

# Conserved charges in deformed 2d CFTs



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Based on [2504.05277, 2511.02007]

with Runkel & Watts,

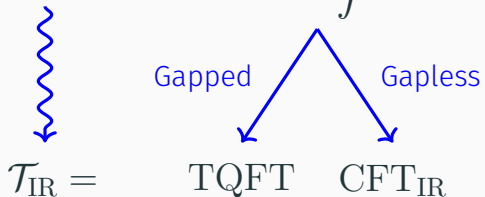
+ to appear [FA, Runkel, Watts, Konechny]<sup>2</sup>

+ partially [2501.07511] & [2601.18667]



# Deformed d dimensional CFT

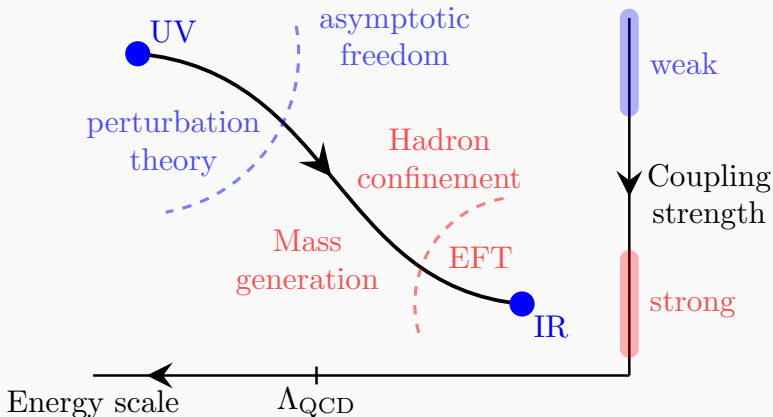
$$\mathcal{T}_{\text{UV}} = \text{CFT}_{\text{UV}} + g_\rho \int \varphi, \quad (\Delta_\rho \lesssim d)$$



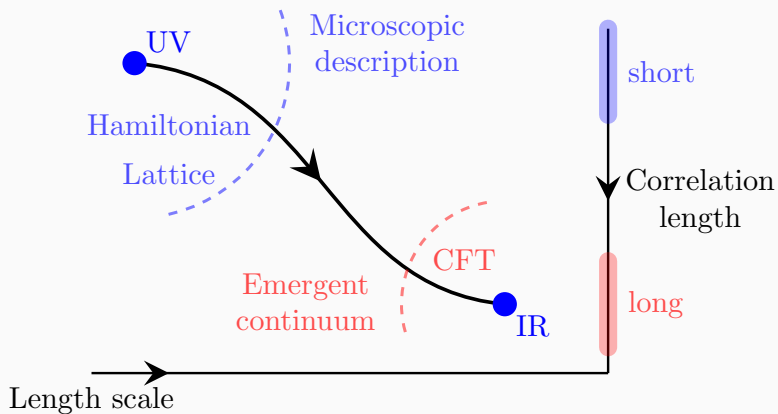
$$\mathcal{T}_{\text{IR}} = \quad \text{TQFT} \quad \text{CFT}_{\text{IR}}$$

Very hard to solve  $\text{CFT}_d$

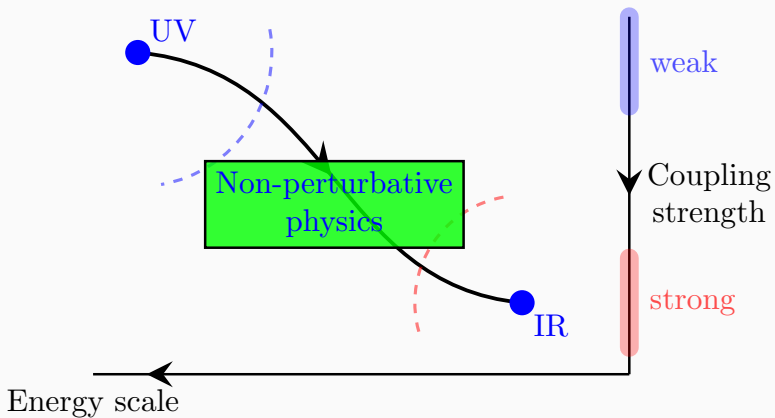
# Asymptotically free gauge theories: gapped example



# Condensed matter systems: gapless example



# The need of non-perturbative physics



## Deformed 2 dimensional CFT


$$\mathcal{T}_{UV} = \text{CFT}_{UV} + g_\rho \int \varphi, \quad (\Delta_\rho \lesssim 2)$$

$\mathcal{T}_{IR} =$       TQFT      CFT<sub>IR</sub>

Hopeful to solve CFT<sub>d=2</sub> (Virasoro algebra)

## Deformed 2 dimensional Minimal Models

$$\mathcal{T}_{UV} = \text{RCFT} + g_\rho \int \varphi, \quad (\Delta_\rho \lesssim 2)$$



$$\mathcal{T}_{IR} = \quad \text{TQFT} \quad \text{CFT}_{IR}$$

Solvable  $\text{CFT}_{UV} = \text{RCFT}$  (Virasoro algebra) !!

# A defect fever

Topological operator = Symmetry

[Gaiotto, Kapustin, Seiberg, Willett]

$$U_g(S^{(d-q-1)}) \quad \begin{array}{c} | \\ \mathcal{V}_\rho(\mathcal{M}^{(q)}) \\ | \end{array} \quad = e^{ig\rho} \quad \begin{array}{c} | \\ \mathcal{V}_\rho(\mathcal{M}^{(q)}) \\ | \end{array}$$

p-form, higher-groups, non-invertible, emanent,  
mixed anomalies, gauging, SymTFT, higher-category ...

