Interplay of Generalised Symmetry and Moduli Spaces in 3d $\mathcal{N} \geq 4$ SCFTs

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- **3d $\mathcal{N} \geq 5$ Theories**
- Reflection groups and moduli spaces
- Mixed anomaly between zero-from and one-form symmetry
- **3d $\mathcal{N} = 4$ Theories**
- Mirror symmetry
- Electric and magnetic one-form symmetry

Main objective: show what role generalised symmetry and their anomaly play on moduli spaces.
[Sebastiano Garavaglia, William Harding, DL, Noppadol Mekareeya 2511.01970]
[Julius F. Grimminger, Amihay Hanany, DL 2503.08791]

Introduction

- In three spacetime dimensions, gauging of a one-form symmetry yields a zero-form symmetry, and vice-versa.
- One-form symmetry and zero-form symmetry can have mixed t'Hooft anomaly.
- It is not clear how the generalised symmetries, especially their anomalies, enter the moduli space data.
- 3d $\mathcal{N} = 5$ theories have both rich generalised symmetries and simple moduli spaces.

Moduli Space of 3d $\mathcal{N} \geq 5$ theories

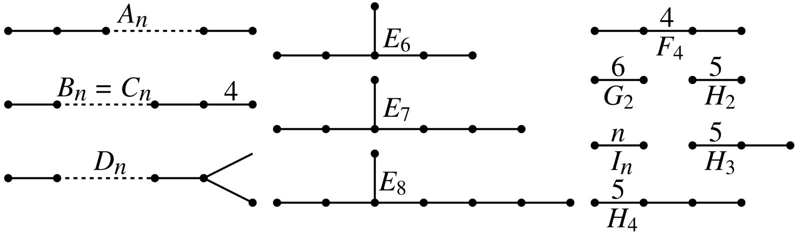
- The moduli spaces of 3d $\mathcal{N} \geq 5$ theories are conjectured to be classified by reflection groups. Their moduli spaces are orbifolds of type $\mathbb{H}^{2N}/\Gamma$, where Γ is a reflection group or its extension by a finite Abelian group.
- $\mathcal{N} = 8$, real reflection groups, which are classified by Coxeter.
- $\mathcal{N} = 6$, complex reflection groups, which are classified by Shephard-Todd. [Yuji Tachikawa, Gabi Zafrir 1908.03346]
- $\mathcal{N} = 5$, quaternionic/symplectic reflection groups, which are classified by Cohen. [Anirudh Deb, Gabi Zafrir 2403.03971]
- $\mathcal{N} = 4$, more subtle but still has clear relation with reflection groups, see discussion later.

Reflection groups

- On a vector space V, a reflection is an involution $\rho \in \mathrm{GL}(V)$, such that $\mathrm{rk}(\mathrm{id} - \rho) = 1$.
- It is called real/complex/quaternionic reflection for $V = \mathbb{R}^N/\mathbb{C}^N/\mathbb{H}^N$.
- Quaternionic reflection can also be defined as symplectic reflection: on a complex vector space with symplectic structure (V, ω), a symplectic reflection is defined by an involution $\rho \in \mathrm{Sp}(V)$, such that $\mathrm{rk}(\mathrm{id} - \rho) = 2$.
- Groups generated by reflections are called reflection groups. Not an abstract group but a matrix group with its representation specified.

Classification of reflection groups: real

- Coxeter diagram: A_N, $B_N = C_N$, D_N, $E_{6,7,8}$, F_4, G_2, I_N, $H_{2,3,4}$



- For example, $\mathcal{W}_{A_N} \cong S_N$ on $\mathbb{R}^N$ are generated by $N-1$ transpositions. Can check the following matrix has rank 1.

$$\mathrm{id} - \begin{pmatrix} 0 & 1 & \cdots \\ 1 & 0 & \\ \vdots & & \ddots \end{pmatrix} \tag{1}$$

Classification of reflection groups: complex

- Real reflections on $\mathbb{C}^N$ are naturally complex reflection.
- A three parameter family: $G(k, p, N)$, where $k|p$. 34 exceptional cases $G_4 - G_{37}$.
- $G(k, p, N) \cong \left(\mathbb{Z}_k^{N-1} \times \mathbb{Z}_{k/p}\right) \rtimes S_N$, where $\mathbb{Z}_k = \langle \mathrm{diag}(\omega_k, \omega_k^{-1}, 1, \ldots, 1)\rangle$, $\mathbb{Z}_{k/p} = \langle \mathrm{diag}(\omega_{k/p}, 1, \ldots, 1)\rangle$ and S_N act as the permutation.
- For example, $G(k, k, 2) \cong \mathrm{Dih}_k$ on $\mathbb{C}^2$ are generated by $\mathrm{diag}(\omega_k, \omega_k^{-1})$ and transposition. A reflection basis of the generators can be chosen, such that the following matrices have rank 1.

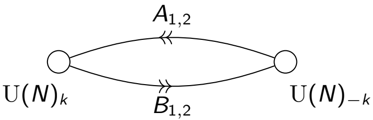
$$\mathrm{id} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathrm{id} - \begin{pmatrix} 0 & \omega_k \\ \omega_k^{-1} & 0 \end{pmatrix} \qquad (2)$$

Classification of reflection groups: symplectic

- Complex reflections on $T^*\mathbb{C}^N$ are naturally symplectic reflection. If $\mathrm{rk}(\mathrm{id}-\rho)=1$, then $\mathrm{rk}(\mathrm{id}-\rho\oplus\rho^{-T})=2$.
- A three parameter family: $G_N(K,H)$, where K is a finite ADE subgroup of SU(2), and H is a normal subgroup of K with the additional restriction that K/H has to be Abelian if $N\geq 3$. 16 exceptional cases.
- $G_N(K,H)\cong(K^{N-1}\times H)\rtimes S_N$. Example will be provided later with 3d $\mathcal{N}=5$ theories.
- Rank 1 symplectic reflection groups are the ADE subgroups of $\mathrm{SU}(2)$.

Moduli space of 3d $\mathcal{N} = 6$ theory as orbifold of complex reflection group

The quiver of equal rank ABJ(M) theory reads:



(3)

- Chiral fields A_i, B_i with conformal dimension 1, where $i = 1, 2$. A_i in bifundamental rep, B_i in anti-bifundamental rep.
- Monopole operators $V_{m,m'}$ with conformal dimension 0, where m is the magnetic flux. To be compatible with CS level $\pm k$, the allowed rep of V are those with highest weights divided by k, i.e. $q^{a_0 k}[a_1 k, a_2 k, \ldots, a_{N-1} k]$ for each $\mathrm{U}(N)$.

Moduli space of 3d $\mathcal{N}=6$ theory as orbifold of complex reflection group

- Vanishing of scalar potential:

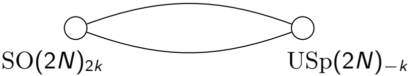
$$A_i, B_i \in \mathfrak{h}_{\mathrm{diag}}. \tag{4}$$

- Diagonal $\mathrm{SU}(N)$ reduces to its Weyl group S_n.
- The monopole operators can composite with A_i, B_i.
- Off-diagonal $\mathrm{U}(1)^N$ reduces to $\mathrm{U}(1)^N/\left(k\mathrm{U}(1)\right)^N \cong \mathbb{Z}_k^N$.
- The moduli space is $\mathbb{C}^{4N}/G(k,1,N)$, where $G(k,1,N)$ is an complex reflection group on $\mathbb{C}^N$ with natural uplift to $\mathbb{C}^{4N} \cong (T^*\mathbb{C}^N)^2$.
- $\left(\mathrm{U}(N+x)_k \times \mathrm{U}(N)_{-k}\right)/\mathbb{Z}_p$ has moduli space $\mathbb{C}^{4N}/G(k,p,N)$.

