# D-branes Wrapped on Fuzzy del Pezzo Surfaces

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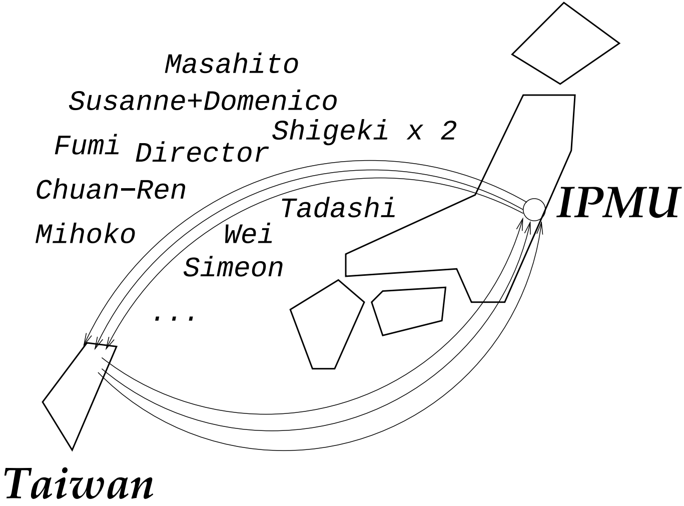
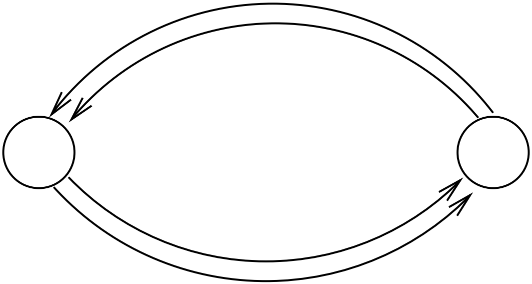
## Based on

- with Kazumi Okuyama (Shinshu U.)
  JHEP 1101:043(2011).
- $(\text{Kazu})^2$, work in progress.

... We have been benefited from IPMU being nearby



... Amusingly, the diagram I just used to describe the impact of IPMU to the research in Taiwan also appears in the description of D-branes on Calabi-Yau manifolds



This kind of diagrams are called quiver diagrams.

... This is quiver (Japanese style), by the way

(Pictures)

Today I will talk about a class of classical vacua in quiver gauge theories which are realized on D-branes probing CY geometries.

Before going into the detail, let me first explain some backgrounds and the motivations for studying them.

# Outline


## Introduction

- From Geometry to Particle Physics via String Theory
- F-GUTs and F(uzz) theory
- Fuzzy Spaces in String Theory

## D-branes wrapped on fuzzy del Pezzo surfaces

- D4-branes wrapped on fuzzy del Pezzo surfaces
  - D-branes probing *CY* and quiver gauge theories
  - KK spectrum on fuzzy $\mathbb{CP}^2$
  - Fuzzy $dP_k$ solutions
- D7-branes wrapped on fuzzy $\mathbb{CP}^2$
  - Adding flavor D7-branes
  - Fuzzy matter localization and the Yukawa couplings
- Summary&Future Directions

## Steps from String Theory to our Universe

1. Find an D-brane configuration which realizes Supersymmetric Standard Model or Supersymmetric Grand Unified Models (GUTs) or else, with closed string background assumed to be stabilized. The closed string background controls the coupling constants in the particle physics model.
2. Show that the closed string background is stable (has no moduli).

   ...

?. Dynamically realize the vacuum in fully non-perturbative formulation of string theory.

While there have been impressive progresses in the Step 1 recently and people discuss at (semi-)realistic level, e.g. possible consequences at the Large Hadron Collider (LHC), our goal today is much more modest.

The vacua we construct may have interesting implications to Particle Physics Phenomenology, in particular to the structure of the Yukawa couplings, but our analysis is still at the primitive stage.