[Frolich, Fuchs, Runkel, Schwiebert, Gaiotto, Seiberg, Tachikawa, Bhardwaj, Chang, Lin, Shao, Wang, Yin, Copetti,  
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Today: QFT<sub>2</sub>  $\mathcal{L}_\rho$




More defects  $\Rightarrow$  More symmetries  $\Rightarrow$  More data

$$\mathcal{T}_{\text{UV}} = \text{CFT}_{\text{UV}} + g \int \varphi, \quad (\Delta_\varphi \lesssim 2)$$
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Often in 2d we know  $\text{top. defects} \subset \text{CFT}_{\text{UV}} \mathcal{D}$  

# Symmetries strike back



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What can we say about defects along RG?

# Invariants under RG flows

$$\text{CFT}_{\text{UV}} + g \int \varphi \rightsquigarrow ?$$

When:  $[\mathcal{D}, \varphi] = 0 \Leftrightarrow$  

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Many constraints from UV data: [Chang, Lin, Shao, Wang, Yin]

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Fixes IR fixed points from RCFTs

[\[Tanaka, Nakayama\]](#)[\[FA, Negro\]](#)[\[FA, Prochazka\]](#) [\[Gaberdiel, Merkens\]](#)

Warm up: top. defects in RCFTs  
To ( $\mathcal{W}$ ) infinity and beyond!

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# A primer on Virasoro Minimal models

## $\mathcal{M}(p, q)$ : Rational 2D CFT, A-series

- Central charge:  $c = 1 - \frac{6(p-q)^2}{pq}$  Unitary iff  $q = p + 1$
- $\frac{(p-1)(q-1)}{2}$  primaries  $\phi_{(r,s)} = \phi_{(p-r, q-s)}$
- weights  $h_{(r,s)}$  and  $C_{rs}^t$  all known
- Fusion Category:  $\phi_\rho \otimes \phi_\sigma = \sum_\delta \mathcal{N}_{\rho\sigma}^\delta \phi_\delta$

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Topological lines = Verlinde Lines  $\{\mathcal{L}_\rho\} \xleftrightarrow{1 \text{ to } 1} \{\varphi\}$

Non-invertible symmetry:  $\mathcal{L}_\rho \times \mathcal{L}_\sigma = \sum_\delta \mathcal{N}_{\rho\sigma}^\delta \mathcal{L}_\delta$

Ward identity:  $\mathcal{L}_\rho \circlearrowleft \phi_\sigma = \begin{bmatrix} S_{\rho\sigma} \\ S_{0\sigma} \end{bmatrix} \dot{\phi}_\sigma$

## Example: the Ising model

### 2D Ising CFT $\mathcal{M}(3, 4)$

$$c = \frac{1}{2}, \quad \text{Primaries : } 1_{0,0}, \epsilon_{\frac{1}{2}, \frac{1}{2}}, \sigma_{\frac{1}{16}, \frac{1}{16}}, \quad \text{TLD : } \mathbf{1}, \eta, \mathcal{N}$$

$$\text{Fusion Algebra } \text{TY}_2: \quad \eta^2 = \mathbf{1}, \quad \mathcal{N} \times \eta = \mathcal{N}, \quad \mathcal{N}^2 = \mathbf{1} + \eta$$

$$\eta \longleftrightarrow \mathbb{Z}_2 \quad \mathcal{N} \longleftrightarrow \text{KW duality defect}$$

Duality in QFT

$$\text{KW : } \text{Ising}_T \longleftrightarrow \text{Ising}_{T-1}$$

Symmetry in CFT

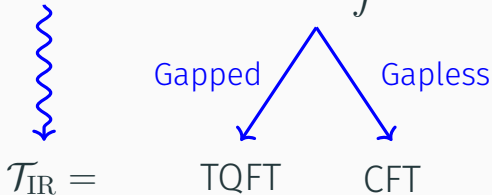
$$\text{KW : } \text{Ising}_{T_c} \curvearrowright$$

TDL honest symmetries of CFT

# Deformation by primary field

Relevant deformation of minimal models

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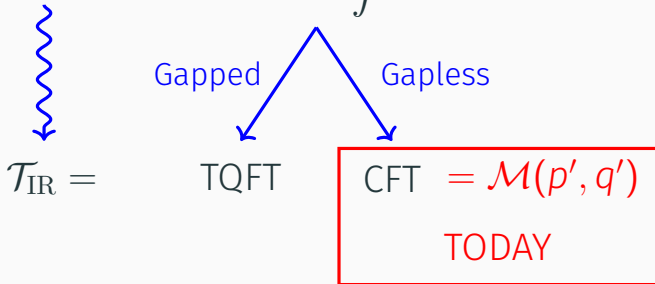
CFT =  $\mathcal{M}(p', q')$

TODAY

# Derfomation by primary field

Relevant deformation of minimal models

$$\mathcal{T}_{UV} = \mathcal{M}(p, q) + g_\rho \int \varphi, \quad (h_\rho < 1)$$



c-theorem with:  $c_{\text{eff}} = 1 - \frac{6}{pq}$  ( $\mathcal{PT}$ -sym) [Ravanini]

We know all\* topological defects!

# Invariants under RG flows

$$\mathcal{M}(p, q) + g_\rho \int \varphi \text{ wavy arrow} ?$$

When:  $[\mathcal{L}_\sigma, \phi_\rho] |\Phi\rangle = 0 \Leftrightarrow$



$=$  


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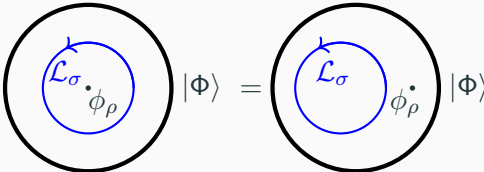
When:  $[\mathcal{L}_\sigma, \phi_\rho] |\Phi\rangle = 0 \Leftrightarrow$  

Then: preserved symmetry along flow &

[Chang, Lin, Shao, Wang, Yin] [Nakayama, Tanaka]

Quantum dimension =  $\langle 0 | \mathcal{L}_\sigma | 0 \rangle = d_\rho = \mathcal{L}_\sigma$  

RG invariant!



