Index for non-relativistic superconformal field theories -- from LLM to ABJM and BL

Yu Nakayama (UC Berkeley),

arxiv:0807.3344[hep-th]
Also work in progress with S. Ryu, M. Sakaguchi and K. Yoshida
Index in theoretical physics

- Index bridges between math and physics
  - Worldsheets string theory
    - Gauss-Bonnet
    - Riemann-Roch
  - Gauge theory
    - Atiya-Singer
    - Fermion zero mode, flux vacua, D-brane...

### Most successful marriage is Witten index

\[ I = \text{Tr}(-1)^F e^{-\beta H} \quad H = \{Q, Q^\dagger\} \]

- Unification of mathematical formulae
  - Path integral of SUSY non-linear sigma model…
Index in mathematics

- Gauss-Bonnet theorem

\[ \chi(M) = \frac{1}{2\pi} \int d^2x \sqrt{g} R \]

= vertex - edge + face

= \sum_{p=0} \left(-1\right)^p \dim H^p(M)

= \sum_{p \text{-form}} (-1)^p e^{-\beta \Delta}

= 2g - 2

\[ \Delta = d^\dagger d + dd^\dagger \]

\[ d^2 = (d^\dagger)^2 = 0 \]

\[ d^\dagger = *d* \]

Euler characteristic is a topological invariant!

- \((-1)^p\) is important!
SUSY and index

- SUSY algebra

\[ \{ Q^\dagger, Q \} = Q^\dagger Q + QQ^\dagger = 2H , \quad Q^2 = (Q^\dagger)^2 = 0 \]

\[ d^\dagger d + dd^\dagger = \Delta , \quad d^2 = (d^\dagger)^2 = 0 \]

- Witten Index

\[ I_w = \text{Tr}(-1)^F e^{-\beta H} \]

Bose Fermi cancellation

When \( H|\Psi\rangle = E|\Psi\rangle \)

Then \( Q|\Psi\rangle \) has same \( E \) but different \( F \)
Gauss-Bonnet from Witten index

Physicists can derive index theorem from path integral

- Consider **supersymmetric quantum mechanics** (non-linear sigma model)

\[
S = \int dt G_{ab}(\phi) \dot{\phi}^a \dot{\phi}^b + G_{ab}(\phi) \dot{\Psi}^a \dot{\Psi}^b + \ldots
\]

- One can show

\[
I_w = \text{Tr}(-1)^F e^{-\beta H} = \chi(M)
\]

- Doing **path integral** by localization method

\[
I_w = \int \mathcal{D}\Phi \mathcal{D}\Psi e^{-S(\Phi,\Psi)} = \int e(M)
\]

\(e(M): \text{Euler Class}\)
Index for superconformal field theory

- Consider SCFT with superconformal charge $S$
  
  \[ \{S, Q\} = \Delta \]

  \[ I_{SCFT} = \text{Tr}(-1)^F e^{-\beta \Delta} \]

- Ex1. 2D NLSM with Calabi-Yau target space
  
  \[ \Rightarrow \text{Index is nothing but elliptic genus} \]

\[ I(\tau, z) = \text{Tr}(-1)^F e^{2\pi i z J_L} q^{L_0} \bar{q}^{\bar{L}_0} \]

\[ \Delta = \bar{L}_0 \]

- Euler characteristic
- Hirzebruch signature
- A-roof genus
Index for superconformal field theory II

- **Ex2 SCFT in 4D** (Romelsberger, Kinney-Maldacena-Minwalla-Raju, Nakayama)

  \[ I(t, y) = \text{Tr}(-1)^F e^{-\beta \Delta} t^{2(E+j_2)} y^{2j_1} \]
  \[ \Delta = 2\{Q^+, Q\} = E - 2j_2 - \frac{3}{2}r \]

- Counting **short multiplets** annihilated by Q

  Encode **geometric information** of conical CY\(_3\) probed by D3-brane (base Sasaki-Einstein space by AdS/CFT)

- **Ex. CY = C\(^3\) (N=4 SYM)**

  \[ I(t, y) = \prod_{n} \frac{(1 - t^{3n} y^{n})(1 - t^{3n} y^{-n})}{(1 - t^{2n})^3} \]

- **Novel mathematical inv?** (counts holomorphic function etc…)
Index for non-relativistic SCFT

But what is the condition to define such an index? Any specific property of SUSY algebra?