## From Geometry to Particle Physics via String Theory

- Superstrings live in 10D.
- We observe only 1+3 space-time dims.
  $\longrightarrow$ Other 6 directions must be compactified.
- Particle contents and parameters of Particle Physics
  $\cdots$ Largely controlled by the 6D geometry.

## From Geometry to Particle Physics via String Theory

- Weak-scale supersymmetry: Attractive explanation for the hierarchy problem (light Higgs mass).
- Phenomenologically, we don't want too much supersymmetry to be left at low energy.
- Calabi-Yau manifold + D-branes keeps $\mathcal{N} = 1$ SUSY in 4D.

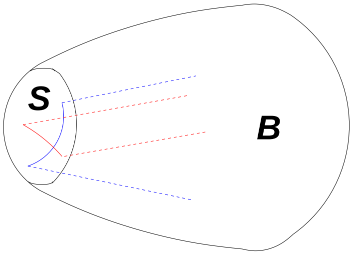
While today's talk is in the framework of type IIB superstring theory, the first motivation for our work came from GUT model buildings in F-theory, in particular, the F(uzz) theory proposal by Heckman and Verlinde.

Let me briefly summarize the relevant materials.

## F-theory GUTs

Donagi-Wijnholt ’08, Beasley-Heckman-Vafa ’08, Hayashi et al. ’08

- Intersecting 7-branes wrapped on cycles in the compactified manifold.
- E-type gauge group → 16 of $SO(10)$
- Decoupling of quantum gravity
  → Shrinkable 4-cycle $S$



Einstein-Hilbert action in 10D

$$\sim \frac{1}{(\ell_{P\,10D})^8} \int_{\mathbb{R}^{1,3}\times B} d^{10}x \sqrt{-g}\, R$$

KK reduction

$$\frac{Vol(B)}{(\ell_{P\,10D})^8} \int_{\mathbb{R}^{1,3}} d^4x \sqrt{-g}\, R$$

4D Planck length

$$(\ell_{P\,4D}) \sim \sqrt{\frac{\ell_{P\,10D}^8}{Vol(B)}}$$

$\Rightarrow \ell_{P\,4D} \to 0$ for $Vol(B) \to \infty$.

## F(uzz) Theory Heckman-Verlinde '10

- F-GUTs based on 7-branes wrapped on "fuzzy" 4-cycle in 6D compact space.
- Natural in the gravity decoupling limit:
  with a flux through the 4-cycle, closed string metric is singular but open string metric may not be.
  c.f. Seiberg-Witten limit for flat space.
- Features:
  - Quantization of the GUT coupling.
  - Structure of the Yukawa couplings from fuzzy intersections of 7-branes.

Low energy effective action on 7-branes

$$\frac{1}{(G_{YM\,8D})^2}\int_{\mathbb{R}^{1,3}\times S} d^8x\,\sqrt{-g}\,F^2+\dots$$

dim. reduction

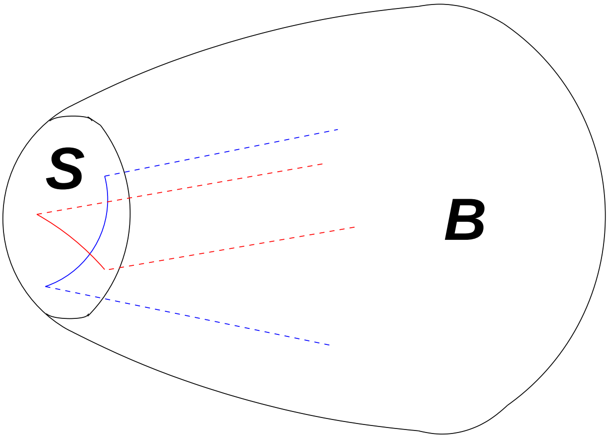
$$\frac{Vol_{open}(S)}{(G_{YM\,8D})^2}\int_{\mathbb{R}^{1,3}} d^4x\,\sqrt{-g}\,F^2+\dots$$