Remarks on 3d $\mathcal{N} = 6$

- There are theories whose moduli space is not an orbifold of complex reflection group but its finite abelian extension.
- For examples, $\left(\mathrm{SU}(N)_k \times \mathrm{SU}(N)_{-k}\right)/\mathbb{Z}_m$, $\Gamma \cong G(k,k,N) \rtimes \mathbb{Z}_{N/m}$ is not an uplift of complex reflection group in general.
- The moduli space of "local oldest" QFT is an orbifold of complex reflection group.

The quiver of equal rank OSp ABJ theory reads:



(5)

- Similar argument can be made to show its moduli space is an orbifold $\mathbb{C}^{4N}/G_N(\hat{D}_k, \mathbb{Z}_{2k})$, where $G_N(\hat{D}_k, \mathbb{Z}_{2k}) \cong (\mathbb{Z}_{2k}^N) \rtimes (\mathbb{Z}_2^{N-1} \rtimes S_N) \cong (\hat{D}_k^{N-1} \times \mathbb{Z}_{2k}) \rtimes S_N$ is an symplectic reflection group $\mathbb{C}^{2N} \cong T^*\mathbb{C}^N$ with natural uplift to $\mathbb{C}^{4N} \cong (T * \mathbb{C}^N)^2$.

Local family of discrete $\mathbb{Z}_2$ gauging

- $\mathbb{Z}_{2,\mathcal{M}}^{[0]}$: magnetic zero-form symmetries, associated with the $\mathrm{SO}(2N)$ gauge node.
- $\mathbb{Z}_{2,\mathcal{C}}^{[0]}$: charge conjugation zero-form symmetries, associated with the $\mathrm{SO}(2N)$ gauge node.
- $\mathbb{Z}_{2,B}^{[1]}$: one-form symmetries, diagonal subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$ center of $\mathrm{SO}(2N) \times \mathrm{USp}(2N)$, off-diagonal $\mathbb{Z}_2$ is screened by the bifundamental matter.
- Their mixed t'Hooft anomaly is described by the following action in a 4d bulk [O.Bergman, F.Mignosa 2412.00184][Y.Tachikawa 1712.09542]:

$$i\pi \int_{M_4} \mathcal{A}_2^B \cup \left[N\mathcal{A}_1^{\mathcal{M}} \cup \mathcal{A}_1^{\mathcal{M}} + k\mathcal{A}_1^{\mathcal{C}} \cup \mathcal{A}_1^{\mathcal{C}} + \mathcal{A}_1^{\mathcal{M}} \cup \mathcal{A}_1^{\mathcal{C}} \right] . \tag{6}$$

- The anomaly depends on even/odd of N and k.

Symmetry webs: D_8

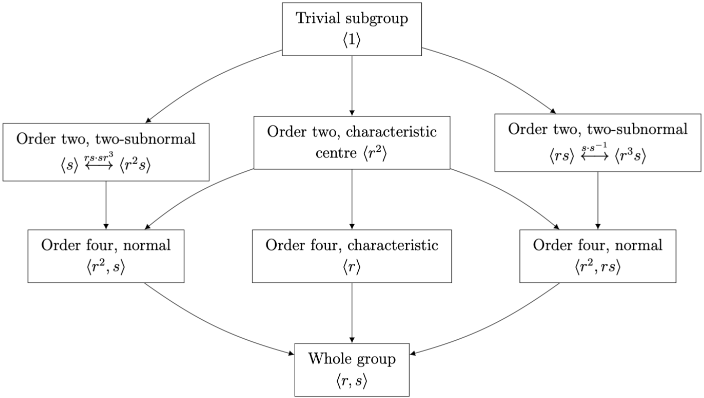


Figure 1: Lattice of subgroups of D_8, where each box represents a distinct conjugacy class of subgroups. Observe that the order two two-subnormal subgroups enjoy an inner automorphism generated by conjugations $x = sxs^{-1} = sxs$, $y = (rs)y(sr^3) = (rs)y(rs)$, where $x \in \langle rs\rangle$ and $y \in \langle s\rangle$.

Symmetry webs: Q_8

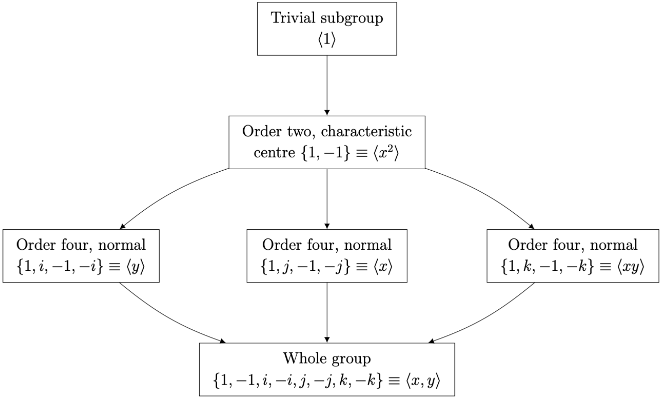
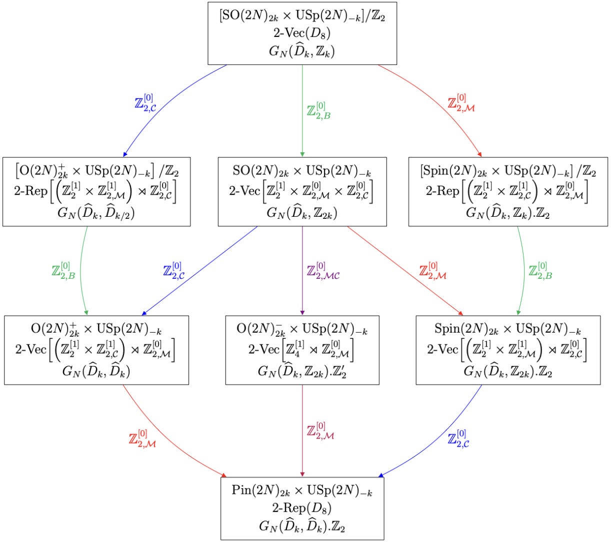
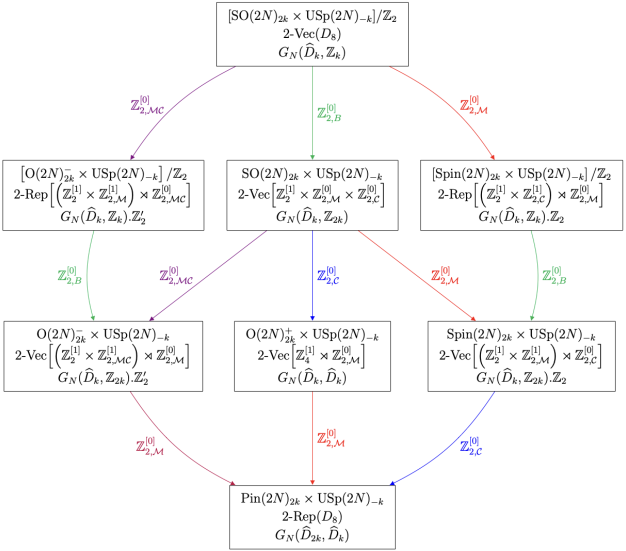


Figure 2: Lattice of subgroups of Q_8, where each box represents a distinct subgroup.

