [a-b, a] &= [-b, a] + [a-b, b]. \end{aligned}$$

Taking the difference,

$$[a, b] + [-b, a] = [a, b-a] + [a, a-b].$$

$[1,0]+[-1,0] \in \mathcal{B}_2(\mathbb{Z}/p\mathbb{Z})$

For $a, b \neq 0$, we have

$$\begin{aligned} [a,b] &= [a-b,b]+[a,b-a] \\ [a-b,a] &= [-b,a]+[a-b,b]. \end{aligned}$$

Taking the difference,

$$[a,b]+[-b,a]=[a,b-a]+[a,a-b].$$

If $b-a=a$, we stop and record:

$$[a,b]+[-b,a]=[a,a]+[a,-a]=[a,a]=[a,0].$$

If $b-a \neq a$, we iterate until $a=b-ma$, i.e., $b=(m+1)a$, where it stops. This is solvable mod p.

$[1,0]+[-1,0]\in\mathcal{B}_2(\mathbb{Z}/p\mathbb{Z})$

We record:

$$[a,b]+[-b,a]=[a,a]+[a,-a]=[a,0]$$

Replacing a by $-a$, and requiring that $b\neq\pm a$,

$$[-a,b]+[-b,-a]=[-a,0],$$

adding these:

$$[a,b]+[-b,a]+[-a,b]+[-b,-a]=[a,0]+[-a,0].$$

$[1, 0] + [-1, 0] \in \mathcal{B}_2(\mathbb{Z}/p\mathbb{Z})$

We record:

$$[a, b] + [-b, a] = [a, a] + [a, -a] = [a, 0]$$

Replacing a by $-a$, and requiring that $b \neq \pm a$,

$$[-a, b] + [-b, -a] = [-a, 0],$$

adding these:

$$[a, b] + [-b, a] + [-a, b] + [-b, -a] = [a, 0] + [-a, 0].$$

These are symmetric in a and b, thus

$$[a, b] + [-b, a] + [-a, b] + [-b, -a] = [b, 0] + [-b, 0].$$

In particular,

$$[a, 0] + [-a, 0] = [b, 0] + [-b, 0]$$

$[1, 0] + [-1, 0] \in \mathcal{B}_2(\mathbb{Z}/p\mathbb{Z})$

We record:

$$[a, b] + [-b, a] = [a, a] + [a, -a] = [a, 0]$$

Replacing a by $-a$, and requiring that $b \neq \pm a$,

$$[-a, b] + [-b, -a] = [-a, 0],$$

adding these:

$$[a, b] + [-b, a] + [-a, b] + [-b, -a] = [a, 0] + [-a, 0].$$

These are symmetric in a and b, thus

$$[a, b] + [-b, a] + [-a, b] + [-b, -a] = [b, 0] + [-b, 0].$$

In particular,

$$[a, 0] + [-a, 0] = [b, 0] + [-b, 0] =: \delta.$$

$[1,0]+[-1,0] \in \mathcal{B}_2(\mathbb{Z}/p\mathbb{Z})$

Consider the sum

$$S := \sum_{a,b,\neq 0, a \neq \pm b} [a,b],$$

We have

$$2S := \sum_{b} \sum_{a \neq \pm b} [a,b] + [-a,b] = (p-3) \cdot \sum_{b} [b,0] = \frac{(p-3)\cdot(p-1)}{2} \cdot \delta,$$

$[1,0]+[-1,0]\in\mathcal{B}_2(\mathbb{Z}/p\mathbb{Z})$

Consider the sum

$$S := \sum_{a,b,\neq 0, a\neq \pm b} [a,b],$$

We have

$$2S := \sum_b \sum_{a\neq\pm b} [a,b]+[-a,b] = (p-3)\cdot\sum_b [b,0] = \frac{(p-3)\cdot(p-1)}{2}\cdot\delta,$$

Apply the blowup relation to each term in S:

$$S = \sum_b \sum_{a\neq\pm b} [a-b,b]+[a,b-a].$$

$[1,0]+[-1,0] \in \mathcal{B}_2(\mathbb{Z}/p\mathbb{Z})$

Relate the two sums to S:

$$\sum_{b,a\neq\pm b}[a-b,b] \quad = \quad S+\sum_b([b,b]-[-2b,b])$$

$$\sum_{b,a\neq\pm b}[a,b-a] \quad = \quad S+\sum_a([a,a]-[-2a,a]).$$

The second sum equals the first, with a and b switched. Thus

$$S = 2S + 2\sum_b [b,b] - \sum_b([-2b,b]+[2b,-b]).$$

$[1,0]+[-1,0]\in\mathcal{B}_2(\mathbb{Z}/p\mathbb{Z})$

Note that

$$0=[-b,b]=[-2b,b]+[-b,2b]$$

so that the last sum vanishes.

