Dimensionally Reducing Generalized Symmetries from (3+1)-Dimensions

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Motivation

- There has been a recent interest in more general symmetry structures: higher group structure, non-invertible symmetries.
- Many examples of these symmetries have been recently studied in various dimensions.
- It is interesting to consider what happens to these symmetries under dimensional reduction.
- Can be used to better understand dimensional reduction.
- Can be used to better understand generalized symmetries.

Outline

1. Introduction:
   - Aspects of dimensional reduction
   - Aspects of generalized symmetries
2. Reducing generalized symmetries from 4d to 2d
3. Example: 4d $\mathcal{N}$=1 SQCD $\rightarrow$ 2d $\mathcal{N}$=(0,2) gauge theory
4. Conclusions

Dimensional reduction

- Take a quantum field theory in D spacetime dimensions. Consider it on the space $R^d \times M$, for M a compact manifold.
- In the limit where the size of M goes to zero, get a new quantum field theory in d spacetime dimensions.
- The properties of the resulting theory depend on the chosen higher dimensional theory and the properties of the compact surface M.
- Can be used as a method to generate new quantum field theories from known ones.
- Leads to relations between the higher and lower dimensional theories and properties of M.

Dimensional reduction: examples

- Dimensional reduction on $M = S^1$:
  - Relation between 6d $(2,0)$ SCFT and 5d MSUSY Yang-Mills [Douglas, 2011; Lambert, Papageorgakis, Schmidt-Sommerfeld, 2011].
  - Can reduce 4d $\mathcal{N}$=1 SCFTs to 3d $\mathcal{N}$=2 SCFTs [Aharony, Razamat, Seiberg, Willett, 2013].
- Dimensional reduction on $M = T^2$:
  - 6d $(2,0)$ SCFT → $\mathcal{N}$=4 super Yang-Mills
- Dimensional reduction on general Riemann surfaces:
  - 6d $(2,0)$ SCFT → 4d $\mathcal{N}$=2 SCFTs, class S construction [Gaiotto, 2017]
- Construction can be used to elucidate properties like the conformal manifold and dualities of the lower dimensional theory.

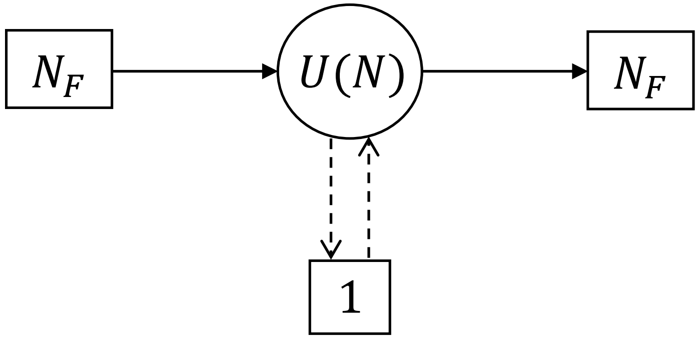
Relation between symmetries

- What is the relation between the symmetries of the two theories?
- Generically, expect the lower dimensional theory to inherit the global symmetries of the higher dimensional parent.
- Furthermore, can predict the ‘t Hooft anomalies of the lower dimensional theory from those of the higher dimensional theory.
- For continuous symmetries, done by integrating the anomaly polynomial of the higher dimensional theory on the compact surface [Benini, Tachikawa, Wecht, 2010]: $\int_M I_{D+2} = I_{d+2}$ ($D, d =$ even integers).
- For discrete symmetries, done by integrating the anomaly theory on the compact surface [Sacchi, Sela, GZ, 2023].
- Here we shall be concerned about the case of more general symmetry structures.

Dimensional reduction from 4d to 2d

- We will be primarily interested in the reduction of 4d $\mathcal{N}$=1 SCFTs on S^2 to give 2d $\mathcal{N}$=(0,2) theories [Gadde, Razamat, Willett, 2015].
- Since S^2 is curved: to preserve some SUSY, must turn on a flux in a $U(1)_R$ R-symmetry such that its curvature cancels the spin connection for some of the supercharges. This allows us to preserve 2 supercharges $\rightarrow$ (0,2) SUSY in 2d.
- There is some freedom in choosing the flux corresponding to a choice of $U(1)_R$ R-symmetry.

