Localized RG flows on composite defects & C-theorem

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Defects are interesting objects in physics

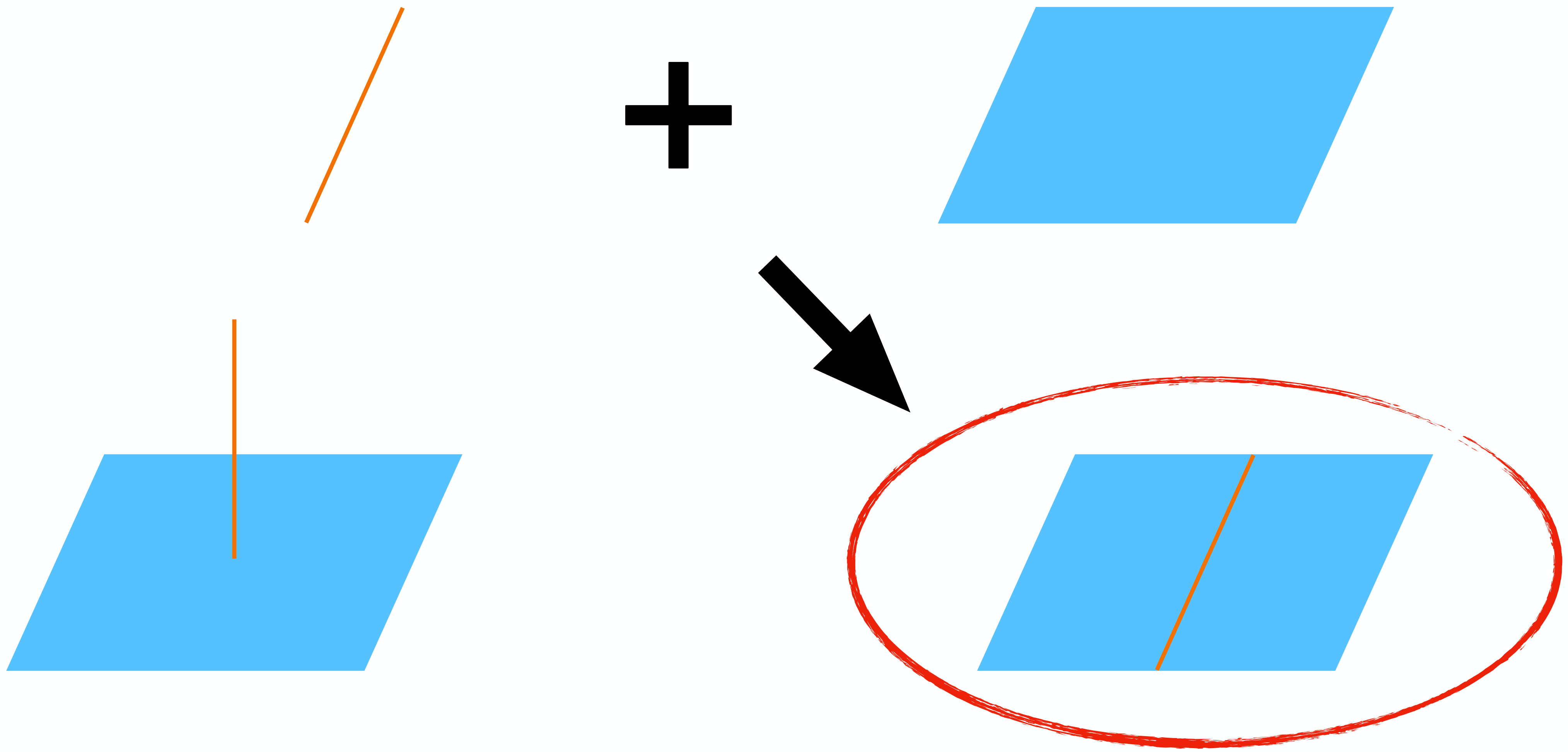
- Line defect: Wilson-'t Hooft line, impurities, cosmic strings, …
- Higher-dimensional defects: D-branes, …
- Topological defect: Skyrmions (2D), Hopfions (3D), …

Normally, only one defect is considered in a physical system. What if there exists more than one defect ? In this situation, perhaps we can do more engineering. The simplest situation is to consider two defects.

- Two of the same dimension
- Two of different dimensions

What is a composite defect ?

We can start by doing some engineering, taking a line defect and a surface defect



When a lower-dimensional defect is embedded inside a higher-dimensional defect, this gives rise to a composite defect.