$[1,0]+[-1,0] \in \mathcal{B}_2(\mathbb{Z}/p\mathbb{Z})$

Note that

$$0 = [-b, b] = [-2b, b] + [-b, 2b]$$

so that the last sum vanishes. As before,

$$\sum_b [b, b] = \sum_b [b, 0] = \frac{(p-1)}{2} \cdot \delta.$$

$[1,0]+[-1,0]\in\mathcal{B}_2(\mathbb{Z}/p\mathbb{Z})$

Note that

$$0=[-b,b]=[-2b,b]+[-b,2b]$$

so that the last sum vanishes. As before,

$$\sum_b [b,b]=\sum_b [b,0]=\frac{(p-1)}{2}\cdot\delta.$$

We find that

$$0=S+(p-1)\delta=\frac{(p-3)(p-1)}{4}\cdot\delta+(p-1)\cdot\delta.$$

It follows that

$$0=\frac{(p-1)(p+1)}{4}\cdot\delta,$$

thus δ is torsion.

$[1,0]+[-1,0]\in \mathcal{B}_2(\mathbb{Z}/p\mathbb{Z})$

Note that

$$0=[-b,b]=[-2b,b]+[-b,2b]$$

so that the last sum vanishes. As before,

$$\sum_b [b,b]=\sum_b [b,0]=\frac{(p-1)}{2}\cdot\delta.$$

We find that

$$0=S+(p-1)\delta=\frac{(p-3)(p-1)}{4}\cdot\delta+(p-1)\cdot\delta.$$

It follows that

$$0=\frac{(p-1)(p+1)}{4}\cdot\delta,$$

thus δ is torsion.

$$\mathcal{B}_n^-(G) \otimes \mathbb{Q} \simeq \mathrm{H}^{\frac{n(n-1)}{2}}(\Gamma(G,n), \mathrm{or}_n^{\otimes n}) = \mathrm{H}_0(\Gamma(G,n), \mathrm{St}_n \otimes \mathrm{or}_n)$$

where

- $$\Gamma(G,n) \subset \mathrm{GL}_n(\mathbb{Z})$$

  is a congruence subgroup,
- or is the orientation (the sign of the determinant), and
- St_n is the Steinberg representation.

Structure

Let G be a nontrivial abelian group. We work $\otimes \mathbb{Q}$ and consider $\mathcal{B}_n(G) \otimes \mathbb{Q}$ in both variables, n and G.

Consider short exact sequences of finite abelian groups

$$0 \to G' \to G \to G'' \to 0$$

and the corresponding short exact sequences of character groups

$$0 \to A'' \to A \to A' \to 0.$$

Let

$$n = n' + n'', \quad n', n'' \geq 1.$$

Multiplication and co-multiplication

We define a $\mathbb{Q}$-bilinear multiplication map

$$\nabla : \mathcal{B}_{n'}(G') \otimes \mathcal{B}_{n''}(G'') \to \mathcal{B}_{n'+n''}(G),$$

given by

$$[a'_1, \dots, a'_{n'}] \otimes [a''_1, \dots, a''_{n''}] \mapsto \sum [a_1, \dots, a_{n'}, a''_1, \dots, a''_{n''}]$$

the sum over all lifts $a_i \in A$ of $a'_i \in A'$, and a''_i are understood as elements of A, via the embedding $A'' \hookrightarrow A$.

Multiplication and co-multiplication

We define a $\mathbb{Q}$-bilinear multiplication map

$$\nabla : \mathcal{B}_{n'}(G') \otimes \mathcal{B}_{n''}(G'') \to \mathcal{B}_{n'+n''}(G),$$

given by

$$[a'_1, \ldots, a'_{n'}] \otimes [a''_1, \ldots, a''_{n''}] \mapsto \sum [a_1, \ldots, a_{n'}, a''_1, \ldots, a''_{n''}]$$

the sum over all lifts $a_i \in A$ of $a'_i \in A'$, and a''_i are understood as elements of A, via the embedding $A'' \hookrightarrow A$.

We also have

$$\nabla^- : \mathcal{B}^-_{n'}(G') \otimes \mathcal{B}^-_{n''}(G'') \to \mathcal{B}^-_{n'+n''}(G).$$

There are also co-multiplication maps

$$\Delta : \mathcal{B}_{n'+n''}(G) \to \mathcal{B}_{n'}(G') \otimes \mathcal{B}^{-}_{n''}(G''),$$

$$\Delta^{-} : \mathcal{B}^{-}_{n'+n''}(G) \to \mathcal{B}^{-}_{n'}(G') \otimes \mathcal{B}^{-}_{n''}(G'').$$

where G'' is nontrivial.

Modular types: structure

$$\mathcal{B}^{-}_{n,prim}(G) = \mathrm{Ker}\left(\mathcal{B}^{-}_{n}(G) \to \bigoplus_{\substack{n'+n''=n \\ n',n''\geq 1 \\ 0\subseteq G' \subsetneq G}} \mathcal{B}^{-}_{n'}(G') \otimes \mathcal{B}^{-}_{n''}(G/G') \right),$$

We have

$$\mathcal{B}_1(G) = \mathcal{B}_{1,prim}(G)$$

for all G; when $G = 1 = \mathbb{Z}/1\mathbb{Z}$ we have

$$\mathcal{B}_1(1) = \mathbb{Q}, \quad \mathcal{B}_n(1) = \mathcal{B}_{n,prim}(1) = 0, \text{ for } n \geq 2.$$

Let G be a cyclic group. Then $\mathcal{B}_n(G) \otimes \mathbb{Q}$ is isomorphic to

$$\bigoplus_{r} \bigoplus_{\substack{n_1+\cdots+n_r=n \\ \mathcal{G}_\bullet \text{ of lengths } r}} \mathcal{B}_{n_1,prim}(gr_1(\mathcal{G}_\bullet)) \otimes \cdots \otimes \mathcal{B}^-_{n_r,prim}(gr_r(\mathcal{G}_\bullet)) \otimes \mathbb{Q},$$

where $\mathcal{G}_\bullet$ is a flag of subgroups of type

$$0 = G_{\leq 0} \subseteq G_{\leq 1} \subsetneq \ldots \subsetneq G_{\leq r} = G, \quad r \geq 1,$$

with strict inclusions, except in the first step.