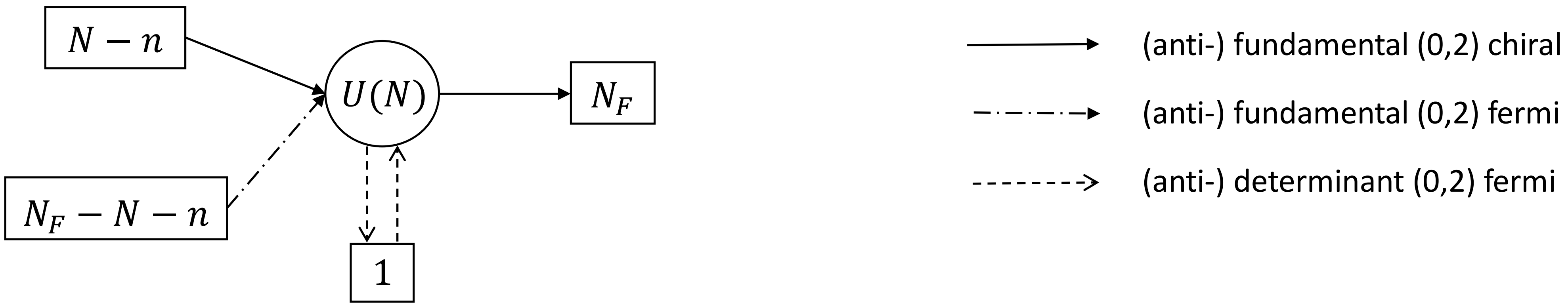
(0,2) triality and Seiberg duality



- A specific example we consider: 4d $\mathcal{N}$=1 $U(N)$ gauge theory* + N_F fundamental flavors. Here we have also added matter in the determinant representation so that the R-symmetry be non-anomalous (dashed arrows).
- This theory has a very rich generalized symmetry structure.
- The S^2 reduction of this theory is known [Gadde, Razamat, Willett, 2015].

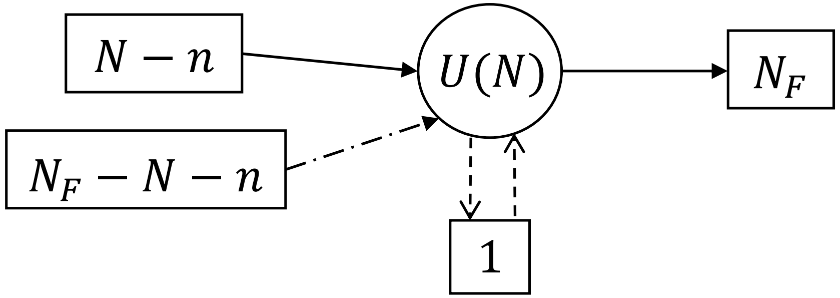
* Globally $U(1) \times SU(N)$.

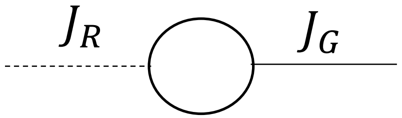
(0,2) triality and Seiberg duality



- When reduced on the 2-sphere: gives a 2d (0,2) gauge theory.
- The 2d gauge theories were studied in [Gadde, Gukov, Putrov, 2013]. Exhibit dual descriptions (related to Seiberg duality of the 4d theory).
- Here the 2d $U(N)$ gauge group comes from the 4d $U(N)$ gauge group.

A puzzle





$Tr(U(1)_R U(1)_G) = 2nN$

- The 2d theory inherit the $U(1)_R$ R-symmetry of the 4d theory.
- However, one can show that the corresponding 2d $U(1)_R$ R-symmetry is anomalous: broken to discrete group $\mathbb{Z}_{2nN}$.
- Why is the symmetry broken in 2d?
- We shall see that this is a consequence of the presence of generalized symmetry structure in the 4d theory.