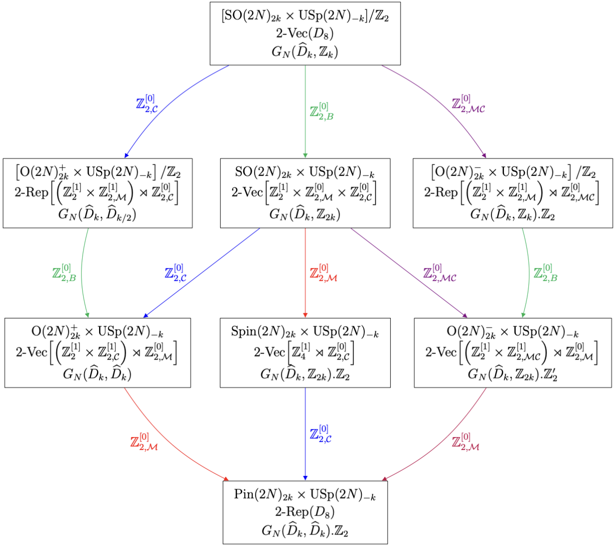
Gauging webs: Even N even k



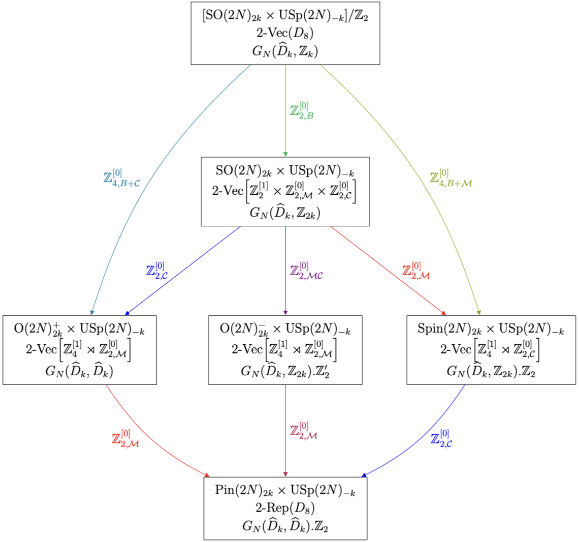
Gauging webs: Even N odd k



Gauging webs: Odd N even k



Gauging webs: Odd N odd k



Anomalous variant

Question

Can $\mathbb{Z}_{2,\mathcal{C}}^{[0]}$, $\mathbb{Z}_{2,\mathcal{M}}^{[0]}$ or $\mathbb{Z}_{2,\mathcal{MC}}^{[0]}$ be gauged after the gauging of $\mathbb{Z}_{2,B}^{[1]}$?

Parity of N	Parity of k	Figure	Anomalous variant
Even	Even	3	$[\mathrm{O}(2N)^{-}_{2k} \times \mathrm{USp}(2N)_{-k}]/\mathbb{Z}_2$ $[\mathrm{Pin}(2N)_{2k} \times \mathrm{USp}(2N)_{-k}]/\mathbb{Z}_2$
Even	Odd	4	$[\mathrm{O}(2N)^{+}_{2k} \times \mathrm{USp}(2N)_{-k}]/\mathbb{Z}_2$ $[\mathrm{Pin}(2N)_{2k} \times \mathrm{USp}(2N)_{-k}]/\mathbb{Z}_2$
Odd	Even	5	$[\mathrm{Spin}(2N)_{2k} \times \mathrm{USp}(2N)_{-k}]/\mathbb{Z}_2$ $[\mathrm{Pin}(2N)_{2k} \times \mathrm{USp}(2N)_{-k}]/\mathbb{Z}_2$
Odd	Odd	6	$[\mathrm{O}(2N)^{\pm}_{2k} \times \mathrm{USp}(2N)_{-k}]/\mathbb{Z}_2$ $[\mathrm{Spin}(2N)_{2k} \times \mathrm{USp}(2N)_{-k}]/\mathbb{Z}_2$ $[\mathrm{Pin}(2N)_{2k} \times \mathrm{USp}(2N)_{-k}]/\mathbb{Z}_2$

Generators of symplectic reflection group

- Let us define the following 2×2 matrices:

$$I = \mathrm{diag}(1,1) \ , \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \quad E_n = \mathrm{diag}(\omega_n, \omega_n^{-1}) \ . \tag{7}$$

- Generators of $\hat{D}_k$:

$$R_1 = \mathrm{diag}(J, J^{-1}, I, \ldots, I) \ , \qquad R_2 = \mathrm{diag}(E_{2k}, E_{2k}^{-1}, I, \ldots, I) \ . \tag{8}$$

- For the $N = 2$ case, we also define a symplectic reflection:

$$R_3 = \mathrm{diag}(E_k, I) \ . \tag{9}$$

Generators of $\mathbb{Z}_2$ gauging

[S.Garavaglia, W.Harding, DL, N.Mekareeya 2511.01970]

Suppose that we gauge a non-anomalous $\mathbb{Z}_{2,S}^{[0]}$ symmetry, where S takes values in $\{B, \mathcal{M}, \mathcal{C}, \mathcal{MC}\}$, in a theory $\mathcal{T}$ whose moduli space is $\mathbb{C}^{4N}/\Gamma$, and obtain a new non-anomalous theory $\mathcal{T}'$ whose moduli space is $\mathbb{C}^{4N}/\Gamma'$. Then, Γ' can be constructed by inserting the extra generator R_S into the set of generators of the group Γ, where we define

$$\begin{aligned} R_B &= \mathrm{diag}(E_{2k}, I, \ldots, I) \ , \\ R_{\mathcal{M}} &= \mathrm{diag}(E_{4k}, E_{4k}, \ldots, E_{4k}) \ , \\ R_{\mathcal{C}} &= \mathrm{diag}(J, I, \ldots, I) \ , \\ R_{\mathcal{MC}} &= R_{\mathcal{M}} R_{\mathcal{C}} \ . \end{aligned} \tag{10}$$

Rank 2 symplectic reflection groups

Group	Theory	Generators	Reduced generators	Order
$G_2(\widehat{D}_k, \mathbb{I})$		P_1, R_1, R_2	P_1, R_1, R_2	$8k$
$G_2(\widehat{D}_k, \mathbb{Z}_k)$	$[\mathrm{SO}(4)_{2k} \times \mathrm{USp}(4)_{-k}]/\mathbb{Z}_2$	P_1, R_1, R_2, R_3	P_1, R_1, R_2, R_3	$8k^2$
$G_2(\widehat{D}_k, \mathbb{Z}_{2k})$	$\mathrm{SO}(4)_{2k} \times \mathrm{USp}(4)_{-k}$	$P_1, R_1, R_2, R_3, R_{\mathcal{B}}$	$P_1, R_1, R_{\mathcal{B}}$	$16k^2$
$G_2(\widehat{D}_k, \widehat{D}_{k/2})$	$[\mathrm{O}(4)^+_4 \times \mathrm{USp}(4)_{-k}]/\mathbb{Z}_2$	$P_1, R_1, R_2, R_3, R_{\mathcal{C}}$	$P_1, R_2, R_{\mathcal{C}}$	$16k^{2*}$
$G_2(\widehat{D}_{2k}, \mathbb{Z}_k)$	$[\mathrm{Spin}(4)_{2k} \times \mathrm{USp}(4)_{-k}]/\mathbb{Z}_2$	$P_1, R_1, R_2, R_3, \widetilde{R}_{\mathcal{M}}$	$P_1, R_1, R_3, \widetilde{R}_{\mathcal{M}}$	$16k^2$
$G_2(\widehat{D}_k, \mathbb{Z}_k).\mathbb{Z}_2'$	$[\mathrm{O}(4)^-_{2k} \times \mathrm{USp}(4)_{-k}]/\mathbb{Z}_2$	$P_1, R_1, R_2, R_3, \widetilde{R}_{\mathcal{MC}}$	$P_1, R_1, \widetilde{R}_{\mathcal{MC}}$	$16k^{2**}$
$G_2(\widehat{D}_k, \widehat{D}_k)$	$\mathrm{O}(4)^+_{2k} \times \mathrm{USp}(4)_{-k}$	$P_1, R_1, R_2, R_3, R_{\mathcal{B}}, R_{\mathcal{C}}$	$P_1, R_{\mathcal{B}}, R_{\mathcal{C}}$	$32k^2$
$G_2(\widehat{D}_{2k}, \mathbb{Z}_{2k})$	$\mathrm{Spin}(4)_{2k} \times \mathrm{USp}(4)_{-k}$	$P_1, R_1, R_2, R_3, R_{\mathcal{B}}, \widetilde{R}_{\mathcal{M}}$	$P_1, R_1, R_{\mathcal{B}}, \widetilde{R}_{\mathcal{M}}$	$32k^2$
$G_2(\widehat{D}_k, \mathbb{Z}_{2k}).\mathbb{Z}_2'$	$\mathrm{O}(4)^-_{2k} \times \mathrm{USp}(4)_{-k}$	$P_1, R_1, R_2, R_3, R_{\mathcal{B}}, \widetilde{R}_{\mathcal{MC}}$	$P_1, R_1, R_{\mathcal{B}}, \widetilde{R}_{\mathcal{MC}}$	$32k^2$
$G_2(\widehat{D}_{2k}, \widehat{D}_k)$	$\mathrm{Pin}(4)_{2k} \times \mathrm{USp}(4)_{-k}$	$P_1, R_1, R_2, R_3, R_{\mathcal{B}}, R_{\mathcal{C}}, \widetilde{R}_{\mathcal{M}}$	$P_1, R_{\mathcal{B}}, R_{\mathcal{C}}, \widetilde{R}_{\mathcal{M}}$	$64k^2$