- Yes! Non-trivial anti-automorphism of algebra.

What is the non-relativistic SCFT?

- NR-limit of M2-brane gauge theory
  - M2-brane mini revolution
  - Chern-Simons matter theory in 1+2 dim

- NR CS-matter theory is used in quantum Hall effects

$$ I = \text{Tr}(-1)^F e^{-\beta \Delta} x^{R-2J} \quad \Delta = -\frac{1}{2} (iD - J + \frac{3}{2}R) $$

- Yet another novel mathematical invariant for CY\(_4\)
What you will hear in the second part of the talk

- Precise definition of my index

- **Representation theories** of non-relativistic superconformal algebra

- Explicit computation of index for **NR CS-matter theory** → Applications to condensed matter physics

- NR-limit of **M2-brane gauge theory** on CY$_4$ → new mathematical invariant for CY$_4$

- What is the **RHS** of the index theorem

\[
I(x) = \int d\Phi d\Psi e^{-S_{eff}(\Phi, \Psi)} \quad \Phi, \Psi : \text{matrix}
\]

Due to localization
Index for Non-relativistic superconformal algebra

Yu Nakayama (UC Berkeley)

now I finished 1/6 of the talk
Index in theoretical physics

- Index bridges between **math** and **physics**
  - **Worldsheet string theory**
    - Gauss-Bonnet
    - Riemann-Roch
  - **Gauge theory**
    - Atiya-Singer
    - Fermion zero mode, flux vacua, D-brane…

**Most successful marriage is Witten index**

\[ I = \text{Tr}(-1)^F e^{-\beta H} \]

\[ H = \{Q, Q^\dagger\} \]

- **Unification of mathematical formulae**
  - Path integral of **SUSY** non-linear sigma model…
Index in theoretical physics II

- Index is robust and *first check* of any duality.
- From AdS-CFT duality to microstate counting of black hole.
- Index (almost) does not depend on any continuous parameter of the theory
  - Exception: wall crossing etc…

Can we do better than Witten index?

- **Yes**, we can! *Superconformal index* (Romelsberger, Kinney-Maldacena-Minwalla-Raju, Nakayama)
- Any other? When? Which SUSY algebra?
  - Index for non-relativistic superconformal theory
Witten index for supersymmetric field theory

- **Witten Index** on $\mathbb{R}^4$ (or $\mathbb{T}^3 \times \mathbb{R}$) captures vacuum structure of the supersymmetric (field) theories

  \[ \mathcal{I}^W = \text{Tr} (-1)^F e^{-\beta H} = \text{Tr}_{H=0} (-1)^F e^{-\beta H} \]

  \[ H = \{ Q^\dagger, Q \} \]

  - **Bose-Fermi cancellation**
    - Only vacuum ($H=0$) states contribute
    - Does not depend on $\beta$

  - **Many applications**
    - Study on vacuum structure
    - Implication for SUSY breaking
    - Derivation of index theorem (geometry)
Non-relativistic (NR) SCFT

- (1+2) dim non-relativistic super conformal field theory
  - Lorentz invariance $\rightarrow$ Galilean invariance
  - Scale invariance appears in massive theory:
    - Schrödinger equation is conformally invariant!
    \[ -\frac{\partial}{\partial t} \psi = \frac{\partial^2}{2m} \psi \]

- Construction from String theory
  - DLCQ of N=4 SYM in (1+3)d $\rightarrow$ (1+2)d NRSCFT
  - NR-limit of CS-matter theory (1+2)d NRSCFT
    - N=2 LLM (Lablanc-Lozano-Min hep-th/9206039)
    - N=6 ABJM (Aharony-Bergman-Jafferis-Maldacena)
    - N=8 BL (Bagger-Lambert)
Non-relativistic Superconformal Algebra I