Generalized symmetries

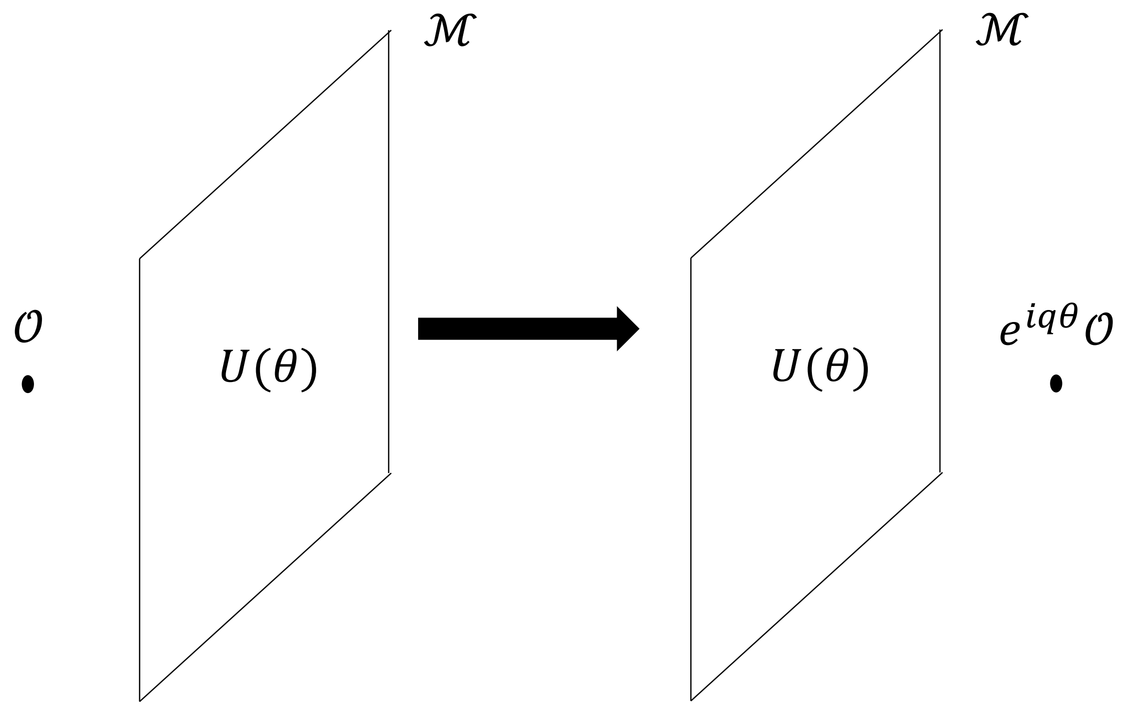
- Renewed interest in symmetries recently.
- Interest spurred by the discovery of novel types of symmetries.
- Here we would be interest with the fate of such symmetries (higher group and non-invertible) upon dimensional reduction.
- Begin with a brief review of generalized symmetries.

Symmetries = Topological operators

- Symmetries can be associated with topological operators.
- Example: $U(1)$. Have conserved current: $d\ *j = 0$.
- Leads to conserved charge:
  $Q = \int j_0\, d^{d-1}x.$
- Can use this to build an operator:

$$U = e^{i\theta Q} = e^{i\theta \oint_{\mathcal{M}} *j}$$

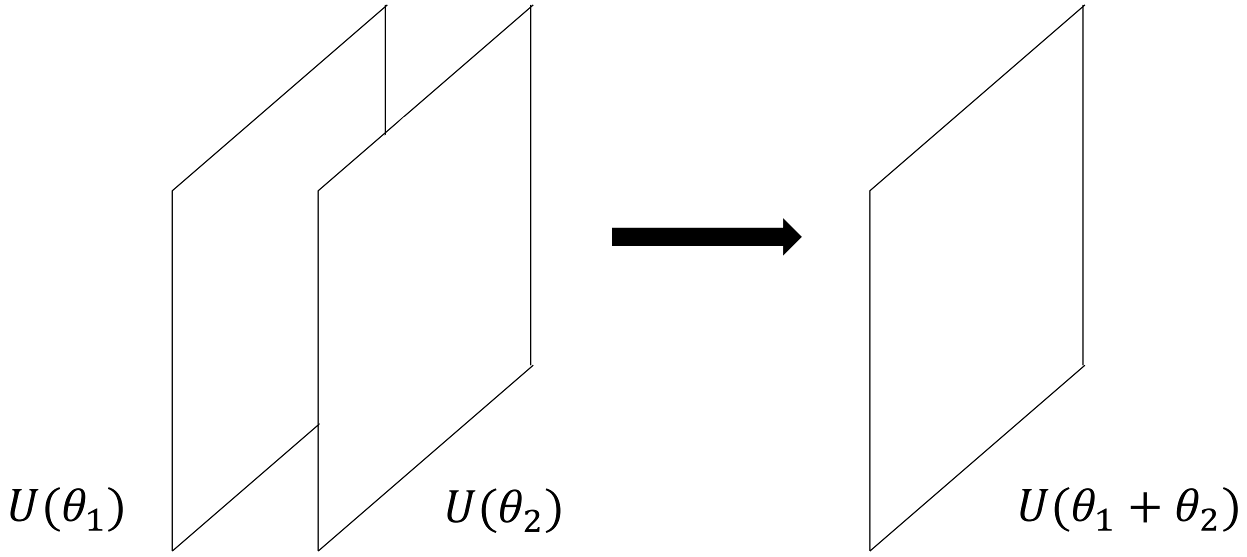
- Current conservation → Operator is topological.
- Works similarly for discrete symmetries.



[Frolich, Fuchs, Runkel, Schweigert, 2009; Kapustin, Seiberg, 2014; Gaiotto, Kapustin, Seiberg, Willett, 2014; …]

Properties of topological operators

- Properties of the topological operators then imply properties of the associated symmetries.



$$U(\theta_1) \otimes U(\theta_2) = U(\theta_1 + \theta_2)$$

- Example: fusion rule → group property.

Generalizations

- The topological operator viewpoint suggests several generalizations of the notion of symmetries.
- Topological operators of higher codimension → higher form symmetries (codimension p+1 operator → p-form symmetry).
- Non-invertible symmetries: symmetries that do not form a group.

$$a \otimes a = b \oplus c \oplus \ldots$$

- Elements don’t necessarily possess an inverse.
- Non-invertible symmetries can be described by a fusion (n-)category.

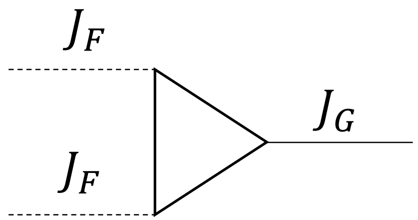
Examples of higher form symmetries

- Pure $SU(N)$ gauge theory: has a $\mathbb{Z}_N$ 1-form symmetry acting on the Wilson lines.
- Similarly for other gauge groups with matter invariant under some part of the center.
- A $U(1)$ gauge theory has $U(1)$ 1-form symmetry acting on the 't Hooft lines.
- Can associate a background 2-form connection, B, to this symmetry, which then couples to the magnetic charges as: $\int B \frac{F}{2\pi}$.

2-group structure

- Given two symmetries, the two can combine to form a larger group.
- Example: two $\mathbb{Z}_2$ symmetries can form a direct product $\mathbb{Z}_2 \times \mathbb{Z}_2$, but can also mix to form a general extension, for example $\mathbb{Z}_4$.
- Similarly, given a 0-form and a 1-form symmetry, the two can form a direct product, but can also mix leading to the analogues of a semi-direct product or a more general extension.
- The resulting structure is referred to as a 2-group structure.