Remarks on the $N = 2$ cases:

* This case applies only when k is even. For odd k, the presence of $R_{\mathcal{C}}$ (along with P_1, R_1, R_2, R_3) implies the presence of R_B, since $R_B = R_{\mathcal{C}}^2 R_3$; thus, the group becomes $G_2(\widehat{D}_k, \widehat{D}_k)$ with order $32k^2$. This implies that $[\mathrm{O}(4)^+_{2k} \times \mathrm{USp}(4)_{-k}]/\mathbb{Z}_2$ is anomalous for odd k.

** This case applies only when k is odd. For even k, the presence of $\widetilde{R}_{\mathcal{MC}}$ (along with P_1, R_1, R_2, R_3) implies the presence of R_B, since we have $R_B = P_1(\widetilde{R}_{\mathcal{MC}})^{-2k} P_1^{-1} (\widetilde{R}_{\mathcal{MC}})^{2k}$; thus the group becomes $G_2(\widehat{D}_k, \mathbb{Z}_{2k}).\mathbb{Z}_2'$ with order $32k^2$. This implies that $[\mathrm{O}(4)^-_{2k} \times \mathrm{USp}(4)_{-k}]/\mathbb{Z}_2$ is anomalous for even k.

Rank 3 symplectic reflection groups

Group	Theory	Generators	Reduced generators	Order
$G_N(\widehat{D}_k, \mathbb{Z}_k)$	$[\mathrm{SO}(2N)_{2k} \times \mathrm{USp}(2N)_{-k}]/\mathbb{Z}_2$	$\{P_j\}, R_1, R_2$	$\{P_j\}, R_1, R_2$	$k \times (4k)^{N-1} \times N!$
$G_N(\widehat{D}_k, \mathbb{Z}_{2k})$	$\mathrm{SO}(2N)_{2k} \times \mathrm{USp}(2N)_{-k}$	$\{P_j\}, R_1, R_2, R_B$	$\{P_j\}, R_1, R_B$	$2k \times (4k)^{N-1} \times N!$
$G_N(\widehat{D}_k, \widehat{D}_{k/2})$	$[\mathrm{O}(2N)^+_{2k} \times \mathrm{USp}(2N)_{-k}]/\mathbb{Z}_2$	$\{P_j\}, R_1, R_2, R_C$	$\{P_j\}, R_2, R_C$	$2k \times (4k)^{N-1} \times N!^*$
$G_N(\widehat{D}_k, \mathbb{Z}_k).\mathbb{Z}_2$	$[\mathrm{Spin}(2N)_{2k} \times \mathrm{USp}(2N)_{-k}]/\mathbb{Z}_2$	$\{P_j\}, R_1, R_2, R_{\mathcal{M}}$	$\{P_j\}, R_1, R_{\mathcal{M}}$	$2k \times (4k)^{N-1} \times N!^\dagger$
$G_N(\widehat{D}_k, \mathbb{Z}_k).\mathbb{Z}_2'$	$[\mathrm{O}(2N)^-_{2k} \times \mathrm{USp}(2N)_{-k}]/\mathbb{Z}_2$	$\{P_j\}, R_1, R_2, R_{\mathcal{MC}}$	$\{P_j\}, R_1, R_{\mathcal{MC}}$	$2k \times (4k)^{N-1} \times N!^{**}$
$G_N(\widehat{D}_k, \widehat{D}_k)$	$\mathrm{O}(2N)^+_{2k} \times \mathrm{USp}(2N)_{-k}$	$\{P_j\}, R_1, R_2, R_B, R_C$	$\{P_j\}, R_B, R_C$	$(4k)^N \times N!$
$G_N(\widehat{D}_k, \mathbb{Z}_{2k}).\mathbb{Z}_2$	$\mathrm{Spin}(2N)_{2k} \times \mathrm{USp}(2N)_{-k}$	$\{P_j\}, R_1, R_2, R_B, R_{\mathcal{M}}$	$\{P_j\}, R_1, R_B, R_{\mathcal{M}}$	$(4k)^N \times N!$
$G_N(\widehat{D}_k, \mathbb{Z}_{2k}).\mathbb{Z}_2'$	$\mathrm{O}(2N)^-_{2k} \times \mathrm{USp}(2N)_{-k}$	$\{P_j\}, R_1, R_2, R_B, R_{\mathcal{MC}}$	$\{P_j\}, R_1, R_B, R_{\mathcal{MC}}$	$(4k)^N \times N!$
$G_N(\widehat{D}_k, \widehat{D}_k).\mathbb{Z}_2$	$\mathrm{Pin}(2N)_{2k} \times \mathrm{USp}(2N)_{-k}$	$\{P_j\}, R_1, R_2, R_B, R_C, R_{\mathcal{M}}$	$\{P_j\}, R_B, R_C, R_{\mathcal{M}}$	$2 \times (4k)^N \times N!$

Rank 3 symplectic reflection groups

* This case is valid only when k is even. For odd k, the presence of $R_{\mathcal{C}}$ (along with $\{P_j\}, R_1, R_2$) implies the presence of R_B, since R_B can be constructed from R_2, $R_{\mathcal{C}}$ and certain P_j; the group becomes $G_N(\widehat{D}_k, \widehat{D}_k)$ with order $N!(4k)^N$. This implies that $[\mathrm{O}(2N)^+_{2k} \times \mathrm{USp}(2N)_{-k}]/\mathbb{Z}_2$ is anomalous for odd k.

† This case is valid only when N is even. For odd N, the presence of $R_{\mathcal{M}}$ (along with $\{P_j\}, R_1, R_2$) implies the presence of R_B, since R_B can be constructed from R_1, $R_{\mathcal{M}}$ and certain P_j; the group becomes $G_N(\widehat{D}_k, \mathbb{Z}_{2k}).\mathbb{Z}_2$ with order $N!(4k)^N$. This implies that $[\mathrm{Spin}(2N)_{2k} \times \mathrm{USp}(2N)_{-k}]/\mathbb{Z}_2$ is anomalous for odd N.