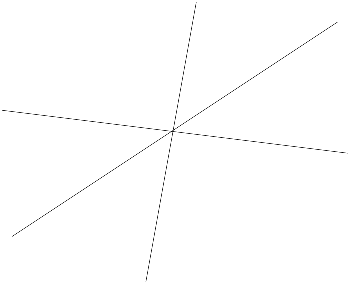
Thus

$$(G_{YM\,4D})^2=\frac{(G_{YM\,8D})^2}{Vol_{open}(S)}$$

$Vol_{open}(S)$ takes discrete values in fuzzy space $S$
($\sim$ size of matrices).

Yukawa couplings appear from intersections of 7-branes.





Ordinary intersection

⇒ Rank one Yukawa

(from one 3-intersection).



Fuzzy intersection

⇒More structures in Yukawa?

c.f. holomorphic NC
Cecotti et al. '09

## F(uzz) Theory (contin.)

- H-V used effective Gauged Linear Sigma Model on 7-branes to describe the fuzzy toric 4-cycle, hoping it to be derived from more microscopic level.
- Currently no microscopic formulation of F-theory.

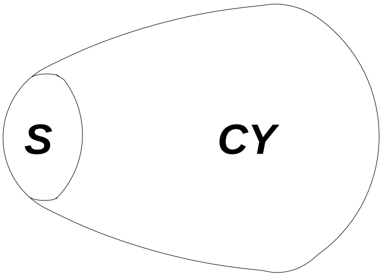
We constructed corresponding configurations in type IIB superstring theory where microscopic formulation is better understood.

## F-theory

- Coupling constant of IIB string is determined by expectation value of the fields called dilaton and axion.
  ⇒ The value of the coupling "constant" may vary place to place.
  ⇒ Perturbative IIB may not be appropriate globally.
- F-theory is a (not yet fully completed) non-perturbative formulation of IIB string, typically describing compactifications to 4D $\mathcal{N} = 1$ SUSY with varying dilaton-axion geometrically.
- Also related to Het etc. by string duality.

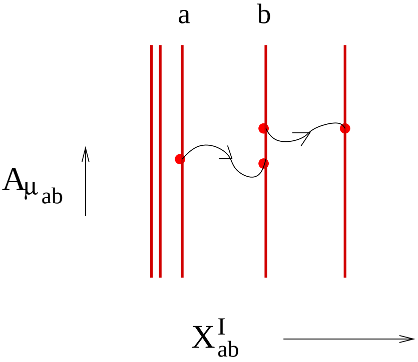
## Our vacua

- IIB string on $\mathbb{R}^{1,3} \times CY_3$
- Local model ($Vol(CY_3) \to \infty$, gravity decouples)
  $\Rightarrow$ 4-cycle $S$ can shrink
  $\Rightarrow$ $S$ is a del Pezzo surface ($\mathbb{CP}^2$ blown up at points)
- D7-branes wrap on *fuzzy* del Pezzo surfaces.



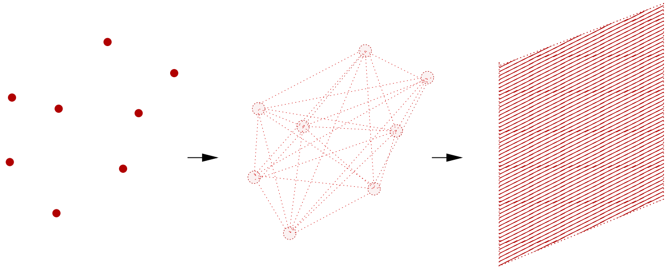
# Fuzzy Spaces in String Theory

## Multiple D-branes and Matrix Coordinates




$X^I_{ab}$: $N \times N$ Hermitian matrices ($N$: # of D-branes).