- **Bosonic part (Schroedinger algebra)**

\[-\frac{\partial}{\partial t} \psi = \frac{\partial^2}{2m} \psi\]

- **Galilean algebra (H,P,G,J,M)**

  \[i[J, P] = -iP , \ i[J, \bar{P}] = i\bar{P}\]
  \[i[J, G] = -iG , \ i[J, \bar{G}] = i\bar{G}\]
  \[i[H, G] = P , \ i[H, \bar{G}] = \bar{P}\]
  \[i[P, \bar{G}] = 2M\]

- **+Dilatation(D)**

  \[i[D, P] = -P , \ i[D, \bar{P}] = -\bar{P} , \ i[D, G] = G , \ i[D, \bar{G}] = \bar{G}\]
  \[i[D, H] = -2H\]

- **+Special conformal transformation (K)** \[\delta t = \epsilon t^2 , \ \delta r_i = \epsilon t r_i\]

\[i[K, \bar{P}] = -\bar{G}\]
\[i[H, K] = D , \ i[D, K] = 2K\]
Non-relativistic Superconformal Algebra II

- Fermionic Part
  - SUSY algebra \((Q_1, Q_2)\)
    \[
    \{Q_1, Q_1^*\} = 2M , \quad \{Q_2, Q_2^*\} = H , \quad \{Q_1, Q_2^*\} = \bar{P} , \quad \{Q_2, Q_1^*\} = P , \\
    i[J, Q_1] = \frac{i}{2} Q_1 , \quad i[J, Q_1^*] = -\frac{i}{2} Q_1^* , \quad i[J, Q_2] = -\frac{i}{2} Q_2 , \quad i[J, Q_2^*] = \frac{i}{2} Q_2^* , \\
    i[\bar{G}, Q_2] = -Q_1 , \quad i[G, Q_2^*] = -Q_1^* , \quad i[D, Q_2] = -Q_2 , \quad i[D, Q_2^*] = -Q_2^* ,
    \]
  - + Superconforal (S)
    \[
    i[K, Q_2] = S , \quad i[H, S^*] = -Q_2^* , \quad i[\bar{P}, S] = -Q_1 , \quad i[J, S] = -\frac{i}{2} S , \\
    \{S, S^*\} = K , \quad \{S, Q_1^*\} = -G , \quad i[D, S] = S , \quad \{S, Q_2^*\} = \frac{i}{2}(iD - J + \frac{3}{2}R) , \\
    i[R, Q_a] = -iQ_a , \quad i[R, S] = -iS ,
    \]

- Grading structure w.r.t. D
  - H: +2, P, Q_2: +1
  - J, D, Q_1: 0,
  - G, S: -1, K: -2
Involutive Anti-automorphism of algebra

There are two different conjugation in SSch

- Quantum mechanics (Dirac conjugation)
  \[ w_0(J) = J^\dagger = J, w_0(P) = P^\dagger = \bar{P}, w_0(G) = G^\dagger = \bar{G}, \]
  \[ w_0(H) = H^\dagger = H, w_0(D) = D^\dagger = D, w_0(K) = K^\dagger = K, w_0(M) = M^\dagger = M \]
  \[ w_0(Q_1) = Q_1^*, w_0(Q_2) = Q_2^*, w_0(S) = S^* \]

- Conformal conjugation (BPZ conjugation)
  \[ w(J) = J, w(P) = \bar{G}, w(G) = \bar{P}, w(H) = -K, \]
  \[ w(K) = -H, w(D) = -D, w(M) = -M, w(R) = R, \]
  \[ w(Q_1) = iQ_1^*, w(Q_2) = iS^*, w(S) = iQ_2^*. \]

- Different conjugation leads to different inner product.
  \[ \langle O\psi|\psi \rangle = \langle \psi|w(O)|\psi \rangle \]
$w_0(O) \text{ vs } w(O): \text{ Two conjugation?}$