$$|\Phi\rangle = |\Phi\rangle$$

# Impose anomaly matching

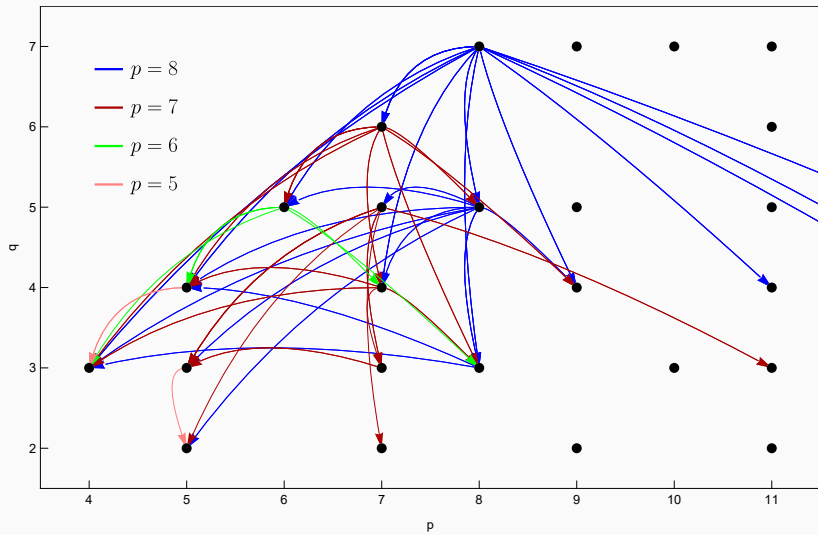
Strategy: Fix  $\mathcal{T}_{UV} = \mathcal{M}(p, q)$

1. For any relevant def. compute all RG invariants
2. Generate all  $\mathcal{M}(p', q')$  with  $c_{\text{eff}}^{\text{IR}} < c_{\text{eff}}^{\text{UV}}$
3. Exclude  $\mathcal{T}_{\text{IR}}$  not fulfilling constraints

Produces putative flows:

$$\mathcal{M}(p, q) \xrightarrow{\phi_{(r,s)}} \mathcal{M}(p', q')$$

# Result



# New flows

$$\text{Result: } \mathcal{M}(p, q) \xrightarrow{\phi_{(1,2k+1)}} \mathcal{M}(p, kp - q)$$

Only known flows:

[Nakayama, Tanaka][FA, Negro]

- $k = 1$ :  $\phi_{(1,3)}$  [Fendley, Saleur, Al. Zamolodchikov][ Al. Zamolodchikov]
- $k = 1/2$ :  $\phi_{(1,2)}, \phi_{(1,5)}$  [Dorey, Dunning, Tateo]
- $k = 3$ :  $\phi_{(1,7)} \quad \mathcal{M}(3, 10) \rightarrow \mathcal{M}(3, 8)$  [Narovlansky, Sun, Tarnopolsky]

Preserve  $\text{Rep}[\text{SU}(2)_{q-2}] \supset \mathbb{Z}_2$  symmetry\*:

$$\{1, \mathcal{L}_{(2,1)}, \dots, \mathcal{L}_{(q-1,1)}\}, \quad \mathcal{L}_{(q-1,1)} \times \mathcal{L}_{(q-1,1)} = 1$$
$$[\mathcal{L}_{(n,1)}, \phi_{(1,2m+1)}] = 0, \quad m = 1, \dots, k$$

\* Only  $\text{Rep}[\text{PSU}(2)_{q-2}]$  for  $k \in \mathbb{Z}/2$

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Suggest a vast generalization [FA, Prochazka]

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# $\mathcal{W}_N$ algebra

Virasoro algebra: generated by modes of spin 2 field  $T(z)$

$\mathcal{W}_N$  algebra: additional currents of spin  $s = 2, \dots, N$

$\{T(z), W^{(3)}, \dots, W^{(N)}\}$  [Zamolodchikov<sup>2</sup>, Fateev, Lukyanov. . .]

$$T(z)W^{(s)}(u) \sim \frac{sW^{(s)}(u)}{(z-u)^2} + \frac{\partial W^{(s)}(u)}{z-u} + \dots$$

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E.g.  $\mathcal{W}^{(3)}$ :

$$W(z)W(u) \sim \frac{c/3}{(z-u)^6} + \frac{2T(u)}{(z-u)^4} + \frac{\partial T(u)}{(z-u)^3} + \frac{1}{(z-u)^2} \left( \frac{3}{10} \partial^2 T(u) + \frac{32}{22+5c} \Lambda(u) \right) + \dots$$

## $\mathcal{W}_N$ Minimal models

Also  $\mathcal{W}_N$  admit minimal truncations:

$\mathcal{W}_2$  minimal models:  $\mathcal{W}_2(p, q)$

- Rational CFT:  $c_{p,q}^{(2)} = (2-1) \left[ 1 - \frac{2(2+1)(p-q)^2}{pq} \right]$
- $\frac{1}{2}(p-1)(q-1)$  primaries
- Unitary iff  $|p-q| = \pm 1$
- Coset construction  $\frac{\widehat{\mathfrak{su}}(2)_\kappa \times \widehat{\mathfrak{su}}(2)_1}{\widehat{\mathfrak{su}}(2)_{\kappa+1}}$   $\kappa = p/(q-p) - 2$

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Can we understand RG flows?