Modular types: structure

- $$\dim \mathcal{B}_{2,prim}(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q} = \dim \mathcal{B}^{-}_{2,prim}(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q}$$

  and is equal to the dimension of the space of cusp forms of weight 2 for $\Gamma_1(N)$,

- $$\dim \mathcal{B}_{3,prim}(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q} = \dim \mathcal{B}^{-}_{3,prim}(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q}$$

  and is equal to the number of certain cuspidal automorphic representations for a congruence subgroup of $\mathrm{GL}_3(\mathbb{Z})$,

Modular types: structure

- $$\dim \mathcal{B}_{2,prim}(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q} = \dim \mathcal{B}^{-}_{2,prim}(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q}$$

  and is equal to the dimension of the space of cusp forms of weight 2 for $\Gamma_1(N)$,

- $$\dim \mathcal{B}_{3,prim}(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q} = \dim \mathcal{B}^{-}_{3,prim}(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q}$$

  and is equal to the number of certain cuspidal automorphic representations for a congruence subgroup of $\mathrm{GL}_3(\mathbb{Z})$,

Computer experiments suggest that, for all $N \geq 1$:

- $$\mathcal{B}_{n,prim}(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q} = \mathcal{B}^{-}_{n,prim}(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q} = 0, \quad n \geq 4,$$

Modular types: structure

Thus we can compute the $\mathbb{Q}$-ranks of $\mathcal{B}_n(\mathbb{Z}/N\mathbb{Z})$ using:

- $n = 1$: Euler function $\phi(N)/2$

Thus we can compute the $\mathbb{Q}$-ranks of $\mathcal{B}_n(\mathbb{Z}/N\mathbb{Z})$ using:

- $n = 1$: Euler function $\phi(N)/2$
- $n = 2$: $\mathrm{g}(X_1(N))$

Modular types: structure

Thus we can compute the $\mathbb{Q}$-ranks of $\mathcal{B}_n(\mathbb{Z}/N\mathbb{Z})$ using:

- $n = 1$: Euler function $\phi(N)/2$
- $n = 2$: $\mathrm{g}(X_1(N))$
- $n = 3$: mysterious dimensions

N	43	51	52	59	63	...	208	211	239
dim	1	1	1	1	2	...	54	7	3

Modular types: structure

Thus we can compute the $\mathbb{Q}$-ranks of $\mathcal{B}_n(\mathbb{Z}/N\mathbb{Z})$ using:

- $n = 1$: Euler function $\phi(N)/2$
- $n = 2$: $\mathrm{g}(X_1(N))$
- $n = 3$: mysterious dimensions

N	43	51	52	59	63	...	208	211	239
dim	1	1	1	1	2	...	54	7	3

- $n = 4$: no primitives, with $N \leq 242$

Modular types

- G a finite abelian group, $A = G^\vee$
- $\mathbf{L} \simeq \mathbb{Z}^n$,
- $\chi \in \mathbf{L} \otimes A$ such that the induced homomorphism

  $$\mathbf{L}^\vee \to A$$

  is a surjection,
- a *basic simplicial cone*, i.e., a strictly convex cone

  $$\Lambda \in \mathbf{L}_{\mathbb{R}}$$

  spanned by a basis of $\mathbf{L}$; $\Lambda \simeq \mathbb{R}^n_{\geq 0}$, for $\mathbf{L} = \mathbb{Z}^n \subset \mathbb{R}^n$.

Modular types

For every equivalence class of triples

$$(\mathbf{L}, \chi, \Lambda),$$

define

$$\psi(\mathbf{L}, \chi, \Lambda)$$

as follows: choose a basis $e_1, \ldots, e_n$ of $\mathbf{L}$, spanning Λ, express

$$\chi = \sum_{i=1}^{n} e_i \otimes a_i, \qquad (1)$$

and put

$$\psi(\mathbf{L}, \chi, \Lambda) = [a_1, \ldots, a_n] \in \mathcal{B}_n(G).$$

Modular types

For every equivalence class of triples

$$(\mathbf{L}, \chi, \Lambda),$$

define

$$\psi(\mathbf{L}, \chi, \Lambda)$$

as follows: choose a basis $e_1, \ldots, e_n$ of $\mathbf{L}$, spanning Λ, express

$$\chi = \sum_{i=1}^{n} e_i \otimes a_i, \tag{1}$$

and put

$$\psi(\mathbf{L}, \chi, \Lambda) = [a_1, \ldots, a_n] \in \mathcal{B}_n(G).$$

The ambiguity in the choices corresponds to the $\mathfrak{S}_n$-action on the basis elements.