Positions of D-branes are described by matrices !



$$[X^I, X^J] = 0 \qquad [X^I, X^J] \neq 0 \qquad [X^I, X^J] = i\,\theta^{IJ}$$

Generically non-commutative !

⇒ Cannot define the positions of the branes precisely and the notion of point becomes fuzzy.

Commuting matrices can be simultaneously diagonalized by $U(N)$. In this case, each diagonal component can be interpreted as a position of a D-brane.

$$X^I = \begin{pmatrix} x_1^I & 0 & \cdots & 0 \\ 0 & x_2^I & 0 & \vdots \\ \vdots & 0 & \ddots & \\ 0 & \cdots & & x_N^I \end{pmatrix}$$

## Example 1. Non-commutative $\mathbb{R}^2$

$$[X^1, X^2] = i\theta^{12} \quad (const.)$$

- Realized by $\infty \times \infty$ matrices.
- Infinitely many Dp-branes with $\theta$
  = D(p+2)-brane with b.g. 2-form field $B_{ij}$, $\theta \sim B^{-1}$.
- Fluctuation around the b.g.
  $\to$ Gauge theory on NC $\mathbb{R}^2$.

## A Side Remark (Advertisement + Inquiry)

- IIA $\leftarrow S^1$ comactification of M-theory (11D)
- 2-form field $B_{ij}$ in IIA $\leftarrow$ 3-form field $C_{ijk}$ in M.
- Constant C-field background
  $\Rightarrow$ Non-commutativity involving 3 coordinates:

$$[X^1, X^2, X^3] = i\theta^{123}$$

- M5-brane with Nambu-Poisson structure
  $= \infty \times$ M2-brane (BLG model)
  Ho-Matsuo '08, Ho-Imamura-Matsuo-Shiba '08
- NP M5-brane has been extensively studied by many ppl in Taiwan String Theory Group, including myself.

## Example 2. Fuzzy Sphere

$$[X^i, X^j] = i\epsilon^{ijk} X^k \quad (i, j, k = 1, 2, 3) \quad (\ast)$$

- Realized by finite size matrices ($\rightarrow S^2$ is compact).
  c.f. quantization of the GUT coupling in F(uzz).
  We will call this case fuzzy.
- Corresponds to a truncation of spherical harmonics.
  $\Rightarrow$ Cannot localize arbitrarily (fuzzy).
- Induced by RR flux.
- Dp-branes with $(\ast)$
  = D(p+2)-brane wrapped on a *fuzzy* sphere.

  (We will *construct* D-branes wrapped on fuzzy spaces in this sense.)

## The 2nd Part

- In the 2nd part of the talk I will explain how to construct fuzzy del Pezzo surfaces in CY as supersymmetric vacua of quiver gauge theories on D-branes probing the CY.
- I will also comment on possible application to the structure of the Yukawa couplings in GUTs.

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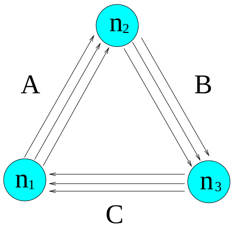
## Remarks

- We are interested in quiver gauge theories on D3-branes filling our 4D space-time.
- However, to put aside the issue of gauge anomaly for time being, we first study D0-branes.

D-branes on CY can be conveniently described by quiver diagrams.

D0-branes on $\mathbb{C}^3/\mathbb{Z}_3$



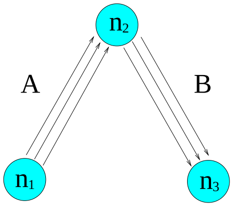

$U(n_1) \times U(n_2) \times U(n_3)$ gauge theory

$$A_i : (n_1, \bar{n}_2), \quad B_i : (n_2, \bar{n}_3), \quad C_i : (n_3, \bar{n}_1),$$
$$(i = 1, 2, 3)$$

F-term condition
($\sim$ relations in the representation of the quiver)

$$A_i B_j = A_j B_i, \quad B_i C_j = B_j C_i, \quad C_i A_j = C_j A_i$$

- We solve it by setting $C_i = 0$ ($i = 1, 2, 3$).
- Represent it by deleting the arrows between the nodes 1 and 3.
- The resulting diagram: Beilinson quiver.