Only known: unitary flows  
 $W_N(p, p+1) \rightarrow W_N(p-1, p)$

# A uniform description: truncation of $\mathcal{W}_\infty$

No canonical embedding  $\mathcal{W}_n[c] \hookrightarrow \mathcal{W}_{n+1}[c]$

- No natural inclusion of DS reductions from  $\widehat{\mathfrak{sl}}_n$
- OPE & currents don't restrict nicely

⇒ Nice inclusions into bigger algebra  $\mathcal{W}_\infty[\lambda, c]$

- Contains currents  $s = 2, 3, \dots, \infty$
- Interpolating family:  $\mathcal{W}_\infty[\lambda = N, c] \simeq \mathcal{W}_N(c)$
- $\mathcal{S}_3$  symmetry:  $\mathcal{W}_\infty[\lambda_1, c] \simeq \mathcal{W}_\infty[\lambda_2, c] \simeq \mathcal{W}_\infty[\lambda_3, c]$

$$\mathcal{W}_\infty[\lambda_1, \lambda_2, \lambda_3], \quad \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0, \quad c = (\lambda_1 - 1)(\lambda_2 - 1)(\lambda_3 - 1)$$

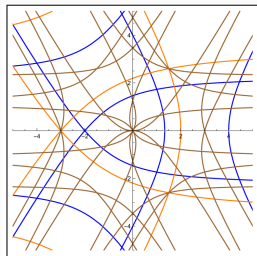
# A uniform description: truncation of $\mathcal{W}_\infty$

$\mathcal{W}_{1+\infty}$  has null vectors:  $\frac{N_1}{\lambda_1} + \frac{N_2}{\lambda_2} + \frac{N_3}{\lambda_3} = 0$

$\Rightarrow$  Quotient by  $\mathcal{I}_{N_1, N_2, N_3}$

$$\mathcal{W}_\infty[\lambda_1, \lambda_2, \lambda_3] \simeq Y_{N_1, N_2, N_3}[c]$$

E.g.  $\mathcal{W}_N \simeq \mathcal{W}_{1+\infty}/\mathcal{I}_{0,0,N}$



Double truncation  $Y_{N_1, N_2, N_3} \cap Y_{M_1, M_2, M_3}$  are RCFTs!

$$W_N(p, q) = Y_{p-N, q-N, 0} \cap Y_{0, 0, N}$$

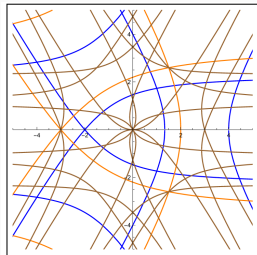
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$$W_N(p, q) = Y_{p-N, q-N, 0} \cap Y_{0, 0, N}$$

Many generalizations: Grassmanian VOA [Prochazka, Eberhardt]

$$\frac{SU(N)_\kappa \times SU(N)_\rho}{SU(N)_{\kappa+\rho}}$$

$$\frac{\mathfrak{g}_\kappa \times \mathfrak{g}_\rho}{\mathfrak{g}_{\kappa+\rho}}$$

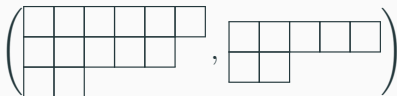
...

# Primaries

[Isomorphism  $Y[\widehat{\mathfrak{gl}}_1]$  or box counting]

Uniform description for representation theory etc. . .

Primaries in  $\mathcal{W}_N(p, q)$  representation  $(A_N, A_N)$  / Relations:



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$$\left( \overbrace{\begin{array}{|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square \\ \hline \square & \square & & & & \\ \hline \end{array}}^{\leq q-N}, \overbrace{\begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & \square & & & \\ \hline \square & \square & & & \\ \hline \end{array}}^{\leq p-N} \right) < N$$

E.g. for  $\mathcal{W}_3(6, 5)$

$$\left\{ \begin{array}{l} (\bullet, \bullet), (\square, \bullet), (\square\square, \bullet), (\square\square, \bullet), (\square\square\square, \bullet), (\square\square\square, \bullet) \\ (\bullet, \square), (\bullet, \square\square), (\bullet, \square\square), (\bullet, \square\square\square), (\bullet, \square\square\square), (\bullet, \square\square\square\square), (\bullet, \square\square\square\square) \\ (\square, \square), (\square\square, \square), (\square\square\square, \square), (\square, \square\square), (\square\square, \square\square), (\square\square\square, \square\square), (\square, \square\square\square), \end{array} \right\}$$

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For Virasoro:  $(N = 2) \phi \sim (\square\square\square\square, \square\square\square\square)$

$$\phi_{(1,2k+1)} \sim (\cdot, \overbrace{\square\square}^{2k}), \quad \mathcal{M}(p, q) \xrightarrow{\phi_{(1,2k+1)}} \mathcal{M}(p, kp - q)$$

Commutates with

$$SU(2)_{q-2} : \left\{ (\cdot, \cdot), (\square, \cdot), \dots, \underbrace{(\overbrace{\square\square\square\square}^{q-2}, \cdot)}_{\mathbb{Z}_2} \right\}$$

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For Virasoro:  $(N = 2) \phi \sim (\square\square\square\square, \square\square\square)$

$$\text{PSU}(2)_{p-2} \ni \phi_{(1, \text{Adj}^k)} \sim (\cdot, \overbrace{\square\square}^{2k}), \quad W_2(p, q) \rightarrow W_2(p, kp - q)$$

Commutates with

$$SU(2)_{q-2} : \left\{ (\cdot, \cdot), (\square, \cdot), \dots, \underbrace{(\overbrace{\square\square\square\square}^{q-2}, \cdot)}_{\mathbb{Z}_2} \right\}$$

$\infty$ -many new flows:  $\mathcal{W}_N(p, q) \xrightarrow{\varphi} \mathcal{W}_N(p, kp - q)$

[FA, Procházka]

- Adjoint-category of deformation  $\varphi \in PSU(N)_{q-N}$
- Preserves large symmetry  $\mathbf{Rep}[SU(N)_{p-N}] \supset \mathbb{Z}_N$   
(Bootstrap?)
- Uniform in rank!  $\Leftarrow$  Truncations of  $\mathcal{W}_\infty$
- **Incredible power of topological lines!**
- Only known with  $k = 1$  [Poghossyan<sup>2</sup>][Lukyanov, Faddeev][Dunning]

## A quick preview: example for $\mathcal{W}_3$

New families: simplest  $\mathcal{W}_3(p, q) \xrightarrow{\varphi} \mathcal{W}_3(p, kp - q)$

$$\varphi \in \text{PSU}(3)_{p-N} = \left\{ \left( \cdot, \underbrace{\begin{array}{|c|c|c|c|c|c|} \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline \end{array}}_m \right), \quad n + m \in 3\mathbb{Z} \right\}$$