Modular types

For every equivalence class of triples

$$(\mathbf{L}, \chi, \Lambda),$$

define

$$\psi(\mathbf{L}, \chi, \Lambda)$$

as follows: choose a basis $e_1, \ldots, e_n$ of $\mathbf{L}$, spanning Λ, express

$$\chi = \sum_{i=1}^{n} e_i \otimes a_i, \tag{1}$$

and put

$$\psi(\mathbf{L}, \chi, \Lambda) = [a_1, \ldots, a_n] \in \mathcal{B}_n(G).$$

The ambiguity in the choices corresponds to the $\mathfrak{S}_n$-action on the basis elements. The blowup relation corresponds to scissors relations on cones. This yields multiplication, co-multiplication, Hecke operators, etc.

Equivariant Burnside group (Kresch-T. 2020)

We work over a field k of characteristic zero (with enough roots of 1). Let

$$\mathrm{Burn}_n(G) = \mathrm{Burn}_{n,k}(G)$$

be the $\mathbb{Z}$-module, generated by symbols

$$(H, Y \circlearrowright K, \beta),$$

where

Equivariant Burnside group (Kresch-T. 2020)

We work over a field k of characteristic zero (with enough roots of 1). Let

$$\mathrm{Burn}_n(G) = \mathrm{Burn}_{n,k}(G)$$

be the $\mathbb{Z}$-module, generated by symbols

$$(H, Y \circlearrowright K, \beta),$$

where

- $H \subseteq G$ is an abelian subgroup, $Y \subseteq Z_G(H)/H$,
- $K = k(F)$, with generically free Y-action, $\mathrm{trdeg}_k(K) = d \leq n$,
- $\beta = (b_1, \ldots, b_{n-d})$, a sequence, up to order, of nonzero elements of $H^\vee$, that generate $H^\vee$.

Equivariant Burnside group: relations

The symbols are subject to **conjugation** and **blowup** relations:

(C): $(H, Y \circlearrowleft K, \beta) = (H', Y' \circlearrowleft K, \beta')$, when

$$H' = gHg^{-1}, \quad Y' = \cdots, \qquad \text{with } g \in G,$$

and β and β' are related by conjugation by g.

Equivariant Burnside group: relations

The symbols are subject to **conjugation** and **blowup** relations:

(C): $\big(H, Y \circlearrowright K, \beta\big) = \big(H', Y' \circlearrowright K, \beta'\big)$, when

$$H' = gHg^{-1}, \quad Y' = \cdots, \qquad \text{with } g \in G,$$

and β and β' are related by conjugation by g.

(B1): $\big(H, Y \circlearrowright K, \beta\big) = 0$ when $b_1 + b_2 = 0$.

Equivariant Burnside group: relations

(B2): $(H, Y \circlearrowright K, \beta) = \Theta_1 + \Theta_2$, where

$$\Theta_1 = \begin{cases} 0, & \text{if } b_1 = b_2, \\ (H, Y \circlearrowright K, \beta_1) + (H, Y \circlearrowright K, \beta_2), & \text{otherwise,} \end{cases}$$

with

$$\beta_1 := (b_1, b_2 - b_1, b_3, \dots, b_{n-d}), \quad \beta_2 := (b_1 - b_2, b_2, b_3, \dots, b_{n-d}),$$

and

Equivariant Burnside group: relations

(B2): $(H, Y \circlearrowright K, \beta) = \Theta_1 + \Theta_2$, where

$$\Theta_1 = \begin{cases} 0, & \text{if } b_1 = b_2, \\ (H, Y \circlearrowright K, \beta_1) + (H, Y \circlearrowright K, \beta_2), & \text{otherwise,} \end{cases}$$

with

$$\beta_1 := (b_1, b_2 - b_1, b_3, \dots, b_{n-d}), \quad \beta_2 := (b_1 - b_2, b_2, b_3, \dots, b_{n-d}),$$

and

$$\Theta_2 = \begin{cases} 0, & \text{if } b_i \in \langle b_1 - b_2 \rangle \text{ for some } i, \\ (\overline{H}, \overline{Y} \circlearrowright K(t), \bar{\beta}), & \text{otherwise,} \end{cases}$$

with

$$\overline{H}^\vee := H^\vee / \langle b_1 - b_2 \rangle, \quad \bar{\beta} := (\bar{b}_2, \bar{b}_3, \dots, \bar{b}_{n-d}), \quad \bar{b}_i \in \overline{H}^\vee.$$

Model case: Blowing up an isolated point (with abelian stabilizer) on a surface.

It will explain the action of $\overline{Y}$ on $\overline{K} = K(t)$.

Equivariant Burnside group

The class

$$[X \circlearrowleft G] \in \mathrm{Burn}_n(G)$$

of a G-variety is computed on a standard model (X, D):

- X is smooth projective, D a normal crossings divisor,
- G acts freely on $U := X \setminus D$,
- for every $g \in G$ and every irreducible component D, either $g(D) = D$ or $g(D) \cap D = \emptyset$.