D-term condition with $C_i = 0$

$$\begin{aligned}
\sum_{i=1}^{3}(A_i A_i^\dagger) &= \zeta_1 \mathbf{1}_{n_1} \\
\sum_{i=1}^{3}(B_i B_i^\dagger - A_i^\dagger A_i) &= \zeta_2 \mathbf{1}_{n_2} \\
\sum_{i=1}^{3}(-B_i^\dagger B_i) &= \zeta_3 \mathbf{1}_{n_3}
\end{aligned}$$

Looks a little bit similar to harmonic oscillators ...

∗ FI parameters $\zeta$ controls the resolution of the orbifold singularity to $\mathbb{CP}^2$.

... Indeed, we can take

$$\mathbb{C}^{n_1} = \mathcal{F}_N^{(3)}, \quad \mathbb{C}^{n_2} = \mathcal{F}_{N+1}^{(3)}, \quad \mathbb{C}^{n_3} = \mathcal{F}_{N+2}^{(3)}$$

where

$$\mathcal{F}_N^{(3)} \;=\; \left\{ |m_1, m_2, m_3\rangle;\; m_1 + m_2 + m_3 = N \right\},$$

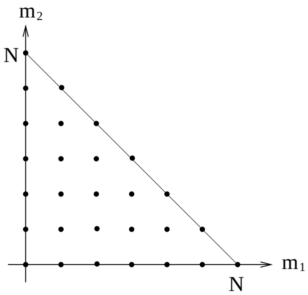
$$|m_1, m_2, m_3\rangle = \frac{(a_1^\dagger)^{m_1}(a_2^\dagger)^{m_2}(a_3^\dagger)^{m_3}}{\sqrt{m_1! m_2! m_3!}} |0\rangle$$

$$[a_i, a_j^\dagger] = \delta_{ij}$$

Remark: bi-fundamental fields are not square matrices.

dim. of these spaces can be counted as

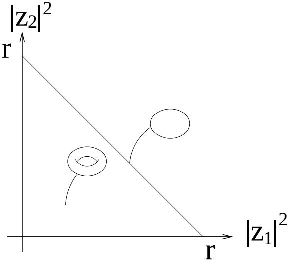
$$\dim \mathcal{F}_N^{(3)} = \frac{1}{2}(N+1)(N+2)$$



Recall the ordinary $\mathbb{CP}^2$:

$$\begin{aligned}\mathbb{CP}^2 &= \left\{(z_1, z_2, z_3);\ z_i \sim \lambda z_i\ (\lambda \in \mathbb{C}^*)\right\} \\ &= \left\{(z_1, z_2, z_3);\ \sum_i |z_i|^2 = r^2, z_i \sim e^{i\theta} z_i\right\}\end{aligned}$$

This can be represented by the toric diagram:




Fibered torus represents two relative phases of $z_i$'s.

The torus degenerates on the edges.

Remaining F- and D-term conditions can be solved by

$$A_i = c a_i \Big|_{\mathcal{F}^{(3)}_{N+1}}, \quad B_i = \tilde{c} a_i \Big|_{\mathcal{F}^{(3)}_{N+2}}$$

$c, \tilde{c}$ are related to the FI-parameters as

$$\zeta_1 = |c|^2(N+3), \quad \zeta_2 = |\tilde{c}|^2(N+4) - |c|^2(N+1),$$
$$\zeta_3 = -|\tilde{c}|^2(N+2)$$

- One can show $\sum_{r=1}^{3} n_r \zeta_r = 0$.