** This case is valid only when (N, k) is (even, odd) or (odd, even). For other parities, the presence of $R_{\mathcal{MC}}$ (along with $\{P_j\}, R_1, R_2$) implies the presence of R_B, since R_B can be constructed from R_1, $R_{\mathcal{M}}R_{\mathcal{C}}$ and certain P_j, and the group becomes $G_N(\widehat{D}_k, \mathbb{Z}_{2k}).\mathbb{Z}_2'$ with order $N!(4k)^N$. This implies that $[\mathrm{O}(2N)^-_{2k} \times \mathrm{USp}(2N)_{-k}]/\mathbb{Z}_2$ is anomalous when N and k are both even or both odd.

Non-equal rank

Theory	N even, k even	N odd, k even	N even, k odd	N odd, k odd
$[\mathrm{SO}(2N)_{2k} \times \mathrm{USp}(2N+2x)_{-k}]/\mathbb{Z}_2$	$G_N(\widehat{D}_k, \mathbb{Z}_{2k})$	$G_N(\widehat{D}_k, \mathbb{Z}_{2k})$	$G_N(\widehat{D}_k, \mathbb{Z}_{2k})$	$G_N(\widehat{D}_k, \mathbb{Z}_{2k})$
$\mathrm{SO}(2N)_{2k} \times \mathrm{USp}(2N+2x)_{-k}$	$G_N(\widehat{D}_k, \mathbb{Z}_{2k})$	$G_N(\widehat{D}_k, \mathbb{Z}_{2k})$	$G_N(\widehat{D}_k, \mathbb{Z}_{2k})$	$G_N(\widehat{D}_k, \mathbb{Z}_{2k})$
$[\mathrm{O}(2N)^+_{2k} \times \mathrm{USp}(2N+2x)_{-k}]/\mathbb{Z}_2$	$G_N(\widehat{D}_k, \widehat{D}_k)$	$G_N(\widehat{D}_k, \widehat{D}_k)$	Anomalous	Anomalous
$[\mathrm{Spin}(2N)_{2k} \times \mathrm{USp}(2N+2x)_{-k}]/\mathbb{Z}_2$	$G_N(\widehat{D}_k, \mathbb{Z}_{2k}).\mathbb{Z}_2$	Anomalous	$G_N(\widehat{D}_k, \mathbb{Z}_{2k}).\mathbb{Z}_2$	Anomalous
$[\mathrm{O}(2N)^-_{2k} \times \mathrm{USp}(2N+2x)_{-k}]/\mathbb{Z}_2$	Anomalous	$G_N(\widehat{D}_k, \mathbb{Z}_{2k}).\mathbb{Z}_2'$	$G_N(\widehat{D}_k, \mathbb{Z}_{2k}).\mathbb{Z}_2'$	Anomalous
$\mathrm{O}(2N)^+_{2k} \times \mathrm{USp}(2N+2x)_{-k}$	$G_N(\widehat{D}_k, \widehat{D}_k)$	$G_N(\widehat{D}_k, \widehat{D}_k)$	$G_N(\widehat{D}_k, \widehat{D}_k)$	$G_N(\widehat{D}_k, \widehat{D}_k)$
$\mathrm{Spin}(2N)_{2k} \times \mathrm{USp}(2N+2x)_{-k}$	$G_N(\widehat{D}_k, \mathbb{Z}_{2k}).\mathbb{Z}_2$	$G_N(\widehat{D}_k, \mathbb{Z}_{2k}).\mathbb{Z}_2$	$G_N(\widehat{D}_k, \mathbb{Z}_{2k}).\mathbb{Z}_2$	$G_N(\widehat{D}_k, \mathbb{Z}_{2k}).\mathbb{Z}_2$
$\mathrm{O}(2N)^-_{2k} \times \mathrm{USp}(2N+2x)_{-k}$	$G_N(\widehat{D}_k, \mathbb{Z}_{2k}).\mathbb{Z}_2'$	$G_N(\widehat{D}_k, \mathbb{Z}_{2k}).\mathbb{Z}_2'$	$G_N(\widehat{D}_k, \mathbb{Z}_{2k}).\mathbb{Z}_2'$	$G_N(\widehat{D}_k, \mathbb{Z}_{2k}).\mathbb{Z}_2'$
$\mathrm{Pin}(2N)_{2k} \times \mathrm{USp}(2N+2x)_{-k}$	$G_N(\widehat{D}_k, \widehat{D}_k).\mathbb{Z}_2$	$G_N(\widehat{D}_k, \widehat{D}_k).\mathbb{Z}_2$	$G_N(\widehat{D}_k, \widehat{D}_k).\mathbb{Z}_2$	$G_N(\widehat{D}_k, \widehat{D}_k).\mathbb{Z}_2$

Non-equal rank

Theory	$N+x$ even, k even	$N+x$ odd, k even	$N+x$ even, k odd	$N+x$ odd, k odd
$[\mathrm{SO}(2N+2x)_{2k} \times \mathrm{USp}(2N)_{-k}]/\mathbb{Z}_2$	$G_N(\widehat{D}_k, \widehat{D}_k)$	$G_N(\widehat{D}_k, \widehat{D}_k)$	$G_N(\widehat{D}_k, \widehat{D}_k)$	$G_N(\widehat{D}_k, \widehat{D}_k)$
$\mathrm{SO}(2N+2x)_{2k} \times \mathrm{USp}(2N)_{-k}$	$G_N(\widehat{D}_k, \widehat{D}_k)$	$G_N(\widehat{D}_k, \widehat{D}_k)$	$G_N(\widehat{D}_k, \widehat{D}_k)$	$G_N(\widehat{D}_k, \widehat{D}_k)$
$\left[\mathrm{O}(2N+2x)^+_{2k} \times \mathrm{USp}(2N)_{-k}\right]/\mathbb{Z}_2$	$G_N(\widehat{D}_k, \widehat{D}_k)$	$G_N(\widehat{D}_k, \widehat{D}_k)$	Anomalous	Anomalous
$[\mathrm{Spin}(2N+2x)_{2k} \times \mathrm{USp}(2N)_{-k}]/\mathbb{Z}_2$	$G_N(\widehat{D}_k, \widehat{D}_k).\mathbb{Z}_2$	Anomalous	$G_N(\widehat{D}_k, \widehat{D}_k).\mathbb{Z}_2$	Anomalous
$\left[\mathrm{O}(2N+2x)^-_{2k} \times \mathrm{USp}(2N)_{-k}\right]/\mathbb{Z}_2$	Anomalous	$G_N(\widehat{D}_k, \widehat{D}_k).\mathbb{Z}_2$	$G_N(\widehat{D}_k, \widehat{D}_k).\mathbb{Z}_2$	Anomalous
$\mathrm{O}(2N+2x)^+_{2k} \times \mathrm{USp}(2N)_{-k}$	$G_N(\widehat{D}_k, \widehat{D}_k)$	$G_N(\widehat{D}_k, \widehat{D}_k)$	$G_N(\widehat{D}_k, \widehat{D}_k)$	$G_N(\widehat{D}_k, \widehat{D}_k)$
$\mathrm{Spin}(2N+2x)_{2k} \times \mathrm{USp}(2N)_{-k}$	$G_N(\widehat{D}_k, \widehat{D}_k).\mathbb{Z}_2$	$G_N(\widehat{D}_k, \widehat{D}_k).\mathbb{Z}_2$	$G_N(\widehat{D}_k, \widehat{D}_k).\mathbb{Z}_2$	$G_N(\widehat{D}_k, \widehat{D}_k).\mathbb{Z}_2$
$\mathrm{O}(2N+2x)^-_{2k} \times \mathrm{USp}(2N)_{-k}$	$G_N(\widehat{D}_k, \widehat{D}_k).\mathbb{Z}_2$	$G_N(\widehat{D}_k, \widehat{D}_k).\mathbb{Z}_2$	$G_N(\widehat{D}_k, \widehat{D}_k).\mathbb{Z}_2$	$G_N(\widehat{D}_k, \widehat{D}_k).\mathbb{Z}_2$
$\mathrm{Pin}(2N+2x)_{2k} \times \mathrm{USp}(2N)_{-k}$	$G_N(\widehat{D}_k, \widehat{D}_k).\mathbb{Z}_2$	$G_N(\widehat{D}_k, \widehat{D}_k).\mathbb{Z}_2$	$G_N(\widehat{D}_k, \widehat{D}_k).\mathbb{Z}_2$	$G_N(\widehat{D}_k, \widehat{D}_k).\mathbb{Z}_2$