Equivariant Burnside group

Passing to a standard model X, define:

$$[X \circlearrowleft G] := \sum_H \sum_F (H, Y \circlearrowright k(F), \beta_F(X)) \in \mathrm{Burn}_n(G),$$

where the sum is over (conjugacy classes of) abelian subgroups $H \subseteq G$, and all $F \subset X$ with generic stabilizer H.

Equivariant Burnside group

Passing to a standard model X, define:

$$[X \circlearrowleft G] := \sum_{H} \sum_{F} (H, Y \circlearrowright k(F), \beta_F(X)) \in \mathrm{Burn}_n(G),$$

where the sum is over (conjugacy classes of) abelian subgroups $H \subseteq G$, and all $F \subset X$ with generic stabilizer H.

The symbols record

- the generic stabilizer H,

Equivariant Burnside group

Passing to a standard model X, define:

$$[X \circlearrowleft G] := \sum_H \sum_F (H, Y \circlearrowright k(F), \beta_F(X)) \in \mathrm{Burn}_n(G),$$

where the sum is over (conjugacy classes of) abelian subgroups $H \subseteq G$, and all $F \subset X$ with generic stabilizer H.

The symbols record

- the generic stabilizer H,
- the induced $Y \subseteq Z_G(H)/H$-action on the function field of the subvariety $F \subset X$, with generic stabilizer H,

Equivariant Burnside group

Passing to a standard model X, define:

$$[X \circlearrowleft G] := \sum_H \sum_F (H, Y \circlearrowright k(F), \beta_F(X)) \in \mathrm{Burn}_n(G),$$

where the sum is over (conjugacy classes of) abelian subgroups $H \subseteq G$, and all $F \subset X$ with generic stabilizer H.

The symbols record

- the generic stabilizer H,
- the induced $Y \subseteq Z_G(H)/H$-action on the function field of the subvariety $F \subset X$, with generic stabilizer H,
- the (generic) eigenvalues of H in the normal bundle along F.

Equivariant Burnside group

Kresch–T. (2020)

The class

$$[X \circlearrowleft G] \in \mathrm{Burn}_n(G)$$

is a well-defined G-equivariant birational invariant.

Equivariant Burnside group

Kresch–T. (2020)

The class

$$[X \circlearrowleft G] \in \mathrm{Burn}_n(G)$$

is a well-defined G-equivariant birational invariant.

Proof: Equivariant Weak Factorization.

Burnside groups: incompressibles

Simplifications arise when we focus on geometric properties of the function fields of strata

Burnside groups: incompressibles

Simplifications arise when we focus on geometric properties of the function fields of strata; there is a distinguished subgroup

$$\mathrm{Burn}_n^{\mathrm{inc}}(G) \subset \mathrm{Burn}_n(G),$$

generated by incompressible divisor symbols, i.e.,

$$\mathfrak{s} = (H, Y \circlearrowright K, \beta), \quad \mathrm{trdeg}_k(K) = n - 1,$$

H is a nontrivial cyclic group and $\beta = (b)$, a single character, generating $H^\vee$

Burnside groups: incompressibles

Simplifications arise when we focus on geometric properties of the function fields of strata; there is a distinguished subgroup

$$\mathrm{Burn}_n^{\mathrm{inc}}(G) \subset \mathrm{Burn}_n(G),$$

generated by incompressible divisor symbols, i.e.,

$$\mathfrak{s} = (H, Y \circlearrowright K, \beta), \quad \mathrm{trdeg}_k(K) = n - 1,$$

H is a nontrivial cyclic group and $\beta = (b)$, a single character, generating $H^\vee$, and such that $\mathfrak{s}$ cannot arise from Θ_2 in relation (**B2**).

Burnside groups: incompressibles

The subgroup

$$\mathrm{Burn}_n^{\mathrm{inc}}(G) \subseteq \mathrm{Burn}_n(G),$$

is a direct summand, freely generated by incompressible divisor symbols (modulo conjugation).

Burnside groups: incompressibles

The subgroup

$$\mathrm{Burn}_n^{\mathrm{inc}}(G) \subseteq \mathrm{Burn}_n(G),$$

is a direct summand, freely generated by incompressible divisor symbols (modulo conjugation).

$n = 1$ Every divisor symbol in incompressible.

Burnside groups: incompressibles

The subgroup

$$\mathrm{Burn}_n^{\mathrm{inc}}(G) \subseteq \mathrm{Burn}_n(G),$$

is a direct summand, freely generated by incompressible divisor symbols (modulo conjugation).

$n = 1$ Every divisor symbol in incompressible.

$n = 2$ A divisor symbol

$$(H, Y \circlearrowright K, \beta), \quad \beta = (b),$$

is compressible if and only if Y is cyclic and $K = k(t)$.

Applications: Birationality of linear actions on $\mathbb{P}^2$

Let $G = C_n \times \mathfrak{S}_3$, and χ be a primitive character of C_n. We have a G-action on

$$\mathbb{P}^2 = \mathbb{P}(\mathrm{I} \oplus V \otimes \chi),$$

where V is the standard 2-dimensional representation of $\mathfrak{S}_3$ and I is the trivial representation of G.