The aims:

- Understand the local RG properties of the composite defect
- Seeking a model giving rise to a conformal composite defect
- Test if a C-theorem still applies for the composite defect

The plans:

- Free vector O(N) model with a composite defect in $d=4-\varepsilon$
- Free vector O(N) model with a composite defect in $d=3-\varepsilon$
- A (weak) C-theorem for the sub defect

Defects ~ Localized interactions

In the model we are considering, defects are represented by the localized interactions. This means that they can flow under local renormalization group (RG) flow. The local deformations are marginal classically. For example, a scalar field has a classical dimension

$$\Delta_\phi = \frac{d}{2} - 1$$

We can build local deformations of the type $\int d^r x \phi^q$

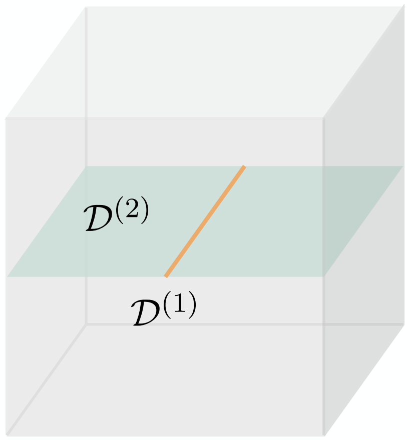
Classically marginal requires that $q\Delta_\phi = r$

d	r	q
4	1	1
4	2	2
3	1	2
3	2	4
...	...	...

$$d = 4 - \varepsilon$$

A single scalar model with composite defect

Model: Line defect + surface defect



$$I = \frac{1}{2}\int d^d x\,(\partial\phi)^2 + \frac{h_0}{2}\int_{\mathbb{R}^2} d^2\hat{y}\,\phi^2 + g_0\int d\tilde{z}\,\phi$$

Epsilon expansion of the bare couplings,

$$h_0 = M^{\epsilon}\left(h + \frac{\delta h}{\epsilon} + \frac{\delta_2 h}{\epsilon^2} + \dots\right),$$
$$g_0 = M^{\epsilon/2}\left(h + \frac{\delta g}{\epsilon} + \frac{\delta_2 g}{\epsilon^2} + \dots\right)$$

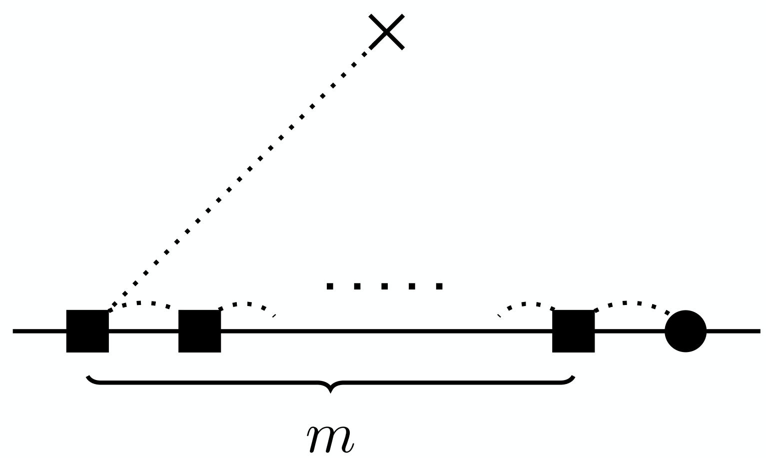
Regularization of the theory, one-point function of the renormalized field should be finite

$$\langle[\phi](x)\rangle = \text{finite}\,, \quad \langle[\phi^2](x)\rangle = \text{finite}\,.$$

All loop calculation

<u>Type 1</u> diagram: Chain type

$$\Gamma_m(x) \;=$$



■ Surface defect

● Line defect

We can use the free field propagator to calculate perturbatively

$$\Gamma_m(x) = (-g_0)\frac{(-h_0/2)^m}{m!}\int \mathrm{d}^r\tilde{z}\int\prod_{i=0}^{m}\mathrm{d}^{2r}\hat{y}_i\,\langle\,\phi(x)\prod_{j=0}^{m}\hat{\phi}^2(\hat{y}_i)\,\tilde{\phi}(\tilde{z})\,\rangle$$

$$= -g_0(-h_0)^m\int\frac{\mathrm{d}^{d-r}\tilde{p}}{(2\pi)^{d-r}}\boxed{f_r^m(|\tilde{p}_{\|,H}|)}\,\frac{e^{-\mathrm{i}\,\tilde{p}x}}{\tilde{p}^2}$$

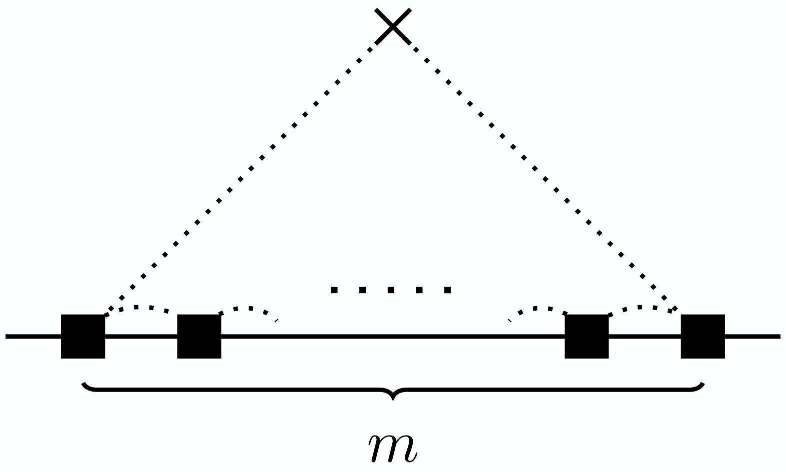
$$\boxed{f_r(|\tilde{p}_{\|,H}|) = (4\pi)^{\frac{\epsilon-2}{2}}\,|\tilde{p}_{\|,H}|^{-\epsilon}\,\Gamma\left(\frac{\epsilon}{2}\right)}$$