Anomaly and moduli space

- On the moduli spaces, only anomaly free theory are present.
- The generators automatically detect non-trivial mixed t'Hooft anomaly between $\mathbb{Z}_{2,B}^{[1]}$ and $\mathbb{Z}_{2,S}^{[0]}$.
- Impose the generator R_S is not always a gauging operation.
- The $\mathbb{Z}_2$ we try to "gauge" gives a $\mathbb{Z}_2$-projective rep on the moduli space.
- Generally, $1 \to A \to G' \to G \to 1$, classified by the $H^2(G, A)$.
- Recovers the D_8 or Q_8 structure.

Anomaly and index

- Index and Hilbert series can still be computed for an anomalous theory.
- The result does not correspond to a well defined geometry object.
- It sums over half-orbit instead of the the full-orbit.
- For example, $[\mathrm{O}(2)_2 \times \mathrm{USp}(2)_{-1}]/\mathbb{Z}_2$, the index reads:

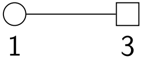
$$I = 1 + x + \left(a^3 g + \frac{g}{a^3} + ag + \frac{g}{a}\right) x^{3/2} + \left(a^4 + \frac{1}{a^4} - a^2 - \frac{1}{a^2} - 1\right) x^2 - \left(a^3 g + \frac{g}{a^3} + ag + \frac{g}{a}\right) x^{5/2} + \dots \,. \tag{11}$$

- Its Higgs (or Coulomb) branch limit gives:

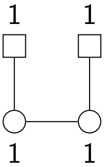
$$\mathrm{HS} = \frac{1 - t^6}{(1-t)(1-t^3)(1-t^4)}. \tag{12}$$

3d $\mathcal{N} = 4$ simple Abelian theories

- Gauge group: $\mathrm{U}(1)^r$.
- Matter fields: n cotangent type hypermultiplets, whose gauge charges can be arranged into a $n \times r$ charge matrix $\mathbf{q} : \mathbb{Z}^r \to \mathbb{Z}^n$.
- For example:



$$\mathbf{q} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{13}$$



$$\mathbf{q} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{pmatrix} \tag{14}$$

Simple charge matrix

- The theory is called simple if $\mathrm{coker}(\mathbf{q})$ has no torsion, i.e. $\mathbb{Z}^r/\mathbf{q}(\mathbb{Z}^n)$ is trivial.
- Equivalently, matrix $\mathbf{q}$ has trivial Smith normal form.
- Smith normal form encodes how much the electric charge lattice is "stretched", thus, it determines the electric one form symmetry (charged by Wlison line).
- Trivial Smith normal form means trivial electric one form symmetry, any Wlison line can be screened by hypermultiplet.
- Two charge matrices related by right multiplication of $\mathrm{SL}^{\pm}(r, \mathbb{Z})$ defines the same theory, $\mathbf{q} \sim \mathbf{q} \cdot A$.
- $\mathbf{q}$ induces an embedding of gauge $\mathrm{U}(1)^r$ into $\mathrm{U}(1)^n$.
- Can always find a $n \times n - r$ matrix $\mathbf{b}$, such that the combined matrix $(\mathbf{q}, \mathbf{b}) \in \mathrm{SL}^{\pm}(n, \mathbb{Z})$. Such matrix $\mathbf{b}$ induces an embedding of flavour $\mathrm{U}(1)^{n-r}$ into $\mathrm{U}(1)^n$.

Higgs and Coulomb branch

- Higgs branch $\mathcal{H}$ is parameterised by vev of the scalars in hypermultiplets, $\mathbb{H}^n///\mathrm{U}(1)^r \cong \mu^{-1}(0)//(\mathbb{C}^\times)^r$. A hypertoric variety.
- Higgs branch $\mathcal{C}$ is parameterised by vev of the scalars in vectormultiplets, or equivalently dressed monopole operators. Classically $(T * \mathbb{C}^\times)^r$, in general receive quantum correction.
- Both Higgs and Coulomb branch are symplectic singularities.
- Higgs and Coulomb branch exchange under 3d mirror symmetry.
- Description of 3d mirror for simple Abelian theory firstly given by [J. de Boer, K. Hori, H. Ooguri, Y. Oz and Z. Yin 1996].

3d mirror symmetry I

- The mirror pair of simple Abelian theory is encoded by the following short exact sequence (Convention in [A. Ballin, T. Creutzig, T. Dimofte and W. Niu 2304.11001]):

$$0 \longrightarrow \mathbb{Z}^r \underset{\mathbf{b}^{\vee,\mathsf{T}}}{\overset{\mathbf{q}}{\rightleftarrows}} \mathbb{Z}^n \underset{\mathbf{b}}{\overset{\mathbf{q}^{\vee,\mathsf{T}}}{\rightleftarrows}} \mathbb{Z}^{n-r} \longrightarrow 0.$$

$$\mathbf{q}^{\vee,\mathsf{T}} \cdot \mathbf{q} = 0_{(n-r)\times r} \qquad (15)$$

$$\mathbf{q}^{\vee,\mathsf{T}} \cdot \mathbf{b} = \mathrm{Id}_{n-r}. \qquad (16)$$

- $\mathcal{A}$ defined by $(\mathbf{q}, \mathbf{b})$, its 3d mirror $\mathcal{A}^\vee$ defined by $(\mathbf{q}^\vee, \mathbf{b}^\vee) = (\mathbf{b}, \mathbf{q})^{-\mathsf{T}}$.
- $\mathcal{H}(A) = \mathcal{C}(A^\vee)$, $\mathcal{C}(A) = \mathcal{H}(A^\vee)$.
- Gale duality, Koszul duality [T. Hausel, B.Sturmfels, T. Braden, A. Licata, N. Proudfoot, B. Webster and . . .].
- Simple theory is closed under 3d mirror symmetry. Both Higgs and Coulomb branch are hypertoric variety.

Star-Polygon Duality of SQED with coprime charges

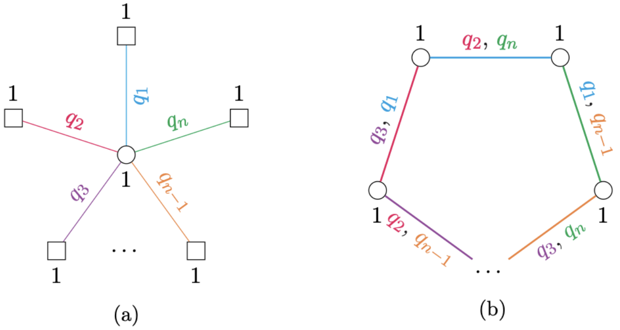


Figure 14: a: The star-shaped quiver description of SQED with charge vector $\mathbf{q} = (q_1, \cdots, q_n)^{\mathsf{T}}$. b: The unframed quiver description of the mirror theory defined by $\mathbf{q}^{\vee}$. One can see the 3d mirror relation gives rise to a Star-Polygon duality.