Two conjugations are related by two different (but equivalent) quantization scheme

- Recall in 2D relativistic CFT on plane
  - Quantization w.r.t. “time” Hamiltonian
    \[ P^\dagger = P, \quad K^\dagger = K \ldots \]
  - Radial quantization w.r.t. “radial” Hamiltonian
    \[ P = L_{-1}, \quad K = L_{+1}, \quad P^\dagger = K \]
- Similar in NRSCA
Two conjugation leads to **two different index**

- **Index from** $w_0 \rightarrow$ **Witten index**
  
  \[ I_0 = \text{Tr}(-1)^F e^{-\beta H}, \]
  
  where $H = \{w_0(Q_2^*), Q_2^*\} = \{Q_2, Q_2^*\}$.
  
  - Does not depend on $\beta$
  - Counts vacua ($H=0$)

- **Index from** $w \rightarrow$ **NEW index for NRSCA!**
  
  \[ I(x) = \text{Tr}(-1)^F e^{-\beta \Delta_x R - 2J}, \]
  
  where $\Delta = i\{S, Q_2^*\} = \frac{-1}{2}(iD - J + \frac{3}{2}R)$
  
  - Does not depend on $\beta$
  - Counts **BPS operators** ($\Delta = 0$)
  - $x$ is chemical potential to distinguish BPS operators
Summary of representation of NRSCA

There are four types of representation in NRSCA

- **Vacuum representation**: Annihilated by $Q_2, Q_2^*$, $d_0 = 0, j_0 = \frac{3}{2}r_0, m$

- **Chiral representation**
  - Annihilated by $Q_2^* \rightarrow d_0 = -j_0 + \frac{3}{2}r_0$
  - Contribute to the index

- **Anti-chiral representation**
  - Annihilated by $Q_2 \rightarrow d_0 = j_0 - \frac{3}{2}r_0$
  - Complex conjugation of chiral representation

- **Long representation**
  - Not annihilated either by $Q_2, Q_2^*$
  - Two chiral representations make one long representation
Some properties of index for NRSCA

\[ I(x) = \text{Tr}(-1)^F e^{-\beta \Delta} x^{R-2J}, \]

where \( \Delta = i\{S, Q^*_2\} = -\frac{1}{2}(iD - J + \frac{3}{2}R) \)

- Index does **not** depend on \( \beta \)
- Protected by any **exactly marginal deformation**
  - Two chiral multiplets combine into one long multiplet → does not change index.
  - All the BPS states (actually infinitely many!) contribute. Know a lot about the NRSCFT (than just vacua)!
  - Independence of the exactly marginal deformation → the **first step** to check any duality.
Example of Index for NRCFTs

Yu Nakayama (UC Berkeley)

now I finished 2/3 of my talk
N=2 Abelian CS-matter theory (LLM)

\[ S = \int dt d^2 x \left[ \frac{\kappa}{2} \partial_t A_i \epsilon_{ij} A_j - A_0 \left( \kappa B + e|\Phi|^2 + e|\psi|^2 \right) + i\Phi^* \partial_t \Phi + i\psi^* \partial_t \psi 
- \frac{1}{2m} |D_i \Phi|^2 - \frac{1}{2m} |D_i \psi|^2 + \frac{e}{2m} B|\psi|^2 + \frac{e^2}{2m\kappa} |\Phi|^4 + \frac{3e^2}{2m\kappa} |\Phi|^2 |\psi|^2 \right] \]

- Construction

Begin with N=2 massive relativistic CS-matter theory

- Take NR-limit

\[ \partial_\mu \partial^\mu \phi + m^2 \phi = 0 \]

\[ \phi = \frac{1}{\sqrt{2m}} (e^{-imt} \Phi + e^{imt} \Phi^\dagger) \]

- We only keep particle \( \Phi(x, t) \)

\[ \rightarrow \text{Schrodinger equation} \quad -i \frac{\partial}{\partial t} \Phi = \frac{\partial^2}{2m} \Phi \]
More on the construction