Summing up all the diagrams gives

$$\langle\phi(x)\rangle = (-g_0)\int\frac{\mathrm{d}^{d-r}\tilde{p}}{(2\pi)^{d-r}}\,\frac{1}{1+h_0 f(|\tilde{p}_{\|,H}|)}\,\frac{e^{-\mathrm{i}\,\tilde{p}x}}{\tilde{p}^2}$$

<u>Type II</u> diagram: Loop type, with only surface defects

$$\tilde{\Gamma}_m(x) \;=$$



$$\tilde{\Gamma}_m(x) = \frac{(-h_0/2)^m}{m!}\int \prod_{i=1}^{m} \mathrm{d}^{2r}\hat{y}_i \langle \phi^2(x) \prod_{j=1}^{m} \hat{\phi}^2(\hat{y}_j)\rangle$$
$$= (-h_0)^m \int \frac{\mathrm{d}^{d-2r}\hat{p}}{(2\pi)^{d-2r}} \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{e^{-ix\hat{p}}}{(\hat{p}+p)^2 p^2} f_r^{m-1}(|p_{\|,H}|)$$

Summing up all the diagrams gives the non-factorizable part

$$\langle \phi^2(x)\rangle = (\langle \phi(x)\rangle)^2 + \boxed{\int \frac{\mathrm{d}^{d-2r}\hat{p}}{(2\pi)^{d-2r}} \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{e^{-ix\hat{p}}}{(\hat{p}+p)^2 p^2} \frac{-h_0}{1+h_0 f_r(|p_{\|,H}|)}}$$

Finite conditions give the renormalized coupling

$$\frac{M^{\epsilon/2}}{g_0} + \frac{h_0}{g_0}\frac{M^{-\epsilon/2}}{2\pi\epsilon} = \frac{1}{g}\,, \qquad \frac{M^{\epsilon}}{h_0} + \frac{1}{2\pi\epsilon} = \frac{1}{h}$$

$$\Downarrow$$

$$\beta_h = \frac{\mathrm{d}h}{\mathrm{d}\ln M} = \frac{h(h-2\pi\epsilon)}{2\pi}\,, \quad \beta_g = \frac{\mathrm{d}g}{\mathrm{d}\ln M} = \frac{g(h-\pi\epsilon)}{2\pi}$$

Comments:

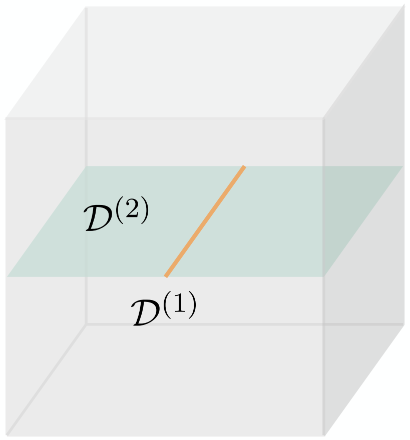
- The beta function for the line coupling is corrected by the surface coupling, but not vice versa
- The localized RG flows on the surface and the line cannot flow to the fixed point simultaneously -> no conformal composite defect
- This analysis applies for free theories in higher even dimensions, they cannot host conformal composite defects