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$$\left\{ \left( \cdot, \cdot \right), \left( \square, \cdot \right), \left( \square \square, \cdot \right), \left( \begin{array}{|c|} \hline \square \\ \hline \end{array}, \cdot \right), \left( \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \cdot \right), \left( \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}, \cdot \right) \right\}$$

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Generalizes nicely to  $\mathcal{W}_N!$

# Generalization to $\mathcal{W}_N$

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Adjoint deformations:

$$\varphi \in \text{PSU}(N)_{p-N} = \left\{ \left( \cdot, \begin{array}{cccccccc} \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square & \square \\ \square & & & & & & & \end{array} \right), \# \text{ boxes} \in N\mathbb{Z} \right\}$$

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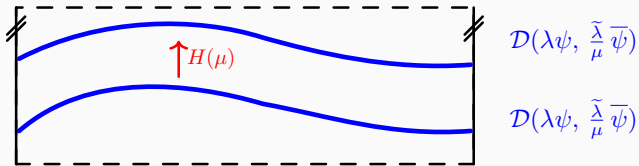
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Generalizes nicely to  $\mathcal{W}_\infty$  (?):

$$\mathcal{W}_\infty(\lambda_2, \lambda_3) \rightarrow \mathcal{W}_\infty(-\lambda_2 - 2(k-1)\lambda_3, \lambda_3)$$

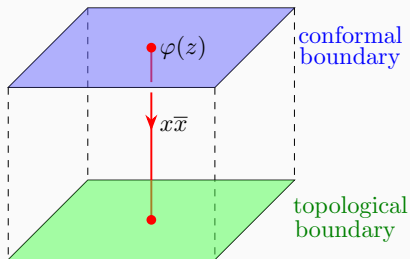
## Translational invariant defects

---



## 3d TFT for 2d RCFT

TFT<sub>d=3</sub> encoding the QFT<sub>d=2</sub>



- Topological ops in QFT: Neumann @ top boundary
- Non topological: Dirichlet boundary conditions

All lines = topological anyons of 3D TFT

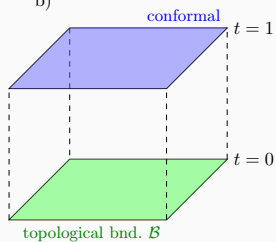
# Chiral TFT

In 2d CFT I like chiral TFT:

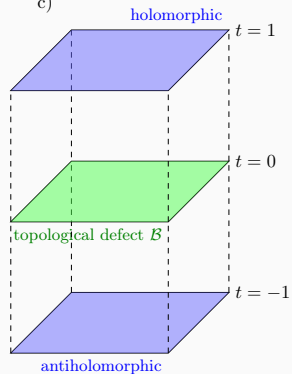
a)



b)



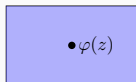
c)



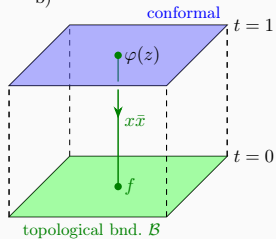
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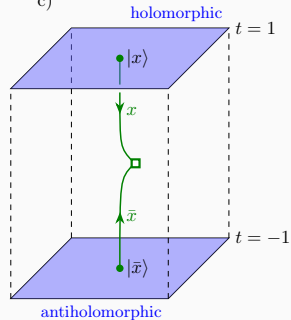
a)



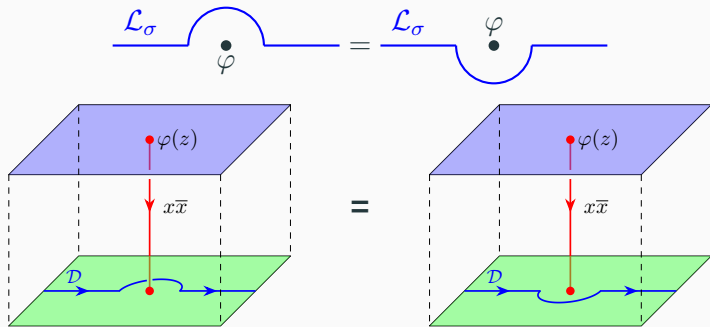
b)



c)



# Example: commutation condition from before



# Commuting defects and where to find them

Hamiltonian of perturbed CFT:  $H(\mu) = H_0 + H_{\text{pert}}(\mu)$

$$H_0 = \frac{2\pi}{L} \left( L_0 + \bar{L}_0 - \frac{c}{12} \right), \quad H_{\text{pert}}(\mu) = 2i\mu \int_0^L \varphi(s) ds .$$

$\mathcal{D}$  **topological** in  $H_0$ :

$$\underbrace{[H_0, \mathcal{D}] = 0}_{\text{Top. in CFT}}, \quad \& \quad \underbrace{[\varphi, \mathcal{D}] = 0}_{\text{Comm. with def.}}$$

$$[\mathbb{T}(\mu), \mathcal{D}] = 0 \implies [H(\mu), \mathcal{D}] = 0$$

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Stronger than **conserved!**

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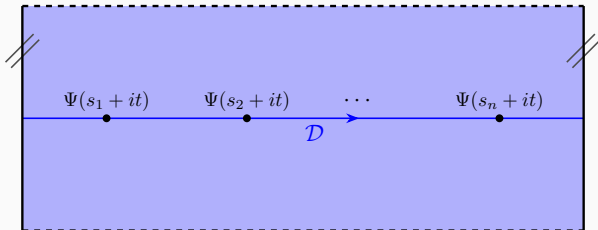
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What kind of animals?