General Abelian theory

[J.Grimminger, A.Hanany, DL 2503.08791]

- Gauge group: $\mathrm{U}(1)^r \times \mathbb{Z}_{l_1} \times \cdots \times \mathbb{Z}_{l_{n-r}}$, where $l_{i+1}|l_i$.
- Matter fields: n cotangent type hypermultiplets, whose gauge charges under $\mathrm{U}(1)^r$ is given by $\mathbf{q}$, and gauge charges under $\mathbb{Z}_{l_1} \times \cdots \times \mathbb{Z}_{l_{n-r}}$ is given by $\mathbf{b}$.
- $\mathbf{q}$ has Smith normal form $\lambda = \mathrm{diag}(k_1, \ldots, k_r)$, where $k_{i+1}|k_i$. Can always find an $\mathrm{SL}^{\pm}(r, \mathbb{Z})$ transformation, $\mathbf{q} \to \mathbf{q}_0 \cdot \lambda$, where $\mathbf{q}_0$ has trivial Smith normal form.
- The combined matrix $(\mathbf{q_0}, \mathbf{b})$ defines a simple Abelian theory. $\mathbf{b}$, induces an embedding $\mathbb{Z}_{l_1} \times \cdots \times \mathbb{Z}_{l_{n-r}} \to \mathrm{U}(1)^n$ the same way as $\mathrm{U}(1)^{n-r} \to \mathrm{U}(1)^n$.
- $(\mathbf{q}, \mathbf{b})$ and $(\mathbf{q}, \mathbf{b}) \cdot P$ defines the same theory for certain P.

Equivalence class

- The matrix P takes a specific form:

$$P = \begin{pmatrix} S_{r\times r} & M_{r\times(n-r)} \\ 0_{(n-r)\times r} & B_{(n-r)\times(n-r)} \end{pmatrix}, \tag{17}$$

- where S is an element of $\mathrm{SL}^{\pm}(r,\mathbb{Z})$, M is an $r\times(n-r)$ integer matrix, and B is an element of a parabolic subgroup of $\mathrm{SL}^{\pm}(n-r,\mathbb{Z})$.
- If some of the ls are equal, the discrete group can be written as $\mathbb{Z}_{l_1}^{r_1}\times\cdots\times\mathbb{Z}_{l_s}^{r_s}$, with $\sum_i r_i = n-r$, then B takes a specific form:

$$B = \begin{pmatrix} S_1 & * & * & * \\ & S_2 & * & * \\ & & \ddots & * \\ & & & S_n \end{pmatrix}, \tag{18}$$

- where S_i is an element of $\mathrm{SL}^{\pm}(r_i,\mathbb{Z})$.

Smith normal form

- Electric Smith normal form: $\lambda = \mathrm{diag}(k_1, \ldots, k_r)$ gives the electric one form symmetry $\mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_r}$, the charged objects are Wilson lines. [F. Apruzzi, L. Bhardwaj, D. S. W. Gould, S. Schafer-Nameki, M. Bullimore, A. E. V. Ferrari, and . . .]
- Magnetic Smith normal form: $\gamma = \mathrm{diag}(l_1, \ldots, l_{n-r})$ gives the magnetic one form symmetry $\mathbb{Z}_{l_1} \times \cdots \times \mathbb{Z}_{l_{n-r}}$, the charged objects are vortex lines.
- The defining data of a general Abelian theory can be arranged into a quadruple: $[\lambda, (\mathbf{q}_0, \mathbf{b}), \gamma]$.
- Call the simple theory defined by $(\mathbf{q}_0, \mathbf{b})$ as $\mathcal{A}_0$, the Higgs and Coulomb branch follows the relation:

$$\mathcal{H}(\mathcal{A}) = \mathcal{H}(\mathcal{A}_0)/\mathbb{Z}_{l_1} \times \cdots \times \mathbb{Z}_{l_{n-r}}, \quad \mathcal{C}(\mathcal{A}) = \mathcal{C}(\mathcal{A}_0)/\mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_r} \tag{19}$$

Statement

For a general Abelian theory $\mathcal{A}$ defined by $[\lambda, (\mathbf{q}_0, \mathbf{b}), \gamma]$, its 3d mirror $\mathcal{A}^\vee$ is given by $[\gamma, (\mathbf{b}, \mathbf{q}_0)^{-\mathsf{T}}, \lambda] = [\lambda^\vee, (\mathbf{q}_0^\vee, \mathbf{b}^\vee), \gamma^\vee]$

$$\mathbb{Z}_E^n \leftrightarrow \mathbb{Z}_M^n$$
$$(\mathcal{H}, \mathcal{C}) \leftrightarrow (\mathcal{C}^\vee, \mathcal{H}^\vee)$$
$$(\mathrm{SU}(2)_{\mathcal{H}}, \mathrm{SU}(2)_{\mathcal{C}}) \leftrightarrow (\mathrm{SU}(2)_{\mathcal{C}^\vee}, \mathrm{SU}(2)_{\mathcal{H}^\vee})$$
$$(G_{\mathcal{H}}, G_{\mathcal{C}}) \leftrightarrow (G_{\mathcal{C}^\vee}, G_{\mathcal{H}^\vee})$$
$$(\text{vortex line}, \text{Wilson line}) \leftrightarrow (\text{Wilson line}^\vee, \text{vortex line}^\vee)$$
$$(\text{magnetic}, \text{electric})\text{1-form symmetry} \leftrightarrow (\text{electric}, \text{magnetic})\text{1-form symmetry}^\vee$$

Operator map

- The GIOs Y takes the form:

$$Y = \prod_{i=1}^{n} X_i^{d_i} \tilde{X}_i^{\tilde{d}_i} \tag{20}$$

- where the gauge invariance condition under $\mathrm{U}(1)^r$:

$$\sum_{i=1}^{n} q_{ia}(d_i - \tilde{d}_i) = 0 \tag{21}$$

- Can further write:

$$Y = \prod_{i=1}^{n} M_i B \tag{22}$$

- where M_i is a meson-like operator:

$$M_i = \begin{cases} X_i^{d_i} \tilde{X}_i^{d_i}, & d_i \leq \tilde{d}_i; \\ X_i^{\tilde{d}_i} \tilde{X}_i^{\tilde{d}_i}, & d_i > \tilde{d}_i, \end{cases} \tag{23}$$

Operator map

- where B is a baryon-like operator:

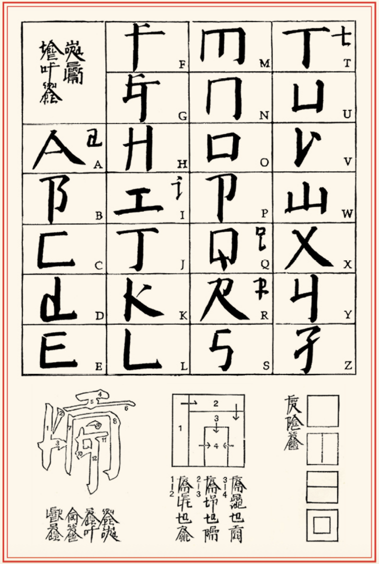
$$B = \prod_{i=1}^{n} B_i, \tag{24}$$

$$B_i = \begin{cases} \tilde{X}_i^{\tilde{d}_i - d_i}, & d_i \leq \tilde{d}_i; \\ X_i^{d_i - \tilde{d}_i}, & d_i > \tilde{d}_i. \end{cases} \tag{25}$$