$$d = 3 - \varepsilon$$

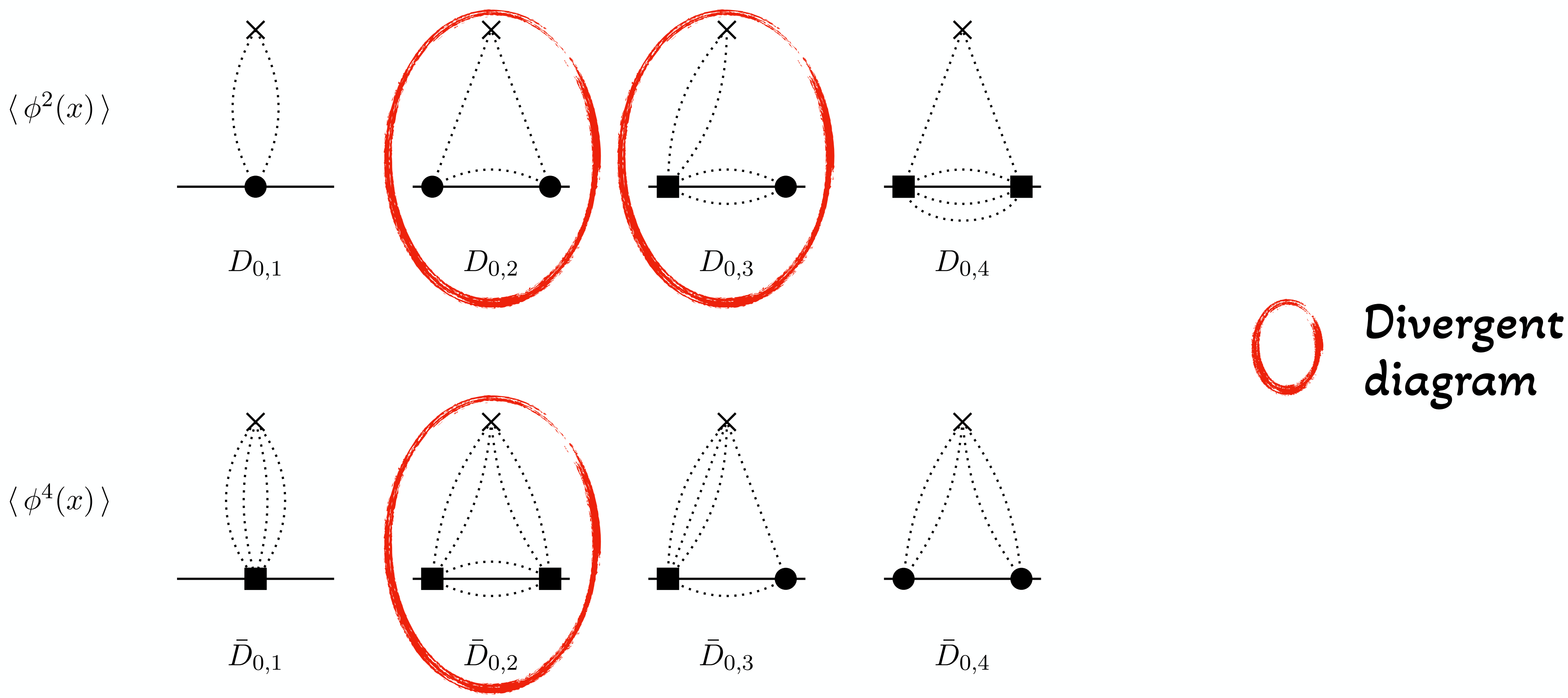
O(N) model with composite defect

Model: surface defect + line defect

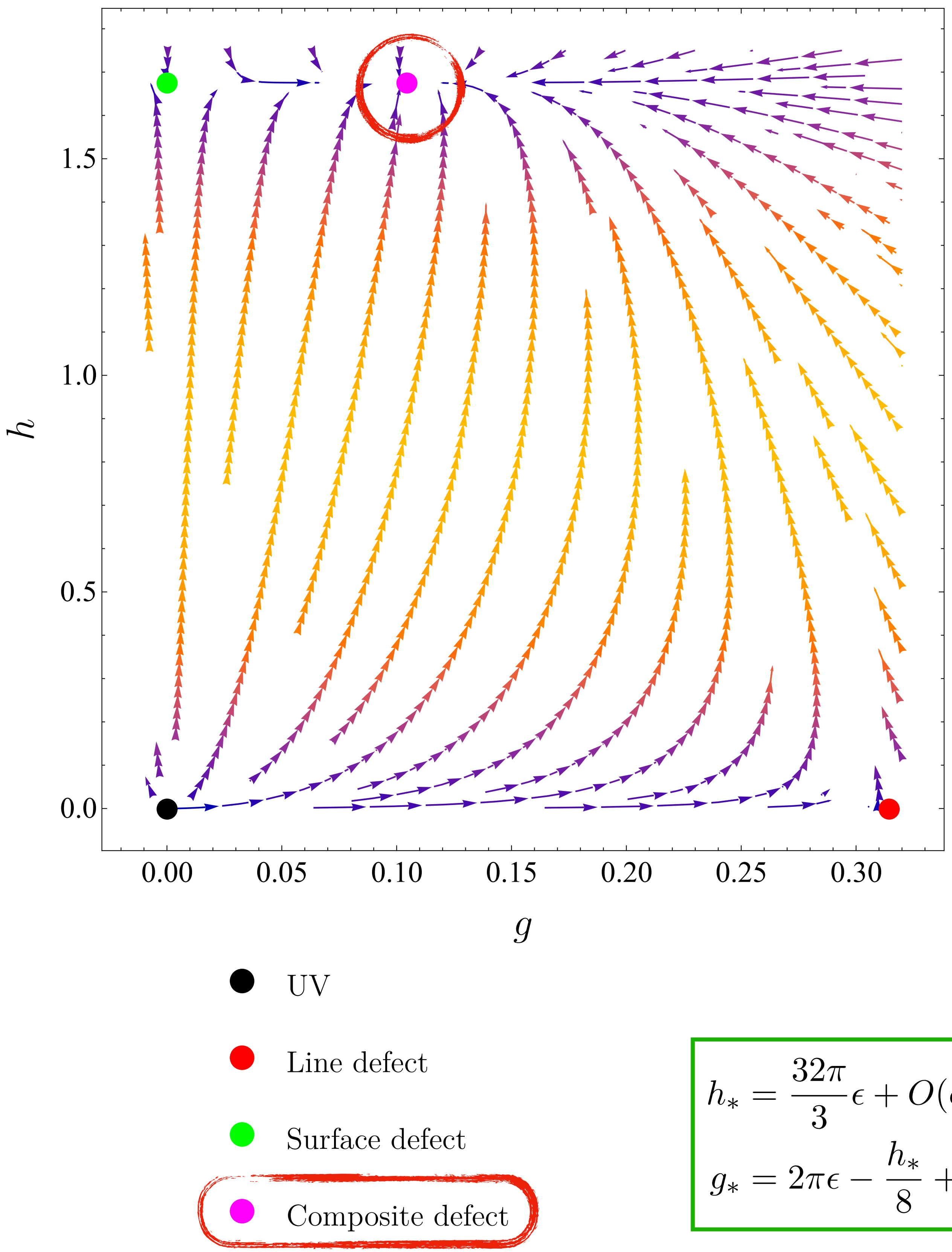
$$I \equiv \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{d}^d x\, \partial_\mu \phi^I \partial^\mu \phi^I + \frac{h_0}{4!}\int_{\mathbb{R}^2} \mathrm{d}^2 \hat{y}\, \left(\phi^I \phi^I\right)^2 + \frac{g_{0,IJ}}{2}\int_{\mathbb{R}} \mathrm{d}\tilde{z}\, \phi^I \phi^J$$