# Perturbed defect operators

Perturb a topological defect  $\mathcal{D}$  by (chiral) defect operators:

$$\underbrace{\psi(z)}_{\text{hol}}, \quad \underbrace{\bar{\psi}(\bar{z})}_{\text{antihol.}}, \quad \psi, \bar{\psi} \in \mathcal{D}, \quad \varphi \sim \bar{\psi}\psi$$



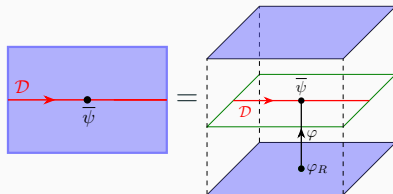
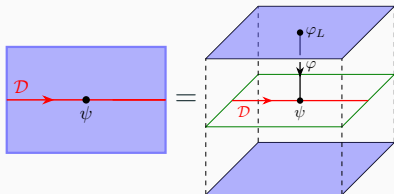
$$\mathcal{D}(\lambda\psi + \tilde{\lambda}\bar{\psi}) = \exp\left(\int_0^L (\lambda\psi(s + it) + \tilde{\lambda}\bar{\psi}(s + it)) ds\right),$$

s coordinate along defect, t position of defect

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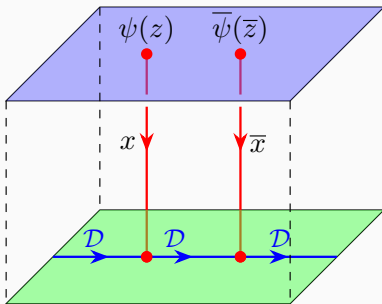
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## Bulk commutation condition

Conservation law in QFT: (*Complicated:*)

$$[H(\mu), \mathcal{D}(\lambda\psi + \tilde{\lambda}\bar{\psi})]_{\text{QFT}} = 0$$

Implied by condition in CFT pert. in  $\lambda, \tilde{\lambda}$ :

$$\mu [\mathcal{D}, \varphi]_{\text{CFT}} = \lambda \tilde{\lambda} [\psi, \bar{\psi}]_{\text{CFT}}$$

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Remember: Topological  $\longleftrightarrow [\mathcal{L}, \varphi] = 0$

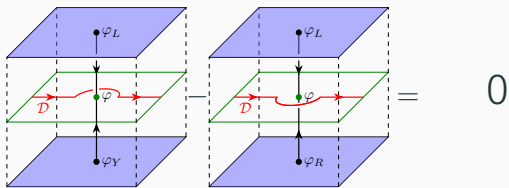
→ “Adiabatically” deform Line & Bulk simultaneously

→ Rigidly translational invariant

→ Sometimes not renormalized (E.g.  $h < \frac{1}{2}$  in  $\mathcal{M}(p, q)$ )

# Condition for translation-invariant

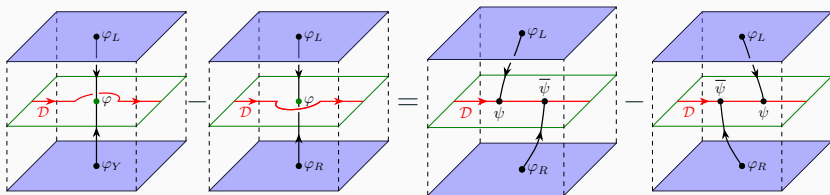
$$\text{Topological: } [\mathcal{D}, \varphi] = 0$$



**Rigid:** deformed bulk, undeformed line

# Condition for translation-invariant

$$\text{Transl. invariant: } \mu [\mathcal{D}, \varphi] = \lambda \tilde{\lambda} [\psi, \bar{\psi}]$$



**Natural:** deformed bulk & line accordingly

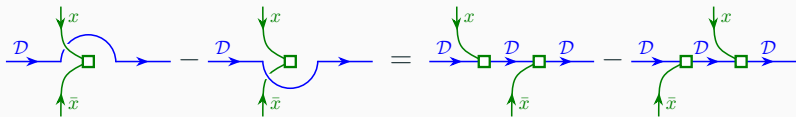
$\mathcal{D}(\lambda\psi + \frac{\tilde{\lambda}}{\mu}\bar{\psi})$  is **(non-local) conserved charge**

# Conditions

Basis for defect fields:

$$\psi = \sum_{a,b \in \mathcal{D}} \kappa_{ab} \begin{array}{c} \text{---} a \text{---} \square \text{---} b \text{---} \\ \uparrow x \\ \downarrow \bar{x} \end{array}, \quad \bar{\psi} = \sum_{a,b \in \mathcal{D}} \tilde{\kappa}_{ab} \begin{array}{c} \text{---} a \text{---} \square \text{---} b \text{---} \\ \downarrow \bar{x} \\ \uparrow x \end{array},$$

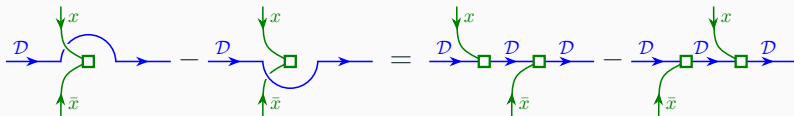
Bulk commutation condition:



The diagram illustrates the bulk commutation condition for defect fields. It shows an equality between two expressions. On the left side, there are two terms separated by a minus sign. Each term consists of a blue horizontal line representing a bulk field  $\mathcal{D}$ . A green square represents a defect. In the first term, a green arrow labeled  $x$  points down to the defect, and a green arrow labeled  $\bar{x}$  points up to the defect. A blue arc connects the two ends of the line, passing over the defect. In the second term, the blue arc passes under the defect. On the right side, there are also two terms separated by a minus sign. Each term consists of a blue horizontal line with two green squares representing defects. In the first term, the left defect has a green arrow labeled  $x$  pointing down and a green arrow labeled  $\bar{x}$  pointing up. In the second term, the right defect has a green arrow labeled  $x$  pointing down and a green arrow labeled  $\bar{x}$  pointing up. The blue line passes through both defects.