  $\Rightarrow$ overall $U(1)$ of the gauge group decouples.
- From the remaining $U(1)$ subgroups, we obtain the following gauge equivalence relations:

$$A_i \sim e^{i(\theta_3-\theta_2)} A_i, \quad B_i \sim e^{(\theta_2-\theta_1)} B_i \quad (i = 1, 2, 3)$$

  $\Rightarrow$ The identifications in the toric description of $\mathbb{CP}^2$.

## D-brane charges

Calculated using mirror symmetry or Beilinson construction of stable vector bundles on $\mathbb{CP}^2$

Douglas-Fiol-Romelsberger '03

$$Q_{\mathrm{D4}} = n_1 - 2n_2 + n_3,\ Q_{\mathrm{D2}} = n_2 - n_1,\ Q_{\mathrm{D0}} = \frac{n_1 + n_2}{2}$$

Our solution has

$$Q_{\mathrm{D4}} = 1, \quad Q_{\mathrm{D2}} = N + 2, \quad Q_{\mathrm{D0}} = \frac{1}{2}(N+2)^2$$

In the large volume limit

$$Z = Q_{\mathrm{D4}} + Q_{\mathrm{D2}}\omega + Q_{\mathrm{D0}}\omega^2 = e^{(N+2)\omega}$$

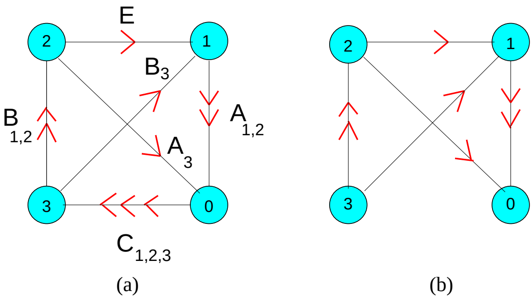
$\omega$: generator of $H^2(\mathbb{CP}^2, \mathbb{Z})$.

Our solution represents a single D4-brane wrapped on $\mathbb{CP}^2$, with $N+2$ unit of magnetic flux in its worldvolume.

## KK spectrum

- So far only algebraic structure of fuzzy $\mathbb{CP}^2$.
- Quadratic fluctuations around the solution should give KK modes on $\mathbb{CP}^2$, with the KK scale given by the FI parameters, if the background represents D4 wrapped on fuzzy $\mathbb{CP}^2$.
- We have checked that the lower excitation modes (4D view)/Higgs mass (1D view) of 1D gauge field reproduces KK modes on $\mathbb{CP}^2$, while higher excitation modes receive corrections due to the fuzziness.
- Full analysis may lead to a fuzzy version of the partially twisted D4-brane worldvolume theory, similar to the one used in F-GUTs.

## Fuzzy $dP_1$




(a) McKay quiver (b) Beilinson-Bondal quiver

Superpotential

$$\begin{aligned} W &= \mathrm{Tr}\Big[A_1EB_2C_3 - A_2EB_1C_3 \\ &\quad +A_2B_3C_1 - A_3B_2C_1 \\ &\quad +A_3B_1C_2 - A_1B_3C_2\Big] \end{aligned}$$

Again set $C = 0$. The F-term condition becomes

$$A_1EB_2 = A_2EB_1, \quad A_2B_3 = A_3B_2, \quad A_3B_1 = A_1B_3$$

D-term condition with $C_i = 0$

$$\begin{aligned}
\zeta_0 \mathbf{1}_{V_0} &= A_1 A_1^\dagger + A_2 A_2^\dagger + A_3 A_3^\dagger \\
\zeta_1 \mathbf{1}_{V_1} &= E E^\dagger + B_3 B_3^\dagger - A_1^\dagger A_1 - A_2^\dagger A_2 \\
\zeta_2 \mathbf{1}_{V_2} &= B_1 B_1^\dagger + B_2 B_2^\dagger - E^\dagger E - A_3^\dagger A_3 \\
\zeta_3 \mathbf{1}_{V_3} &= -B_1^\dagger B_1 - B_2^\dagger B_2 - B_3^\dagger B_3
\end{aligned}$$