Applications: Birationality of linear actions on $\mathbb{P}^2$

Let $G = C_n \times \mathfrak{S}_3$, and χ be a primitive character of C_n. We have a G-action on

$$\mathbb{P}^2 = \mathbb{P}(\mathrm{I} \oplus V \otimes \chi),$$

where V is the standard 2-dimensional representation of $\mathfrak{S}_3$ and I is the trivial representation of G. Then

$$[X \circlearrowleft G]^{\mathrm{inc}} = (C_n, \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^1), (\chi)) + (C_n, \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^1), (-\chi)).$$

Applications: Birationality of linear actions on $\mathbb{P}^2$

Let $G = C_n \times \mathfrak{S}_3$, and χ be a primitive character of C_n. We have a G-action on

$$\mathbb{P}^2 = \mathbb{P}(\mathrm{I} \oplus V \otimes \chi),$$

where V is the standard 2-dimensional representation of $\mathfrak{S}_3$ and I is the trivial representation of G. Then

$$[X \circlearrowleft G]^{\mathrm{inc}} = (C_n, \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^1), (\chi)) + (C_n, \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^1), (-\chi)).$$

If $\chi \neq \pm\chi'$ then the corresponding actions are not G-birational.

Applications: Birationality of linear actions on $\mathbb{P}^2$

Let $G = C_n \times \mathfrak{S}_3$, and χ be a primitive character of C_n. We have a G-action on

$$\mathbb{P}^2 = \mathbb{P}(\mathrm{I} \oplus V \otimes \chi),$$

where V is the standard 2-dimensional representation of $\mathfrak{S}_3$ and I is the trivial representation of G. Then

$$[X \circlearrowleft G]^{\mathrm{inc}} = (C_n, \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^1), (\chi)) + (C_n, \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^1), (-\chi)).$$

If $\chi \neq \pm\chi'$ then the corresponding actions are not G-birational.

Birational rigidity techniques do not work well in this case, since $X^G \neq \emptyset$.

Applications: quadric threefolds

Consider $X \subset \mathbb{P}^4$ given by

$$x_1^2 + \cdots + x_5^2 = 0,$$

with an action of $G \subset W(\mathsf{D}_5)$, permuting the variables and changing signs.

Applications: quadric threefolds

Consider $X \subset \mathbb{P}^4$ given by

$$x_1^2 + \cdots + x_5^2 = 0,$$

with an action of $G \subset W(\mathsf{D}_5)$, permuting the variables and changing signs.

The action is linearizable if $X^G \neq \emptyset$.

Consider $X \subset \mathbb{P}^4$ given by

$$x_1^2 + \cdots + x_5^2 = 0,$$

with an action of $G \subset W(\mathrm{D}_5)$, permuting the variables and changing signs.

The action is linearizable if $X^G \neq \emptyset$. Linearizable actions of abelian groups have fixed points; thus we assume that

- $X^H \neq \emptyset$, for all abelian $H \subseteq G$, and
- $X^G = \emptyset$.

Applications: quadric threefolds

Consider $X \subset \mathbb{P}^4$ given by

$$x_1^2 + \cdots + x_5^2 = 0,$$

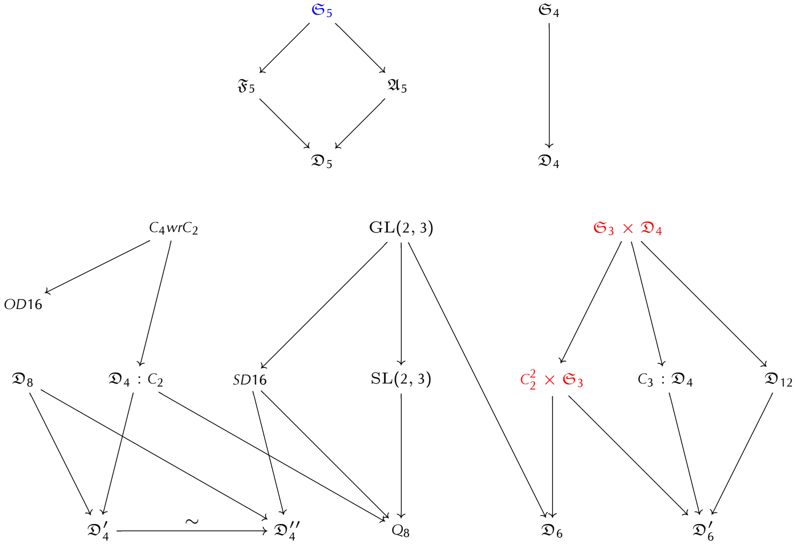
with an action of $G \subset W(\mathsf{D}_5)$, permuting the variables and changing signs.

The action is linearizable if $X^G \neq \emptyset$. Linearizable actions of abelian groups have fixed points; thus we assume that

- $X^H \neq \emptyset$, for all abelian $H \subseteq G$, and
- $X^G = \emptyset$.

Then G is one of the following...

Applications: quadric threefolds



Theorem (Cheltsov-Sarikyan-Zhuang, 2023)

Let $X \subset \mathbb{P}^4$ be a smooth quadric over $k = \mathbb{C}$:

$$x_1^2 + \cdots + x_5^2 = 0,$$

with the $\mathfrak{S}_5$-action given by permutations of variables. This action is not linearizable.