- **No direct relation** between relativistic SCFT in (1+2) dim and NRSCFT in (1+2) dim!
  - No non-relativistic limit for “unparticle” traveling at the speed of light!
  - Add mass term → take NR limit

- **Group theoretically** $\text{Sch}_{2+1}$ is a subgroup of $\text{SO}(2,4) \leftarrow (1+3)$ dim relativistic conformal algebra
  - Starting point of the AdS/NRCFT correspondence
  - (1+2) dim NRCFT may be dual to 5-dimensional gravity theory with light cone-compactification (Son, MIT group)
    - DLCQ of N=4 SYM as (1+2) NRSCFT?
Applications

- **M2-brane mini revolution**
  - Add mass term ⇔ Add background 4-from flux
  - NR limit ⇔ near horizon (Penrose) limit? (contraction of algebra)

  New arena for solvable(?) string theory, gauge gravity duality! Note: non-relativistic system is easier!

- **Quantum Hall effect**
  - CS-matter theory is effective field theory for “anyon”: theoretical basis of (fractional) quantum hall effect.
  - Derivation of Laughlin’s wavefunction
N=2 Abelian CS-matter theory (continued)

\[ S = \int dt d^2x \left[ \frac{\kappa}{2} \partial_t A_i \epsilon_{ij} A_j - A_0 \left( \kappa B + e |\Phi|^2 + e |\Psi|^2 \right) + i \Phi^* \partial_t \Phi + i \Psi^* \partial_t \Psi \right. \]

\[ \left. - \frac{1}{2m} |D_i \Phi|^2 - \frac{1}{2m} |D_i \Psi|^2 + \frac{e}{2m} B |\Psi|^2 + \frac{e^2}{2m \kappa} |\Phi|^4 + \frac{3e^2}{2m \kappa} |\Phi|^2 |\Psi|^2 \right] \]

- **Conformally invariant** (all order in perturbation theory)
- **SUSY generator**
  
  \[ Q_1 = \int d^2x \Phi^* \Psi \]
  
  \[ Q_2 = \int d^2x \Phi^* D_+ \Psi \]

\[ S = \int d^2x (t \Phi^* D_+ \Psi + x_+ \Phi^* \Psi) \]

- **Index does not depend on exactly marginal deformation**
  - we can take \( e \rightarrow 0 \) limit (free theory)
- **Count BPS states!**

<table>
<thead>
<tr>
<th>Letters</th>
<th>( U(1) )</th>
<th>( J )</th>
<th>( R )</th>
<th>( iD )</th>
<th>( R - 2J )</th>
</tr>
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<td>( \Phi )</td>
<td>1</td>
<td>0</td>
<td>2/3</td>
<td>-1</td>
<td>2/3</td>
</tr>
<tr>
<td>( \psi^* )</td>
<td>-1</td>
<td>-1/2</td>
<td>1/3</td>
<td>-1</td>
<td>4/3</td>
</tr>
<tr>
<td>( P )</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>2</td>
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</table>
Computation of index

\[ I(x) = \text{Tr}(-1)^F e^{-\beta \Delta} x^{R-2J}, \]

where \( \Delta = i\{S, Q^*\} = -\frac{1}{2}(iD - J + \frac{3}{2}R) \)

How to collect gauge invariant BPS (\( \Delta = 0 \)) state efficiently?

- **Idea:** integrate over the gauge orbit (U(1) holonomy)

  \[ \Phi \rightarrow \Phi e^{i\theta} \]
  \[ \psi^* \rightarrow \psi^* e^{-i\theta} \]

- **Composite operator has charge:**

  \[ \Phi^2 \propto e^{2i\theta}, \Phi \psi^* \propto 1, \psi^2 D \psi^* \psi^* \propto 1 \ldots \]