Our solution

$$A_i = c a_i, \quad B_i = \tilde{c} a_i \quad (i = 1, 2, 3), \quad E = \sqrt{|c|^2 + |\tilde{c}|^2}\, a_4,$$

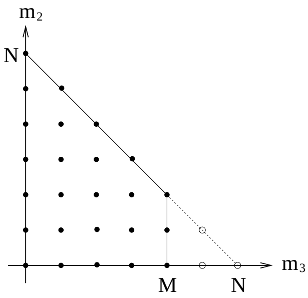
Chan-Paton spaces at the nodes:

$$V_3 = \mathcal{F}_{N+2,M+1}, \quad V_2 = \mathcal{F}_{N+1,M+1}, \quad V_1 = \mathcal{F}_{N+1,M}, \quad V_0 = \mathcal{F}_{N,M}$$

where

$$\mathcal{F}_{N,M} = \left\{ |m_1, m_2, m_3, m_4\rangle, \quad \sum_{i=1}^{3} m_i = N, \quad m_3 + m_4 = M \right\}$$

## Quantized toric diagram of $dP_1$

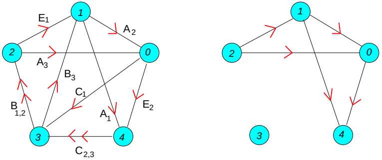



Fuzzy $\mathbb{CP}^1$:

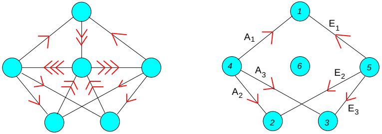
$$\mathcal{F}_{N-M}(\mathbb{CP}^1) = \left\{ |m_1, m_2, M\rangle,\ m_1 + m_2 = N - M \right\}$$

replaces the point $z_1 = z_2 = 0$ of $\mathbb{CP}^2$.

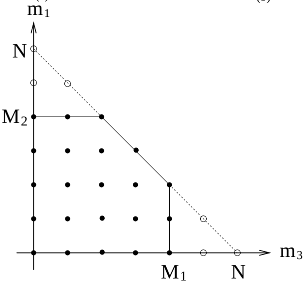
We constructed solutions for D4-branes on $dP_{2,3}$ in a similar (but less general) way.

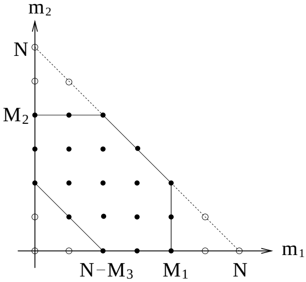



(a) (b)



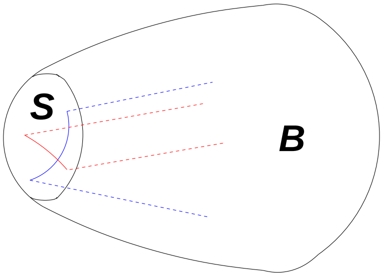

(a) (b)



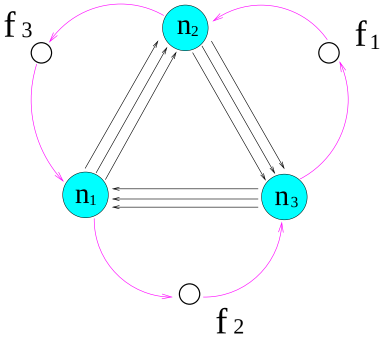



## D7-branes wrapped on fuzzy $\mathbb{CP}^2$

- If we start from D3-branes instead of D0-branes, the classical solutions represent D7-branes wrapped on fuzzy del Pezzo surfaces.
- Gauge anomaly can be canceled by adding flavor D7-branes which extend in the non-compact directions in the *CY* (in the gravity decoupling limit).