Continuous 2-group structure in 4d

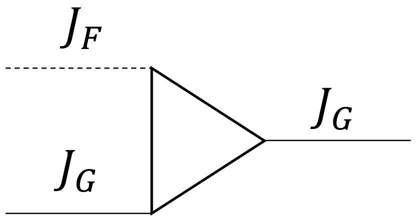


$$Tr(U(1)_F^2 U(1)_G) = \kappa$$

- Example: 4d $U(1)_G$ gauge theory with a $U(1)_F$ global symmetry and the above mixed gauge-flavor anomaly.
- Causes anomalous variation of the action under background flavor gauge transformation ($A_F \rightarrow A_F + d\lambda$): $\delta S \sim \kappa \int \lambda \frac{dA_F\, dA_G}{(2\pi)^2}$.
- Due to the coupling $\int B\ \frac{dA_G}{2\pi}$, can be canceled by shifting $B \rightarrow B - \lambda\kappa \frac{dA_F}{2\pi}$.
- Leads to a 2-group structure between $U(1)_F$ and 1-form magnetic symmetry [Cordova, Dumitrescu, Intriligator, 2018]:

$$A_F \rightarrow A_F + d\lambda\, , B \rightarrow B + d\Lambda - \lambda\kappa \frac{dA_F}{2\pi}$$

Non-invertible symmetries: chiral anomaly



$$Tr(U(1)_G^2 U(1)_F) = \kappa \qquad\qquad U(1)_F \to \mathbb{Z}_\kappa$$

- What about examples of non-invertible symmetries?
- One example is a $U(1)$ flavor symmetry “broken” by an ABJ anomaly [Choi, Lam, Shao, 2022; Cordova, Ohmori, 2022].
- Here the $U(1)$ axial symmetry is broken to an invertible $\mathbb{Z}_\kappa$ by the anomaly.
- However, it was argued that axial rotations by a rational angle are still symmetries, albeit non-invertible.

Reduction of generalized symmetries

- It is interesting to ask what happens to the aforementioned generalized symmetries upon dimensional reduction on a surface Σ.
- Specifically, we shall consider reduction in the presence of fluxes:
  - Fluxes in flavor symmetries (2-group): $\int_{\Sigma} c_1(F) = m_F$
  - Fluxes in gauge symmetries (2-group and non-inv.): $\int_{\Sigma} c_1(G) = m_G$
  - For gauge symmetry flux: mean looking at the theory expanded around a vacuum with fixed flux.
- Can study this using the relation to anomalies.

Reduction of generalized symmetries: anomaly polynomial viewpoint

- Can integrate the anomaly polynomial for anomalies involving gauge symmetries.
- Reduction on Σ of 2-group in the presence of flavor fluxes:

$$\int_{\Sigma} c_1(F) = m_F \quad , \qquad \frac{\kappa}{2}\int_{\Sigma\times M_4} c_1^2(F)c_1(G) \rightarrow m_F\kappa \int_{M_4} c_1(F)c_1(G) \rightarrow Tr(U(1)_F U(1)_G) = m_F\kappa$$

- Resulting 2d anomalies imply flavor symmetry broken to a discrete group: $U(1)_F \rightarrow \mathbb{Z}_{\kappa m_F}$.

Reduction of generalized symmetries: anomaly polynomial viewpoint

- Similarly, can consider Reduction on Σ of non-invertible symmetry in the presence of gauge fluxes:

$$\int_{\Sigma} c_1(G) = m_G \quad , \qquad \frac{\kappa}{2} \int_{\Sigma\times M_4} c_1^2(G) c_1(F) \to m_G \kappa \int_{M_4} c_1(G) c_1(F)$$

- Resulting 2d anomaly again imply flavor symmetry broken to a discrete group: $U(1)_F \to \mathbb{Z}_{\kappa m_G}$.
- Can similarly consider the case of 2-group with gauge flux.
- However, anomalies involving gauge symmetries should not be good physical observables, so how should we understand this?
- Can more properly understand this as the reduction of the generalized symmetries.