Diagrams up to the second order



O(1) model: only a single scalar



Comments:

- In this dimension, both conformal line and surface defects can be hosted
- A conformal composite defect exists !!!
- At the CCD fixed point, line coupling is modified by the surface one

$$h_* = \frac{32\pi}{3}\epsilon + O(\epsilon^2)\,,$$

$$g_* = 2\pi\epsilon - \frac{h_*}{8} + O(\epsilon^2) = \frac{2\pi}{3}\epsilon + O(\epsilon^2)$$

O(N) breaking pattern: O(N) -> O(m) * O(N-m) on the line

$$g_{IJ} = \mathrm{diag}\,(\underbrace{g\,,\,g\,,\cdots,\,g}_{m},\,\underbrace{g'\,,\,g'\,,\cdots,\,g'}_{N-m})$$

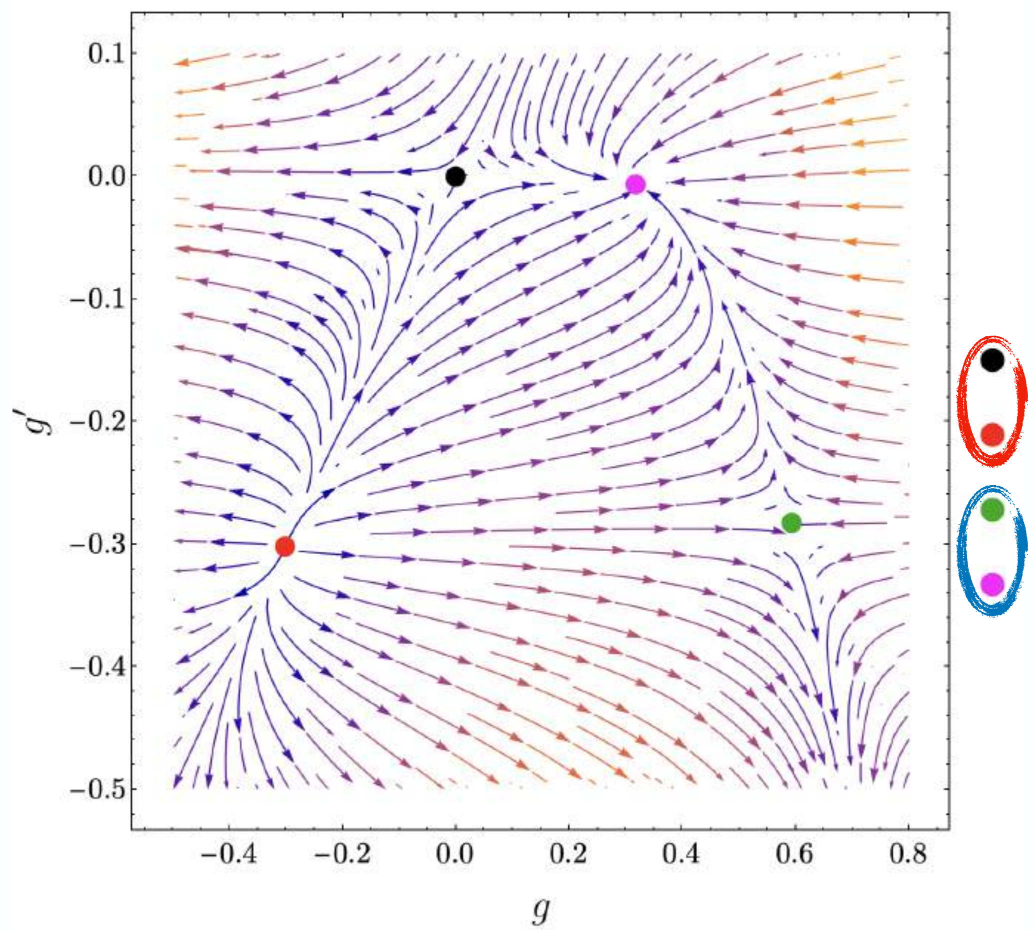
Beta functions:

$$\beta_h = -2\epsilon\, h + \frac{N+8}{48\pi}\, h^2 + (\text{higher order})$$

$$\beta_g = -\epsilon\, g + \frac{1}{2\pi} g^2 + \frac{1}{48\pi}(m\, g + (N-m)g' + 2g)h + \cdots\,,$$
$$\beta_{g'} = -\epsilon\, g' + \frac{1}{2\pi} g'^2 + \frac{1}{48\pi}(m\, g + (N-m)g' + 2g')h + \cdots\,.$$