Consider $X_4 \subset \mathbb{P}^5$ given by

$$\sum_{1\leq i<j<k<l\leq 6} x_i x_j x_k x_l = \sum_{i=1}^{6} x_i = 0,$$

it carries an action of $\mathfrak{S}_6$.

Applications: Burkhardt quartic (with Cheltsov and Zhijia Zhang, 2023)

Consider $X_4 \subset \mathbb{P}^5$ given by

$$\sum_{1 \le i < j < k < l \le 6} x_i x_j x_k x_l = \sum_{i=1}^{6} x_i = 0,$$

it carries an action of $\mathfrak{S}_6$. Then the action of any G containing

$$H := \langle (12) \rangle$$

is not linearizable.

Applications: Burkhardt quartic (with Cheltsov and Zhijia Zhang, 2023)

Consider $X_4 \subset \mathbb{P}^5$ given by

$$\sum_{1\leq i<j<k<l\leq 6} x_i x_j x_k x_l = \sum_{i=1}^{6} x_i = 0,$$

it carries an action of $\mathfrak{S}_6$. Then the action of any G containing

$$H := \langle (12) \rangle$$

is not linearizable. Indeed, the fixed locus of H is a quartic with 12 singular points, a K3 surface S.

Applications: Burkhardt quartic (with Cheltsov and Zhijia Zhang, 2023)

Consider $X_4 \subset \mathbb{P}^5$ given by

$$\sum_{1\le i<j<k<l\le 6} x_i x_j x_k x_l = \sum_{i=1}^{6} x_i = 0,$$

it carries an action of $\mathfrak{S}_6$. Then the action of any G containing

$$H := \langle (12) \rangle$$

is not linearizable. Indeed, the fixed locus of H is a quartic with 12 singular points, a K3 surface S. The symbol

$$(H, Y \circlearrowright k(X), (1))$$

is incompressible (for any Y).

Applications: Burkhardt quartic (with Cheltsov and Zhijia Zhang, 2023)

Consider $X_4 \subset \mathbb{P}^5$ given by

$$\sum_{1 \le i < j < k < l \le 6} x_i x_j x_k x_l = \sum_{i=1}^{6} x_i = 0,$$

it carries an action of $\mathfrak{S}_6$. Then the action of any G containing

$$H := \langle (12) \rangle$$

is not linearizable. Indeed, the fixed locus of H is a quartic with 12 singular points, a K3 surface S. The symbol

$$(H, Y \circlearrowright k(X), (1))$$

is incompressible (for any Y). Such symbols do not arise for linear actions.

Applications: cubic fourfolds

Kresch–Hassett–T. 2020

There exists a rational cubic 4-fold with a nonlinearizable action of

$$G = C_6.$$

Applications: cubic fourfolds

Kresch–Hassett–T. 2020

There exists a rational cubic 4-fold with a nonlinearizable action of

$$G = C_6.$$

Böhning–von Bothmer–T. 2023

There exists a rational cubic 4-folds with nonlinearizable but stably linearizable action of $\mathfrak{F}_7$.

Applications: Birational characters for (projective) linear actions

Theorem (Kresch-T. 2022)

Explicit algorithm to compute

$$[\mathbb{P}(V) \circlearrowleft G] \in \mathrm{Burn}_n(G)$$

for (projective) linear actions.

Applications: Birational characters for (projective) linear actions

Theorem (Kresch-T. 2022)

Explicit algorithm to compute

$$[\mathbb{P}(V) \circlearrowleft G] \in \mathrm{Burn}_n(G)$$

for (projective) linear actions.

Based on an equivariant version of De-Concini–Procesi compactifications of subspace arrangements.

Theorem (Kresch-T. 2022)

Explicit algorithm to compute

$$[\mathbb{P}(V) \circlearrowleft G] \in \mathrm{Burn}_n(G)$$

for (projective) linear actions.

Based on an equivariant version of De-Concini–Procesi compactifications of subspace arrangements.

This has been implemented in **Magma** by Kaiqi Yang and Zhijia Zhang.

Applications: Birational characters for (projective) linear actions

There are two projective linear actions of $G = \mathfrak{S}_6$ on $\mathbb{P}^3$, with classes

$$\begin{aligned}[\mathbb{P}^3 \circlearrowleft G] = (&C_1, \mathfrak{S}_6 \circlearrowright k(\mathbb{P}^3), ()) \\ &+ (C_2, \mathfrak{A}_4 \circlearrowright k(\mathbb{P}^2), (1)) + (C_2', \mathfrak{A}_4 \circlearrowright k(\mathbb{P}^2), (1)) \\ &+ (C_2'', C_2^2 \circlearrowright k(\mathbb{P}^2), (1)) + (C_3, \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^2), (1)) \\ &+ (C_3^2, 1 \circlearrowright k, ((1,1),(1,2),(2,0))),\end{aligned}$$

respectively,

$$\begin{aligned}[\mathbb{P}^3 \circlearrowleft G] = (&C_1, \mathfrak{S}_6 \circlearrowright k(\mathbb{P}^3), ()) \\ &+ (C_2, \mathfrak{A}_4 \circlearrowright k(\mathbb{P}^2), (1)) + (C_2', \mathfrak{A}_4 \circlearrowright k(\mathbb{P}^2), (1)) \\ &+ (C_2'', C_2^2 \circlearrowright k(\mathbb{P}^2), (1)) + (C_3', \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^2), (2)) \\ &+ (C_3^2, 1 \circlearrowright k, ((0,2),(2,0),(2,2))).\end{aligned}$$