# Conditions

Bulk commutation condition:



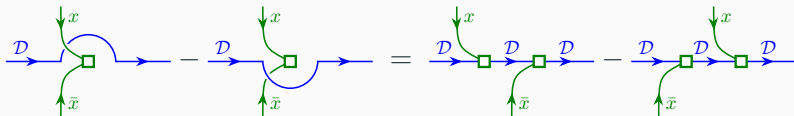
Using standard TQFT rules:

$$\delta_{ac} \left( \mathbf{R}^{(xa)b} - \frac{1}{\mathbf{R}^{(ax)b}} \right) \mathbf{F}_{1b}^{(ax\bar{x})a} = \delta_{b \in \mathcal{D}} \kappa_{ab} \tilde{\kappa}_{bc} - \sum_{d \in \mathcal{D}} \mathbf{F}_{db}^{(xa\bar{x})c} \tilde{\kappa}_{ad} \kappa_{dc}$$

$$\forall a, c \in \mathcal{D}, \quad \forall b \in a \otimes x$$

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When solution exists  $\Rightarrow$  Defect is conserved

# Solutions in Minimal models

Minimal model $M(p, q)$	perturbing bulk field $\varphi$	weights $h = \bar{h}$ of $\varphi$	topological defect solving (4)
$q \geq 3$	(1, 2)	$h_{1,2} = \frac{3}{4}t - \frac{1}{2}$	$(1, 1) \oplus (1, 2)$
$q \geq 4$	(1, 3)	$h_{1,3} = 2t - 1$	(1, 2)
$q \geq 6$	(1, 5)	$h_{1,5} = 6t - 2$	(1, 3)
$q = 9, 10, 18$	(1, 7)	$h_{1,7} = 12t - 3$	(1, 5)

And many others... E.g. (1,9) on (1,5), (1,6) (1,7), etc...

$$\boxed{[\mathcal{D}(\lambda, \mu/\lambda), \mathcal{D}(\lambda', \mu/\lambda')] = 0, \quad \lambda \neq \lambda'}$$

Infinitely many non-local conserved charges!  
(Complicated condition)

(1, 7) deformation has no local conserved charges [BLZ]

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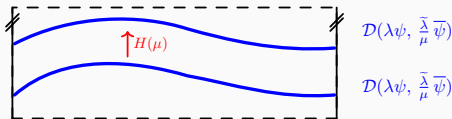
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But NEW  $\infty$  non-local conserved charges!!

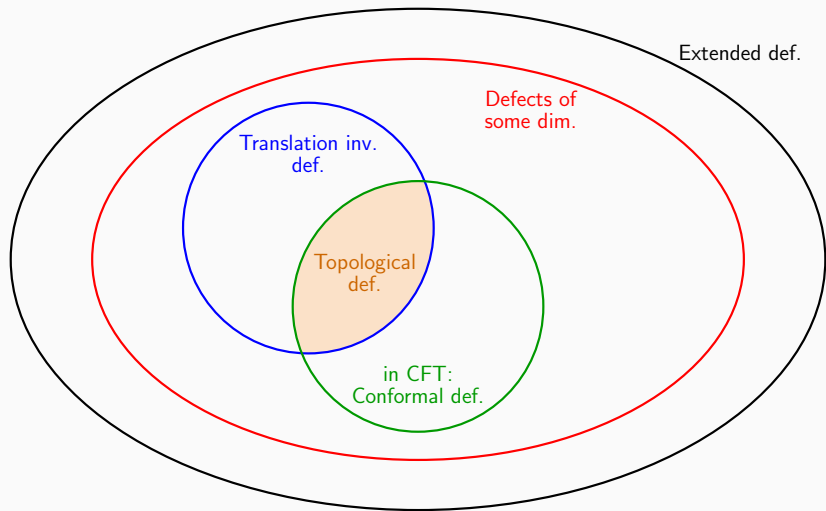
## Translational invariant defects:

$$\left[ \mathcal{D}(\lambda\psi + \frac{\tilde{\lambda}}{\mu}\bar{\psi}), H(\mu) \right] = 0$$

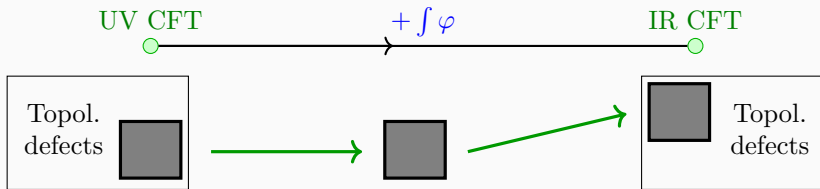


- Admit non-singular fusion!
- $\mathcal{D}_{\text{top}} \subseteq \mathcal{D}_{\text{trans.inv.}}^{\text{per}} \subseteq \mathcal{D}_{\text{trans.inv.}}$
- **Generalize topological defects:**  $\sim \mathcal{F}$  Fusion category
- Trans. Invariant: Drinfeld-Yetter Module ( $\text{Rep}[\mathcal{F}]$ )
- Non-local conserved charges in QFT

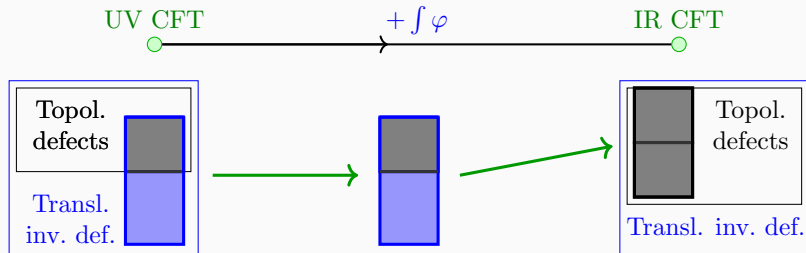
# General picture



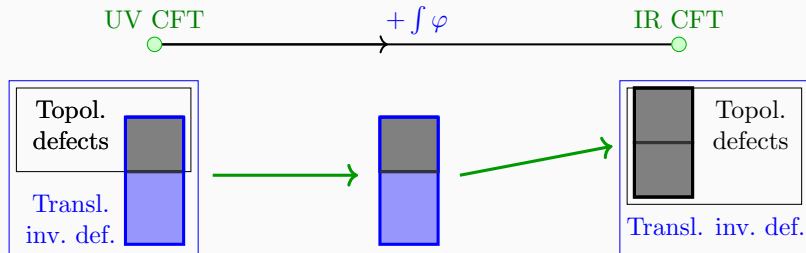
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Emergent symmetries in the IR?