RG flows w/ fixed surface coupling

$$h_* = \frac{96\,\pi}{N+8}\,\epsilon + O(\epsilon^2)$$



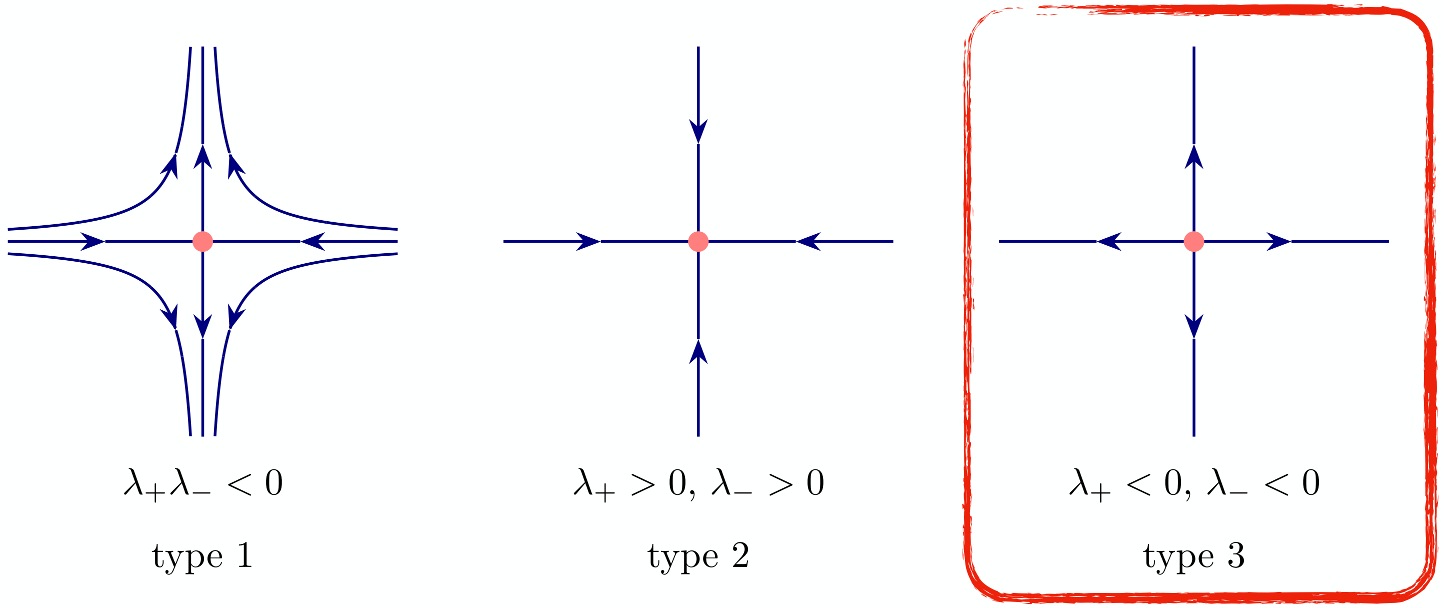
O(N) symmetric

O(N) broken

Comments:

- Four CCD fixed points, 2 * O(N) symmetric + 2 * O(N) broken
- Unitarity (real couplings) requires N >= 23
- Local RG analysis shows that the trivial one is the most UV among the four for N > 4. We can perturb the coupling around the fixed point, obtaining the eigenvalues for the phase space (g,g')

$$\lambda_+ = \frac{N-4}{N+8}\epsilon\ , \qquad \lambda_- = -\frac{N+4}{N+8}\epsilon$$



A mixing of composite operators

We consider two composite operators of the same classical dimension on the line defect

$$\Phi \equiv \frac{1}{\sqrt{m}} \sum_{\alpha=1}^{m} (\phi^\alpha)^2 \ , \qquad \Psi \equiv \frac{1}{\sqrt{N-m}} \sum_{i=m+1}^{N} (\phi^i)^2 \ , \qquad \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = Z_S \begin{pmatrix} [\Phi] \\ [\Psi] \end{pmatrix}$$

Apparently, once the surface coupling is turned on, at the quantum level, the two operators stop to be orthogonal to be each other, as the wavefunction renormalization matrix acquires off-diagonal entries. We can define a matrix for the anomalous dimension

$$\gamma \equiv Z_S^{-1} \frac{\mathrm{d}}{\mathrm{d}\log M} Z_S \Big|_{\text{fixed point}}$$

Its eigenvalues give rise to the anomalous dimensions of two new defect operators

$$\gamma_\pm = \frac{48(g_* + g'_*) + (N+4)h_* \pm \sqrt{192\Delta_g(12\Delta_g + mh_*) - 96N\Delta_g h_* + N^2 h_*^2}}{96\pi}$$