Reduction of generalized symmetries: flavor flux

- Consider the reduction of a 2-group in the presence of flavor fluxes.
- The 2-group transformation law (under 0-form gauge trans.):

$$A_F \to A_F + d\lambda\,, B \to B - \lambda\kappa \frac{dA_F}{2\pi}$$

- When reduced on the surface, we get: $$\int_\Sigma B \to \int_\Sigma B - \lambda\kappa \int_\Sigma \frac{dA_F}{2\pi} = \int_\Sigma B - \lambda\kappa m_F$$

- Holonomy of 1-form symmetry, $\exp(\mathrm{i} \int_\Sigma B)$, is a parameter in the theory so must remain invariant.
- This leads to the breaking of the flavor symmetry: $U(1)_F \to \mathbb{Z}_{\kappa m_F}$.

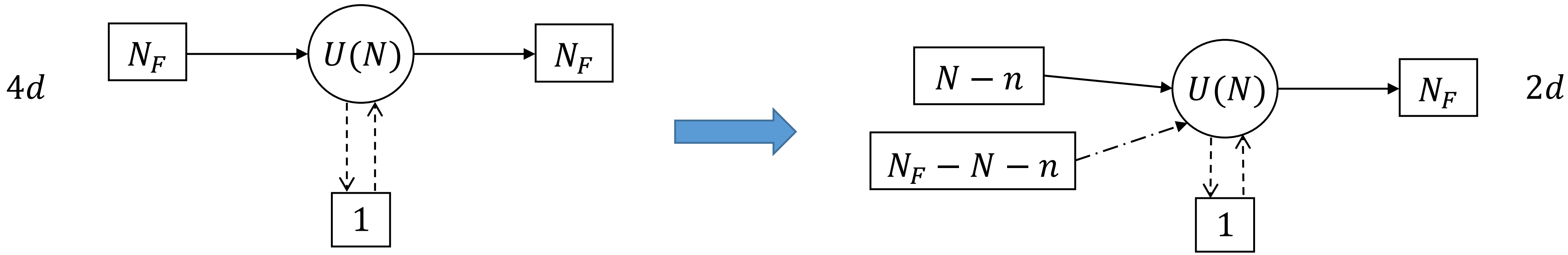
Reduction of generalized symmetries: gauge flux

- Can similarly discuss the case with gauge flux: $\int_\Sigma c_1(G) = m_G$
- Reducing a 2-group in the presence of gauge flux, 2-group reduces to a 't Hooft anomaly for the zero form symmetry:

$$A_F \to A_F + d\lambda\, , B \to B - \lambda\kappa \frac{dA_F}{2\pi} \text{ in 4d} \to Tr(U(1)_F^2) = \kappa m_G \text{ in 2d}$$

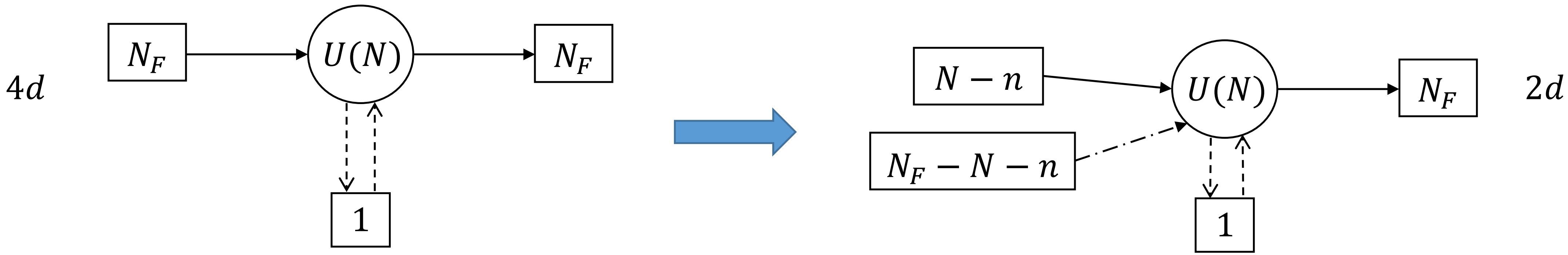
- Reducing a non-invertible chiral symmetry, non-invertible symmetry becomes a discrete $\mathbb{Z}_{m_G\kappa}$ invertible symmetry.
- Can study this by directly reducing the topological defect.
- Can also study this by reducing the symmetry TQFT [Antinucci, Benini, 2024; Brennan, Sun, 2024].