Example of the quiver with flavor D7




- Flavor D7-branes give rise to matter curves on the fuzzy/gauge D7-brane in the fuzzy vacua.

Flavor D7 wrap on $z_1 = 0$

$$\text{interaction} \sim \langle q_r | A_1 | q_{r+1} \rangle$$

In the fuzzy vacuum

$$\text{mass term} \sim \langle q_r | a_1 | q_{r+1} \rangle$$

The massless modes $|q_0\rangle$

$$a_1 |q_0\rangle = 0 \rightarrow |q_0\rangle \sim |0, m_2, m_3\rangle$$

This may be interpreted as an analogue of *fuzzy* matter (de)localization around $z_1 = 0$ in F(uzz) theory. Our construction is closer to the microscopic formulation.

Fuzzy version of the matter curves $z_1 = 0$ and $z_1 = \beta z_2$:

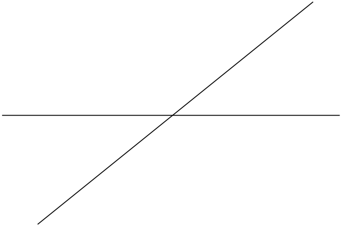
$$\begin{aligned} \ker(a_1) &= \left\{ |0, m_2, m_3\rangle, \quad m_2 + m_3 = N \right\} \\ \ker(b_1) &= \left\{ (b_2^\dagger a_3)^m |0, 0, N\rangle, \quad 0 \le m \le N \right\} \end{aligned}$$

where

$$b_1 = \frac{a_1 - \beta a_2}{\sqrt{1 + |\beta|^2}}, \quad b_2 = \frac{\bar{\beta} a_1 + a_2}{\sqrt{1 + |\beta|^2}}, \quad [b_i, b_j^\dagger] = \delta_{ij}$$

Their overlap is given by

$$\langle 0, m, N - m | 0, m', N' - m' \rangle_b = \left( \frac{1}{\sqrt{1 + |\beta|^2}} \right)^m \delta_{m,m'} \delta_{N,N'}$$



ordinary overlap



fuzzy overlap

The Yukawa couplings at low energy arise from the Yukawa couplings of the original theory. Schematically,

$$q\phi q \to q_0\phi_0 q_0$$

0 denotes the massless modes in the fuzzy $dP$ vacuum.



The fuzziness would give structures to the Yukawa couplings.

Working ...

- We have not constructed viable Yukawa couplings in the original theory which lead to realistic Yukawa in the fuzzy vacuum.
- Gauge groups other than $U(N)$ do not get along very well with fuzzy spaces. There are also restrictions on the representations, both from string theory and the fuzziness.
- There are many things one can try at this stage.

# Summary

- We constructed classical supersymmetric vacua of quiver gauge theories on Dp-branes probing *CY* which represent D(p+4)-branes wrapped on fuzzy del Pezzo surfaces.
- We explicitly solved both **D-term** and F-term conditions.

  ⇒ Can study physics around the vacua.
- Studied $\mathbb{CP}^2$ case in detail and examined D-brane charges and KK modes.

# Future Directions

- We did not attempt to classify fuzzy del Pezzo solutions in our first paper. Systematic construction may be given by brane tilings (for toric case).
- Fuzziness does not get along very well with other gauge groups ($SO, Sp, E$) which are useful in constructing realistic GUT models.
- However, no need to stick to the D7 fuzzy worldvolume interpretation since we are already working in 4D.

# Future Directions

- Lesson in the broader context:

  4D point of view

  GUT $\leftarrow$ Higgs from even larger product gauge group to *explain* Yukawa couplings from higher energy scale.
- May try (or not) to be closer to F-GUTs.
- Starting from D3-branes probing F-theory.

*Thank You !*



*... And be Fortune with You !*