Considering the non-trivial O(N) symmetric fixed point, $\gamma_\pm \Big|_{P_1} = \frac{12 - N \pm N}{8+N}\epsilon$

Subdefect C-theorem

C - theorems

We can define a function, normally called a c-function after Zamolodchikov. This function has monotonic properties along the RG flow. In 2d, it coincides with the central charge s.t. it depicts the d.o.f. of the theory at different energy scales and UV has more d.o.f. than the IR. There are three versions of C-theorem

- Weak version: C-function is non-increasing compared at the UV and the IR fixed points connected by an RG flow;
- Strong version: C-function is non-increasing along the RG flow;
- Strongest version: the RG flow is a gradient flow of the C-function.

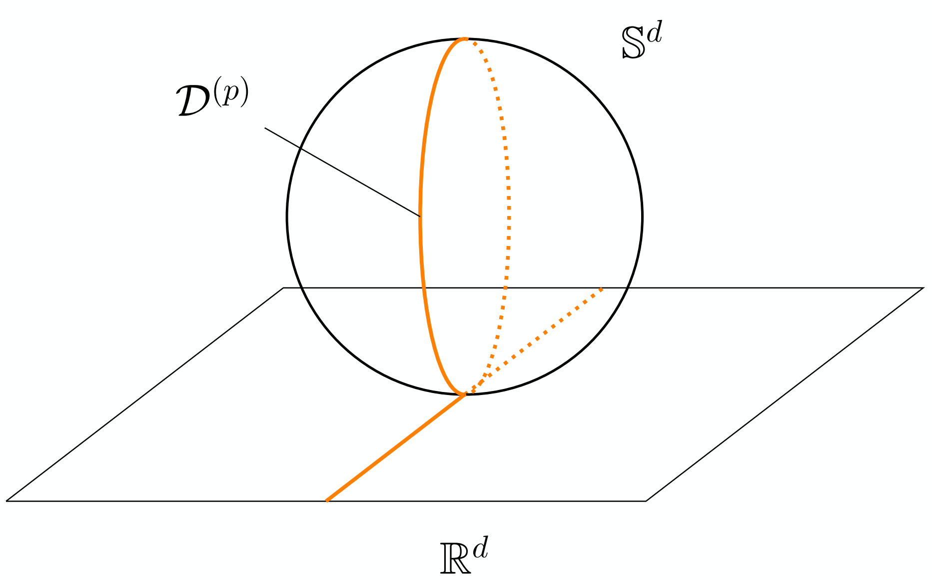
A conjecture of the sub-defect C - theorem

Conjecture. In a unitary CFT$_d$ with a composite defect $\mathcal{D}^{(p_1,\cdots,p_n)} = \cup_{i=1}^n \mathcal{D}^{(p_i)}$ consisting of n sub-defects of p_i dimensions satisfying $0 < p_1 < p_2 < \cdots < p_n < d$ and $\mathcal{D}^{(p_1)} \subset \mathcal{D}^{(p_2)} \subset \cdots \subset \mathcal{D}^{(p_n)}$, let $Z^{(p_1,\cdots,p_n)} \equiv \langle \mathcal{D}^{(p_1,\cdots,p_n)} \rangle$ be the partition function on a d-sphere. Then, the function $\mathcal{C}$ defined by

$$\mathcal{C} \equiv \sin\left(\frac{\pi p_1}{2}\right) \log \left| \frac{Z^{(p_1,p_2,\cdots,p_n)}}{Z^{(p_2,\cdots,p_n)}} \right| \tag{4.1}$$

does not increase under any localized RG flow on the sub-defect $\mathcal{D}^{(p_1)}$ of the lowest dimension,

$$\mathcal{C}_{\mathrm{UV}} \geq \mathcal{C}_{\mathrm{IR}} \ . \tag{4.2}$$



Argument from conformal perturbation theory

Let us consider the following deformation localized on the submost defect,

$$I^{(p_1,\cdots,p_n)} = I^{(p_1,\cdots,p_n)}_{\text{DCFT}} + \tilde{\lambda}_0 \int \mathrm{d}^{p_1}\tilde{x}\,\sqrt{\tilde{g}}\,\widetilde{\mathcal{O}}(\tilde{x})\ , \qquad \boxed{\widetilde{\Delta} = p_i - \varepsilon}$$

Adopting the Wilsonian renormalization procedure, we can find the

renormalized coupling,

$$\begin{aligned}\tilde{\lambda}(\mu) &= \tilde{\lambda}_0 - \frac{\tilde{\lambda}_0^2}{2}\int_{\mu_{\text{UV}}^{-1}<|\tilde{x}_1-\tilde{x}_2|<\mu^{-1}} \mathrm{d}^{p_1}\tilde{x}_2\,\frac{\widetilde{C}}{|\tilde{x}_1-\tilde{x}_2|^{\widetilde{\Delta}}} + \cdots \\ &= \tilde{\lambda}_0 - \tilde{\lambda}_0^2\,\frac{\pi^{\frac{p_1}{2}}\,\widetilde{C}}{\varepsilon\,\Gamma\left(\frac{p_1}{2}\right)}\left[\mu^{-\varepsilon} - \mu_{\text{UV}}^{-\varepsilon}\right] + \cdots\ .\end{aligned}$$