There are two projective linear actions of $G = \mathfrak{S}_6$ on $\mathbb{P}^3$, with classes

$$\begin{aligned}[\mathbb{P}^3 \circlearrowleft G] = {} & (C_1, \mathfrak{S}_6 \curvearrowright k(\mathbb{P}^3), ()) \\ & + (C_2, \mathfrak{A}_4 \curvearrowright k(\mathbb{P}^2), (1)) + (C_2', \mathfrak{A}_4 \curvearrowright k(\mathbb{P}^2), (1)) \\ & + (C_2'', C_2^2 \curvearrowright k(\mathbb{P}^2), (1)) + (C_3, \mathfrak{S}_3 \curvearrowright k(\mathbb{P}^2), (1)) \\ & + (C_3^2, 1 \curvearrowright k, ((1,1), (1,2), (2,0))),\end{aligned}$$

respectively,

$$\begin{aligned}[\mathbb{P}^3 \circlearrowleft G] = {} & (C_1, \mathfrak{S}_6 \curvearrowright k(\mathbb{P}^3), ()) \\ & + (C_2, \mathfrak{A}_4 \curvearrowright k(\mathbb{P}^2), (1)) + (C_2', \mathfrak{A}_4 \curvearrowright k(\mathbb{P}^2), (1)) \\ & + (C_2'', C_2^2 \curvearrowright k(\mathbb{P}^2), (1)) + (C_3', \mathfrak{S}_3 \curvearrowright k(\mathbb{P}^2), (2)) \\ & + (C_3^2, 1 \curvearrowright k, ((0,2), (2,0), (2,2))).\end{aligned}$$

These differ in $\mathrm{Burn}_3(G)$; thus, the actions are not birational.

Equivariant Burnside group: structure

Let us examine the crucial relation

(B2): $(H, Y \circlearrowright K, \beta) =$

$$(H, Y \circlearrowright K, \beta_1) + (H, Y \circlearrowright K, \beta_2) + (\overline{H}, \overline{Y} \circlearrowright K(t), \bar{\beta}).$$

Equivariant Burnside group: structure

Let us examine the crucial relation

(B2): $(H, Y \circlearrowright K, \beta) =$

$$(H, Y \circlearrowright K, \beta_1) + (H, Y \circlearrowright K, \beta_2) + (\overline{H}, \overline{Y} \circlearrowright K(t), \bar{\beta}).$$

Observation: This relation preserves various geometric properties of the function field K, e.g.,

- dimensions of MRC quotients

Equivariant Burnside group: structure

Let us examine the crucial relation

(B2): $(H, Y \circlearrowright K, \beta) =$

$$(H, Y \circlearrowright K, \beta_1) + (H, Y \circlearrowright K, \beta_2) + (\overline{H}, \overline{Y} \circlearrowright K(t), \bar{\beta}).$$

Observation: This relation preserves various geometric properties of the function field K, e.g.,

- dimensions of MRC quotients
- unramified cohomology, $\mathrm{H}^1(Y, \mathrm{Pic}(D))$,

Equivariant Burnside group: structure

Let us examine the crucial relation

(B2): $(H, Y \circlearrowright K, \beta) =$

$$(H, Y \circlearrowright K, \beta_1) + (H, Y \circlearrowright K, \beta_2) + (\overline{H}, \overline{Y} \circlearrowright K(t), \bar{\beta}).$$

Observation: This relation preserves various geometric properties of the function field K, e.g.,

- dimensions of MRC quotients
- unramified cohomology, $\mathrm{H}^1(Y, \mathrm{Pic}(D))$,
- other stable birational invariants.

This means that the Burnside group admits direct sum decompositions

$$\mathrm{Burn}_n(G) = \oplus \ldots$$

Equivariant Burnside group: structure

Let us examine the crucial relation

(B2): $(H, Y \circlearrowright K, \beta) =$

$$(H, Y \circlearrowright K, \beta_1) + (H, Y \circlearrowright K, \beta_2) + (\overline{H}, \overline{Y} \circlearrowright K(t), \bar{\beta}).$$

Observation: This relation preserves various geometric properties of the function field K, e.g.,

- dimensions of MRC quotients
- unramified cohomology, $\mathrm{H}^1(Y, \mathrm{Pic}(D))$,
- other stable birational invariants.

This means that the Burnside group admits direct sum decompositions

$$\mathrm{Burn}_n(G) = \oplus \ldots$$

The incompressibles we discussed give just one of the direct summands.

Summary

- Ideas from motivic integration led to the introduction of new invariants in birational geometry,

Summary

- Ideas from motivic integration led to the introduction of new invariants in birational geometry,
- Burnside groups have a rich algebraic structure, to be investigated,

Summary

- Ideas from motivic integration led to the introduction of new invariants in birational geometry,
- Burnside groups have a rich algebraic structure, to be investigated,
- There are now many examples of nonbirational actions of finite groups; and we continue to explore the range of applicability of these new invariants.