Application: sphere reduction from 4d to 2d



- Can study this in the context of the sphere reduction of 4d $\mathcal{N}$=1 theories to 2d $\mathcal{N}$=(0,2) theories.
- Here we shall consider the 4d $\mathcal{N}$=1 $U(N)$ SQCD, that we previously introduced.
- The sphere reduction of this theory is the 2d (0,2) $U(N)$ gauge theories that were previously introduced.

Application: sphere reduction from 4d to 2d



- The $U(N)$ version has a rich generalized symmetry structure.
- Has a 2-group structure involving the $U(1)$ magnetic 1-form and various flavor symmetries.
- Has Non-invertible symmetry: global symmetry acting on the two determinant fields broken only by the $U(1)$ gauge anomaly.

Resolution of the puzzle

- One interesting observation: the 4d R-symmetry and magnetic 1-form symmetry form a 2-group: $I_6 \supset -nNc_1^2(R)c_1(G)$.
- Recall that to preserve supersymmetry we need to turn on a magnetic flux in the R-symmetry.
- As such, the R-symmetry should be broken to a discrete group in 2d!
- The resolution of the puzzle: R-symmetry is broken due to the 2-group structure.
- Can match the 2d $U(1)_G U(1)_R$ anomaly with the sphere reduction of the 4d $U(1)_G U(1)_R^2$ anomaly.

Interesting implications

- 2-group leads to breakdown of flavor symmetry in the presence of flavor fluxes: seen example of this for 4d $\mathcal{N}$=1 SQCD.
- 2-group leads to additional ‘t Hooft anomalies in the presence of gauge flux.
- In general: reduction on S^2 gives a direct sum of different 2d theories associated with different gauge fluxes [Gadde, Razamat, Willett, 2015]. Now see that these have different ‘t Hooft anomalies in the presence of a 2-group!
- Non-invertible axial symmetry $\rightarrow$ invertible $\mathbb{Z}_{m_G \kappa}$ 2d symmetry.
- Order of symmetry depends on magnetic flux: different sectors have different global symmetry in the presence of a non-invertible symmetry!
- In the example of 4d $\mathcal{N}$=1 SQCD: can match the 4d and 2d anomalies, including those involving gauge symmetries, for various choices of flux.

Overview of the results

- Here we discuss the fate of 2-group and non-invertible structure upon reduction to 2d.
- In the paper we also discuss the fate of 2-group and non-invertible structure upon reduction to 3d.
- Short overview of the results on that front:
  - In the presence of matter: generalized symmetry structure trivializes once the KK tower is integrated out (Lagrangian examples, no twist).
  - Without matter, generalized symmetry structure can survive (example: Maxwell on a circle).
  - Discuss implication of these for $\mathcal{N}$=1 $U(N)$ SCQD.

Conclusions

- Dimensional reduction can be a useful tool to study QFT problems.
- Understanding the relation between symmetries under dimensional reduction may further our understanding of both dimensional reduction and generalized symmetries.
- Have studied the fate of 2-group and non-invertible symmetries under dimensional reduction, particularly the reduction from 4d to 2d, in the presence of fluxes.
- Have seen that these generalized structures, even when not present in 2d, can leave an imprint on the resulting 2d theories in terms of additional anomalies or breaking to a discrete group.
- Expect results to apply in more general cases: other compactifications, discrete 2-group, more general non-invertible symmetries.

Open questions

- More general reductions from 4d: on more general surfaces, with twisting by discrete symmetries.
- Dimensional reduction from other dimensions:
  - 6d: Little string theories generally possess 2-group structure. Can we see these effects in their reduction to 4d?
  - 5d: Many 5d SCFTs possess 2-group structure. Can we see these effects in their reduction to 3d?
- Understanding the reduction from the symmetry TQFT.

Thank you