Expressing in terms of the dimensionless parameter

$$\tilde{g}(\mu) = \tilde{g}_0\left(\frac{\mu_{\text{UV}}}{\mu}\right)^{\varepsilon} - \tilde{g}_0^2\,\frac{\pi^{\frac{p_1}{2}}\,\widetilde{C}}{\varepsilon\,\Gamma\left(\frac{p_1}{2}\right)}\left[\left(\frac{\mu_{\text{UV}}}{\mu}\right)^{2\varepsilon} - \left(\frac{\mu_{\text{UV}}}{\mu}\right)^{\varepsilon}\right] + \cdots$$

Then the difference of the partition functions up to third order

$$\begin{aligned}\delta\log\,Z^{(p_1,\cdots,p_n)}(\tilde{g}) &\equiv \log\,Z^{(p_1,\cdots,p_n)}(\tilde{g}) - \log\,Z^{(p_1,\cdots,p_n)}(0) \\ &= \frac{2\,\pi^{p_1+1}}{\sin\left(\frac{\pi p_1}{2}\right)\,\Gamma(p_1+1)}\left[-\frac{\varepsilon}{2}\,\tilde{g}^2 + \frac{\pi^{\frac{p_1}{2}}\,\tilde{C}}{3\,\Gamma\left(\frac{p_1}{2}\right)}\,\tilde{g}^3 + O(\tilde{g}^4)\right]\end{aligned}$$

This gives rise to the C-function

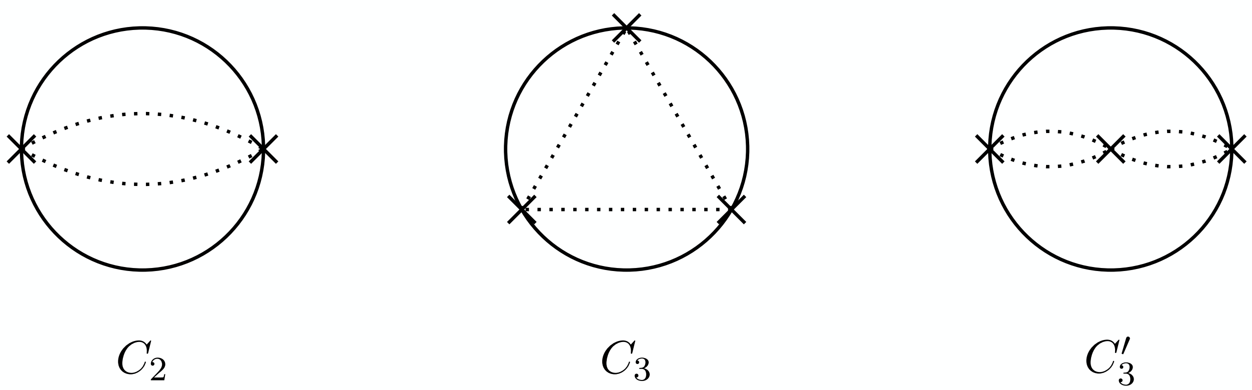
$$\mathcal{C}(\tilde{g}_*) - \mathcal{C}(\tilde{g}=0) = -\frac{\pi\,\Gamma\left(\frac{p_1}{2}\right)^2}{3\,\Gamma(p_1+1)}\,\frac{\varepsilon^3}{\tilde{C}^2} + O(\varepsilon^4)$$

Perturbative test in the d = 3 - ε model

The conjectured sub-defect C-function for our model becomes

$$\mathcal{C} = \ln \frac{Z(h_0, g_0, g_0')}{Z(h_0, 0, 0)} = C_2 + C_3 + C_3' + (\text{higher order})$$

Diagrammatically,



$$\mathcal{C} = -\frac{m\, g^3 + (N-m)\, g'^3}{192\,\pi} + \frac{m\, g\, \beta_g + (N-m)\, g'\, \beta_{g'}}{32}$$

Comments:

- Stationary around the fixed points, $\frac{\partial \mathcal{C}}{\partial g} = -\frac{m}{16}\,\beta_g\,, \qquad \frac{\partial \mathcal{C}}{\partial g'} = -\frac{N-m}{16}\,\beta_{g'}$
- Agrees with Local RG analysis, comparing the two O(N) symmetric fixed points $\mathcal{C}(P_1) - \mathcal{C}(P_0) = \frac{\pi^2\, N\,(N-4)^3}{24\,(N+8)^3}\epsilon^3 + O(\epsilon^4)$

Future questions:

- Other models supporting conformal composite defects ? In d=4 - ε , there is still room for other combinations

- Introducing bulk interactions ?

[Jensen & O'bannon '15]
[Cuomo, Komargodski & Raviv-Moshe '18]
[Wang '21]

- Proofs for the sub-defect C-theorems, perhaps for 1,2,4 dimensions ?

ありがとう