- **Integration over U(1) holonomy only picks up singlets!**

  \[ \int d\theta \Phi^2 e^{2i\theta} + \Phi \psi^* = \Phi \psi^* \]

\[ I(x) = \text{Tr}(-1)^F x^{R-J} = \int \frac{d\theta}{2\pi} \prod_{m=0}^{\infty} \frac{1 - x^{\frac{4}{3}+2m} e^{i\theta}}{1 - x^{\frac{2}{3}+2m} e^{-i\theta}} \]

\[ = 1 - x^2 - 2x^4 - 2x^6 - 2x^8 + x^{12} + 5x^{14} + 7x^{16} + \ldots .\]
Non-relativistic limit of N=6 ABJM

More nontrivial example: NR-limit of ABJM theory

- ABJM model is M2 brane at $C_4/Z_k$ orbifold.
- Massive deformation and take NR-limit
- Gauge group is $U(N) \times U(N)$
- Several possible non-relativistic limit (choice of particle – anti-particle).
  - $Q_2 : 2$
  - $Q_1 : 10$
  - $S : 2$
- Today I focus on N=2 NRSCA subalgebra
Index for Non-relativistic ABJM

We take $N \to \infty$, $k \to \infty$ limit to compute index.

Index DOES depend on $N$ (and $k$ for finite $N$), but in the large $N$ limit $k$ dependence must vanish

Counting gauge invariant BPS states reduces to matrix integral (see Sundborg, Nakayama, KMMR)

\[
I(x) = \int dU_1 dU_2 \exp \left( \sum_n \frac{1}{n} f_{12}(x^n) \text{Tr} U_1^n \text{Tr} (U_2^\dagger)^n + \frac{1}{n} f_{21}(x^n) \text{Tr} (U_1^\dagger)^n \text{Tr} U_2^n \right)
\]

\[
f_{12} = f_{21} = \frac{2x^{2/3} - 2x^{4/3}}{1 - x^2}
\]

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<td>$\Phi_{12}$</td>
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<td>-1</td>
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</table>
Matrix integral for index

- Index computation for NR ABJM reduces to matrix integral.
- The origin of this matrix integral is 1-loop computation on the cylinder (only remaining zero mode is Polyakov loop!).

Large N matrix integral is possible by **saddle point approx.**

\[
I(x) = \int dU_1 dU_2 \exp \left( \sum_n \frac{1}{n} f_{12}(x^n) \text{Tr} U_1^n \text{Tr}(U_2^\dagger)^n + \frac{1}{n} f_{21}(x^n) \text{Tr}(U_1^\dagger)^n \text{Tr} U_2^n \right)
\]

\[
I(x) = \prod_{n=1}^{\infty} \frac{(1 + x^{2n} + x^{4n})^2}{(1 + x^{2n} - x^{4n})(1 + x^{2n} + 3x^{4n})}.
\]

\[
= 1 + 4x^{4/3} - 8x^2 + 24x^{8/3} - 56x^{10/3} + 156x^4 - 408x^{14/3} + 1076x^{16/3} + \cdots.
\]

- Interesting in number theory? Call for Ramanujan…
Discussion

Yu Nakayama (UC Berkeley)

Now I’m in additional time…
Further investigation

- More examples?
  - What is maximally non-relativistic SCA?
  - NR limit of ABJM has 14 SUSY
  - Known NRSCA with 24 SUSY
  - More indices for extended SUSY!

- NR limit of N=8 BL theory?
  - Problem with Majorana fermions…

AdS/NRCFT is the obvious next step

- Can we compute index for DLQC of N=4 SYM?
- Gravity side? KK decomposition of SUGRA background
- Non-relativistic limit of M2-brane background?

- More applications in condensed matter?
A conjecture

- Wave-function of the universe = Laughlin’s wave-function

\[ \psi = \prod_{i<j} (z_i - z_j)^{2k+1} \exp \left[ -\frac{1}{4} eB(z_1^2 + \cdots + z_N^2) \right] \]

- U(1) theory: very strongly coupled universe
- k: Fraction of Landau level \[ \nu = \frac{1}{2k + 1} \]
- k = 0: free-fermion wave-function…

- Maybe exactly obtainable in SUSY CS theory…