# Where do those new defect flow?

New defects = New constraints

[FA, Runkel, Watts, Konenchy][ Runkel '07]

*Warm up* Chiral deformation of  $\mathcal{M}(p, q)$ :

E.g. chiral  $\psi = \psi_{(1,3)}(z)$ ,  $\mathcal{D}_s(\lambda) := \mathcal{L}_{(1,s)}(\lambda\psi)$



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Chiral def breaks  $\mathcal{L}_{(1,s)}$ !

$\mathcal{D}(\lambda\psi)$  satisfy Y-systems! [Runkel][FA, Runkel, Watts, Konechny]<sup>2</sup>



$$\mathcal{D}_2(\lambda)\mathcal{D}_s(\zeta^{\pm s}\lambda) = \mathcal{D}_{s-1}(\zeta^{\pm(s+1)}\lambda) + \mathcal{D}_{s+1}(\zeta^{\pm(s-1)}\lambda), \quad \zeta = e^{i\pi p/q}$$

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$$\mathcal{D}_2(\lambda)\mathcal{D}_s(\zeta^{\pm s}\lambda) = \mathcal{D}_{s-1}(\zeta^{\pm(s+1)}\lambda) + \mathcal{D}_{s+1}(\zeta^{\pm(s-1)}\lambda), \quad \zeta = e^{i\pi p/q}$$

Implies Hirota:  $\mathcal{D}_s(\zeta\lambda)\mathcal{D}_s(\zeta^{-1}\lambda) = 1 + \mathcal{D}_{s-1}(\lambda)\mathcal{D}_{s+1}(\lambda)$

# Where do those new defect flow?

New defects = New constraints

[FA, Runkel, Watts, Konenchy][ Runkel '07]

Warm up Chiral deformation of  $\mathcal{M}(p, q)$ :

E.g. chiral  $\psi = \psi_{(1,3)}(z)$ ,  $\mathcal{D}_s(\lambda) := \mathcal{L}_{(1,s)}(\lambda\psi)$

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★ In the IR:  $\mathcal{D}_s(r^\alpha e^{i\theta}) \sim \mathcal{D}_\sigma^\infty \exp(a_\sigma(\theta)r^\alpha)$   $\alpha = \frac{1}{2(1-p/q)}$

★ We can solve asymptotically!

## Solutions of Hirota in $\mathcal{M}(p, q)$

$$\mathcal{D}_s(\zeta\lambda)\mathcal{D}_s(\zeta^{-1}\lambda) = 1 + \mathcal{D}_{s-1}(\lambda)\mathcal{D}_{s+1}(\lambda)$$

Form asymptotic  $\mathbb{R}^{\max}$  semi-ring:  $\mathcal{D}_s(\lambda) \rightarrow \mathcal{D}_\sigma^\infty \exp(a_\sigma(\theta))$

- $\mathcal{M}(2, -)$

- $\mathcal{M}(3, -)$

- etc ...

- We (Gerard & Anatoly) check with TCSA!

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$$\mathcal{D}_s(\lambda) \rightarrow \mathbf{Id} \exp(a_s \cos(\overline{\alpha\theta})), \quad a_s(\theta) = a_s \frac{\sin((s-1)\pi p/q)}{\sin(\pi p/q)}$$

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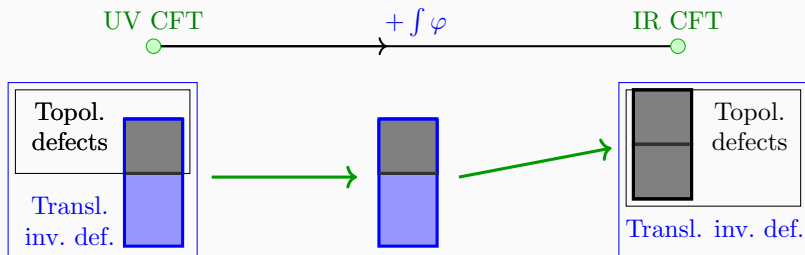
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- $\mathcal{M}(3, -)$  non trivial IR topological defects:

$$\mathcal{D}_s(\lambda) \rightarrow \mathbf{Id} \exp(a_s \cos(\overline{\alpha\theta})) + \mathcal{D}_{q-1} \exp\left(a_{q-s} \cos(\overline{\alpha(\theta - \pi)})\right)$$

- etc ...
- We (Gerard & Anatoly) check with TCSA!

# Explicit realization



Explicit realization in  $\mathcal{M}(3, -)$  example

1.  $\mathcal{D}_{(1,s)}$  broken by chiral  $\phi_{(1,3)} = \psi_{(h,0)}$
2. Survives along flow as  $\mathcal{D}(\lambda\psi)$
3. Re-emergence of topological symmetry in deep IR:

$$\mathcal{D}(\lambda\psi) \quad \rightsquigarrow \quad \text{Id} \quad \text{or} \quad \mathcal{D}_{(1,q-1)}$$

News from 2d CFTs!! [Belavin, Polyakov Zamolodchikov, 1984]

- **New flows via non-invertible symmetries**

→ From Virasoro to  $\mathcal{W}_\infty$  and beyond!



? Fermionic generalization? Fermionic  $\mathcal{W}_\infty$ ? Fermionic flows?

[Tachikawa, Nakayama, Hsieh][Kulp]

? Truncations  $Y_{N,M,L} \cap Y_{0,P,Q}$ , Grassmanian VOA [Eberhardt, Prochazka], compact irrational CFT [Antunes, Behan, Rong], WZW? [Levine. . . ] etc...

- **Beyond the topological framework!**

- **New Transl invariant defects!** Higher dim?

- **Great responsibilities!** Dynamical constraints?

→ Defects in IR from solving Y-systems New exact solvability?

- From 2d to 4d and back WIP [Ambrosino, Gaiotto, Sacchi] (Ask me!)

? Modular invariant classification for  $\mathcal{W}_N$ ? [Gannon] [WIP: FA, Behan]

Thanks for your attention!