- Their contribution to the Hilbert series are:

$\mathcal{H}$	Hilbert Series	$\mathcal{C}^\vee$
F-term relations	$(1-t^2)^r$	Casimirs
Meson-like operators	$\frac{1}{(1-t^2)^n}$	
Baryon-like operators	$\sum_{m_j \in \mathbb{Z}^{n-r}} z_j^{m_j} t^{\sum_{i=1}^{n} \left\| \sum_{j=1}^{n-r} \tau_{ji} m_j \right\|}$	Bare monopole operators

Operator map



Higgs branch Hilbert series as monopole formula: Simple theory

- Coulomb branch Hilbert series is given by monopole formula:

$$\mathrm{HS}_{\mathcal{C}} = \frac{1}{(1-t^2)^r} \sum_{(m_1,\dots,m_r)\in\mathbb{Z}^r} \left(\prod_{a=1}^{r} z_a^{m_a} \right) t^{2\Delta(m_1,\dots,m_r)}, \tag{26}$$

- Coulomb branch Hilbert series is given by Weyl integral:

$$\mathrm{HS}_{\mathcal{H}} = \oint_{|x_a|=1} \left(\prod_{a=1}^{r} \frac{dx_a}{x_a} \right) \mathrm{PE} \left(\sum_{i=1}^{n} \left(\prod_{a=1}^{r} x_a^{q_{ia}} \right) \left(\prod_{j=1}^{n-r} z_j^{b_{ij}} \right) t + c.c - rt^2 \right), \tag{27}$$

Higgs branch Hilbert series as monopole formula: General theory

- For general theory, Coulomb branch Hilbert series is still given by (26). Alternatively, one can also use:

$$\mathrm{HS}_{\mathcal{C}} = \frac{1}{k_1 \cdots k_r} \frac{1}{(1-t^2)^r} \sum_{(s_1,\ldots,s_r)\in\mathbb{Z}_{k_1}\times\ldots\mathbb{Z}_{k_r}} \sum_{(m_1,\ldots,m_r)\in\mathbb{Z}^r} \left(\prod_{a=1}^{r} (\omega_{k_a}^{s_a} z_a)^{m_a} \right) t^{2\Delta_0(m_1,\ldots,m_r)}, \tag{28}$$

- Higgs branch Hilbert series is given by Molien sum and Weyl integral:

$$\mathrm{HS}_{\mathcal{H}} = \frac{1}{l_1 \cdots l_{n-r}} \sum_{(s_1,\ldots,s_{n-r})\in\mathbb{Z}_{l_1}\times\cdots\times\mathbb{Z}_{l_{n-r}}} \oint_{|x_a|=1} \left(\prod_{a=1}^{r} \frac{dx_a}{x_a} \right)$$
$$\times \mathrm{PE}\left(\sum_{i=1}^{n} \left(\prod_{a=1}^{r} x_a^{q_{ia}} \right) \left(\prod_{j=1}^{n-r} (\omega_{l_j}^{s_j} z_j)^{b_{ij}} \right) t + c.c - rt^2 \right). \tag{29}$$

Higgs branch Hilbert series as monopole formula

- From the operator map, instead of a Weyl integral, one can write down the Hilbert series for simple theory as a infinite sum [S.Cremonesi, A.Hanany, A.Zaffaroni 1309.2657]:

$$\mathrm{HS}_{\mathcal{H}} = \frac{1}{(1-t^2)^{n-r}} \sum_{m_j \in \mathbb{Z}^{n-r}} z_j^{m_j} t^{\sum_{i=1}^{n} |\sum_{j=1}^{n-r} \tau_{ji} m_j|} = \mathrm{HS}_{\mathcal{C}^\vee} \tag{30}$$

- which can be naturally identified with the monopole formula of the mirror theory.
- For general theory:

$$\mathrm{HS}_{\mathcal{H}} = \frac{1}{(1-t^2)^{n-r}} \sum_{m_j \in \mathbb{Z}^{n-r}} z_j^{l_j m_j} t^{\sum_{i=1}^{n} |\sum_{j=1}^{n-r} l_j \tau_{ji} m_j|} = \mathrm{HS}_{\mathcal{C}^\vee} \tag{31}$$

One form symmetry on Higgs/Coulomb branch

- Same Higgs/Coulomb, different Coulomb/Higgs.
- SQED with n charge 1 hypers, SQED with n charge q hypers share same Higgs branch a_{n-1}. However, the Coulomb branch are different, the former is A_{n-1} while the later A_{qn-1}.
- Same geometry, different deformation.
- SQED with q charge 1 hypers, SQED with 1 charge q hypers share same Coulomb branch A_{q-1}. However, the former has $q-1$ deformation parameter (the relative masses) but the later has none.
- Freely acting $\mathbb{Z}_q$, remembered by the moduli stack.

Abelianisation map

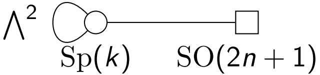
- For a given 3d $\mathcal{N}=4$ theory $\mathcal{T}$, with gauge group G and matter rep $\mathcal{R}$, can find a theory $\mathcal{A}$, with gauge group $T(G)$ and matter rep $\mathcal{R}\ominus T^*(\mathfrak{g}/\mathfrak{h})$. [M.Bullimore, T.Dimofte, D.Gaiotto 1503.04817]
- Their Coulomb branch follows relation:

$$\mathcal{C}(\mathcal{T})\cong\mathcal{C}(\mathcal{A})/\mathcal{W}_{\mathcal{T}}. \qquad (32)$$

- Similar for $3d$ $\mathcal{N}\geq 5$. Remember how the Weyl group enters the reflection group action.
- Coulomb/Higgs branch of 3d $\mathcal{N}=4$ as a quotient of hypertoric variety by reflection group? Too optimistic, $\mathcal{R}\ominus T^*\mathfrak{g}$ can be negative, also examples like e_8 minimal orbit.

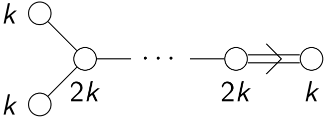
Further problem

- 3d $\mathcal{N} = 4$ B-type ADHM theory admits parity anomaly.



(33)

- Its purposed "3d mirror" through brane system [S.Cremonesi, G.Ferlito, A.Hanany, N.Mekareeya 1408.6835]:



(34)

- Higgs branch $\mathcal{H}$ is moduli space of k B_n instanton.
- Coulomb branch $\mathcal{C}$ is ill defined.

Accidental identification

- Can still work out the monopole formula. For $k = 1$,

$$\mathrm{HS} = \frac{1 - t^{4n-2}}{(1 - t^4)(1 - t^{2n-3})(1 - t^{2n-1})}. \tag{35}$$

- $D_{n+\frac{1}{2}} \cong A_{2n-2}/\mathbb{Z}_2$ singularity? Not well defined.
- Take $n = 5$, same with the Higgs branch limit $[\mathrm{O}(2)_2 \times \mathrm{USp}(2)_{-1}]/\mathbb{Z}_2$.
- A surprise?

Conclusion and Outlook

- Start from a moduli space of a non-anomalous 3d $\mathcal{N} \geq 4$, a well-defined quotient lead us to a non-anomalous relative.
- Mirror description of 3d $\mathcal{N} = 4$ with electric and magnetic one form symmetries swapped.
- "Anomalous" index/Hilbert series might still be useful.
- Consistent definition of non-simply laced Higgs branch?
- Reflection groups and 3d $N \leq 4$ theories?
- Generalised symmetries and anomalies of 3d $N \leq 4$ theories?

Thank you!